LECTURE NOTES
ON

DIGITAL SIGNAL PROCESSING

III B.Tech II semester (JNTUH-R15)

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- **Introduction**:- Introduction to digital Signal Processing, Descrete time Signals & Sequences, Linear Shift Invariant Systems, Stability & Casuality, linear Constant Coefficient difference equations, Frequency domain representation of Discrete time Signals & Systems.
- **Realization of Digital filter**:- Applications of Z-transforms, Solution of difference Equations of digital filters, System Function, Stability Criterion, Frequency response of stable systems, Realization of digital filters - direct, Canonic, Cascade & Parallel forms.

A. Introduction

1) Signals, Systems & Signal Processing:-

Signal:- A Signal is defined as any physical quantity that varies with time, Space or any other independent variable or variables.

→ Mathematically, we describe a signal as a function of one or more independent Variables.

Eg:- $S_1(t) = 5t$

$S(x, y) = 3x + 2xy + 10y^2$

→ However, there are cases where such a functional relationship is unknown or too highly Complicated to be of any practical use.

→ Eg. of natural (such) signals are :

Speech Signal, EEG (Electro encephalogram),
ECG (Electro Cardiogram), temperature etc.

→ System :-

def1 :- A system is defined as an entity that acts on
Input signal & transforms it into an Output signal.
def2 :- A system is defined as a physical device that
on a signal.
Performs an operation
set of elements
def3 :- A system is defined as a
connected together
a fundamental blocks which are
in response to an Input
and Produces an Output

Signal.
Single Input and single - Output
→ Systems may be
multi - Input & multi-Output system.
systems &

→ Signal Processing :-
method of extracting Information
→ Signal Processing is a
type of
from the signal which depends on the
nature of information it carries.
Signal and the
type of signal, we have
→ Depends on the
(i) Analog Signal Processing (ii) Digital Signal
Processing

→ Signal processing is any operation that changes the characteristics of a signal. These characteristics include the amplitude, shape, phase and frequency content of the signal.

→ <u>Analog Signal Processing System</u> :- The system that process the analog signal is known as analog signal processing system.

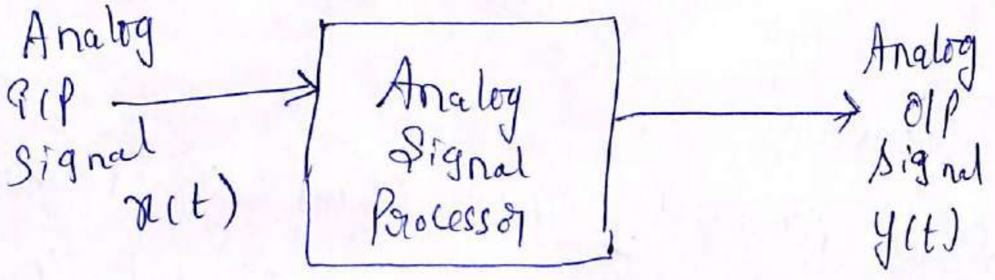


Fig: Analog Signal Processing System

→ <u>Digital Signal Processing System</u> :- The system that process the digital signal is known as digital signal processing system.

Fig: Block diagram of a digital signal processing system.

→ Most of the Signals encountered in Science and engineering are analog in nature. i.e the Signals are functions of a Continuous Variable. Such as time or Space and usually take on values in a continuous range.

(i) A/D Converter:- To Perform the Processing digitally there is a need for an interface between the analog Signal and the digital processor. This interface is called an analog to digi (A/D) Converter.

(ii) Digital Signal processor:- The digital Signal processor may be a large programmable digital Computer or a Small microprocessor programmed to Perform the desired operations on the i/p Signal. It may also be a hardwired digital Processor configured to perform a Specified set of Operations on the input signal. Programmable machines Provide the flexibility to change the Signal processing operations through a change in the Software, whereas hardwired machines are difficult to reconfigure.

→ Consequently, Programmable signal processors are in very common use. DSP-1-3

→ On the other hand, when signal processing operations are well defined, a hardwired implementation of the operations can be optimized, resulting in a cheaper signal processor and runs faster than programmable processors.

(iii) D/A Converter:-

→ In some applications where the digital O/P from the DSP (digital signal processor) is to be given to the user in analog form. Eg: Speech communication.

→ ∴ To provide the O/P signal in analog form, an interface called the digital to analog (D/A) converter is required.

→ In some applications like radar systems, there is no need of D/A converter.

→ Advantages & Limitations of Digital over Analog Signal

Advantages :-

① Flexibility in configuration :- A DSP System allows flexibility in reconfiguring the digital Signal processing operations simply by changing the program. Reconfiguration of an analog system usually implies a redesign of the hardware followed by testing & Verification to see that it operates properly.

② Accuracy :- Tolerances in analog circuit components make it extremely difficult for the system designer to control the accuracy of an analog signal processing system. On the other hand, a digital signal processor provides much better control of accuracy requirements.

③ Data Storage :- Digital Signals ~~processors~~ are easily stored on magnetic media (tape or disk) w/o loss of flexibility and can be processed off-line in a remote laboratory.

④ Cheaper :- In many applications, digital realization is comparatively cheaper than its analog part

(5) Implementation of Sophisticated algorithms :-
The DSP allows to implement sophisticated algorithms when compared to analog part.

(6) Applicability to VLF Signals :- The VLF (very low freq) signals can be easily processed using DSP compared to analog system.

(7) Time Sharing :- DSP allows the sharing of a given processor among a no of signals by time sharing thus reducing the cost of a processing of a signal.

Limitations :-

(1) System Complexity :- System complexity increases in DSP because of devices such as A/D & D/A converters & their filters.

(2) Bandwidth limited by Sampling rate :- Signals having extremely wide bandwidth require fast sampling rate A/D converters & fast digital signal processors.

(3) Power Consumption :- DSP processors require more power than analog ones.

→ Comparission :-

S.NO	Feature	Analog Processing	Digital Processing
1	Speed	Fast	Moderate
2.	cost	Low to Moderate	Moderate
3.	Flexibility	Low	high
4.	Performance	Moderate	High
5.	Self-Calibrating	NO	Yes
6.	Data-logging capability	NO	Yes
7.	Adaptive Capability	Limited	Yes

→ Classification of Signals :-

Signals

- Continuous-time Signals
- Discrete-time Signals
- digital Signal.

(i) Continuous-time Signals :- The Signals that are defin for every instant of time are known as Continu time Signals. They are denoted by x(t).

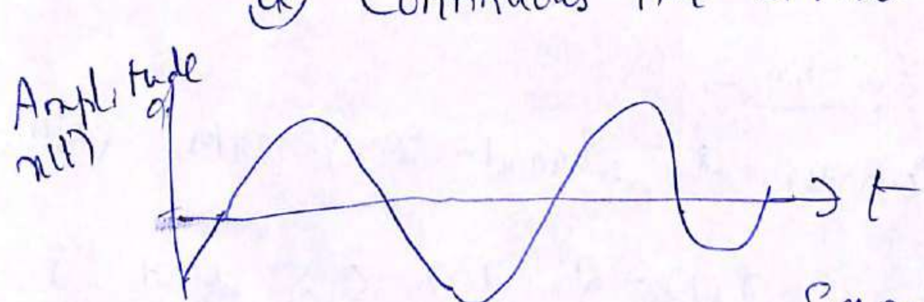
(ii) Discrete-time Signals :- The signals that are del at discrete instants of time are known as discrete-time Signals. The are continuous in amplitude and discrete in time. These are denoted by $x(n)$.

(iii) <u>Digital Signal</u>:- The signals that are discrete in time and quantized in amplitude are digital signals.

<u>Eg</u>:-

(i)

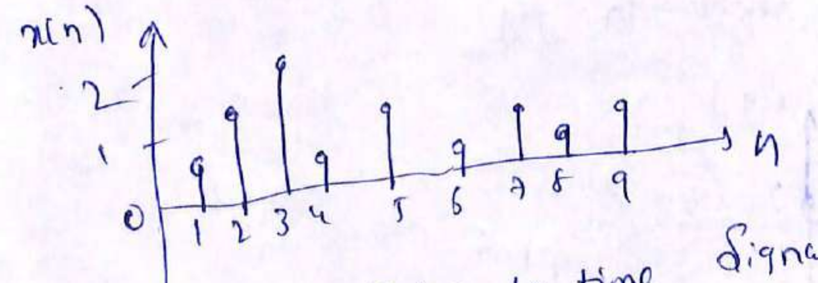
(a) Continuous time Signal (ECG Signal)



(b) Continuous - time Signal.

Continuous in both time & Amplitude.

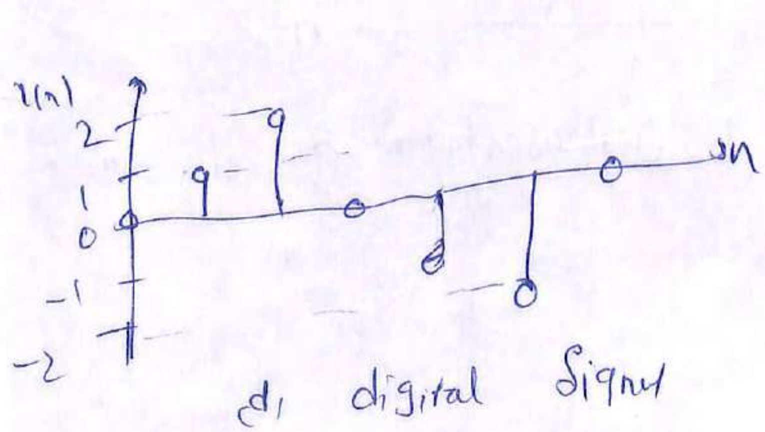
(ii)



(c) Discrete-time Signal

→ Discrete in time & Continuous in amplitude.

(iii)



(d) digital Signal

→ Discrete in both time & amplitude.

(3). Representation of Discrete-time Signals :-

→ Discrete time Signals are defined only at discrete instants of time.

→ There are four different types of representations for discrete-time signals. They are :

1. Graphical representation.
2. Functional "
3. Tabular "
4. Sequence "

1. Graphical representation :-

Let us consider a Signal $x(n)$ with values

$x(-1) = 1$; $x(0) = 2$; $x(1) = 2$; $x(2) = 0.5$ and $x(3) = 1$

This discrete-time Signal can be represented graphically as shown in fig!

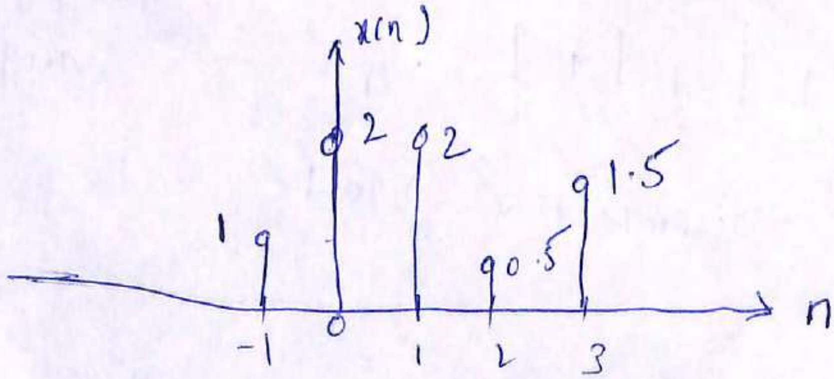


fig: Graphical representation of discrete-time S

(ii) Functional Representation:-

$$x(n) = \begin{cases} 1 & \text{for } n = -1 \\ 2 & \text{for } n = 0, 1 \\ 0.5 & \text{for } n = 2 \\ 1.5 & \text{for } n = 3 \\ 0 & \text{o.w} \end{cases}$$

(iii) Tabular representation:-

n	-1	0	1	2	3
$x(n)$	1	2	2	0.5	1.5

(iv) Sequence representation:-

$x(x) = \{ \cdots 0,$

$$x(n) = \{1, \underset{\uparrow}{2}, 2, 0.5, 1.5\}$$

Elementary Discrete-time Signals:-

1. Unit Step Sequence
2. Unit ramp Sequence
3. Unit Parabolic Sequence
4. Unit impulse Sequence
5. Sinusodial Sequence
6. Real exponential Sequence
7. Complex exponential Sequence.

1. <u>Unit Step Sequence</u> :-

$$u(n) = 1 \quad \text{for } n \geq 0$$
$$= 0 \quad \text{for } n < 0$$

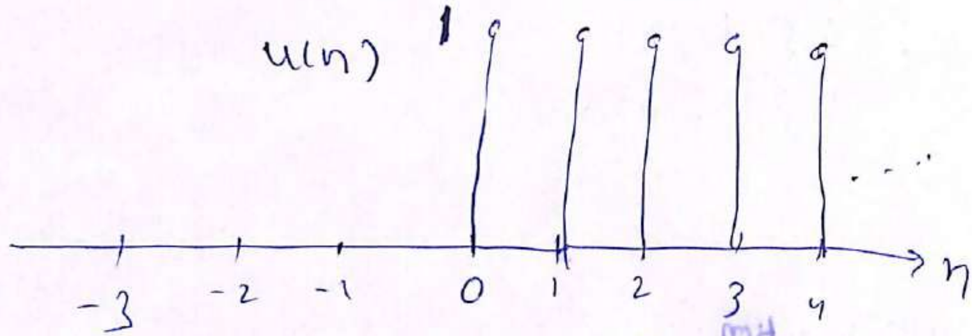
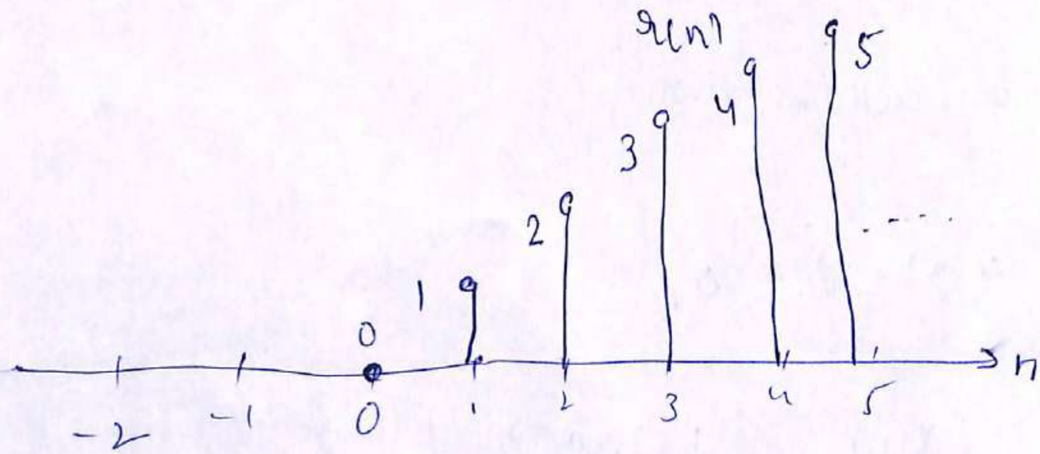


fig: Unit Step Sequence

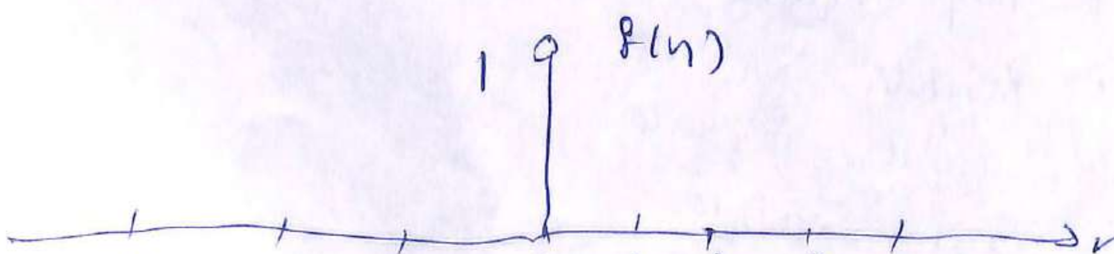
2. <u>Unit ramp Sequence</u> :-

$$r(n) = n \quad \text{for } n \geq 0$$
$$= 0 \quad \text{for } n < 0$$



3. <u>Unit Sample (Unit impulse) Sequence</u> :-

$$\delta(n) = 1 \quad \text{for } n = 0$$
$$= 0 \quad \text{for } n \neq 0$$



(4) Exponential Sequence:-

$$x(n) = a^n \quad \forall n$$

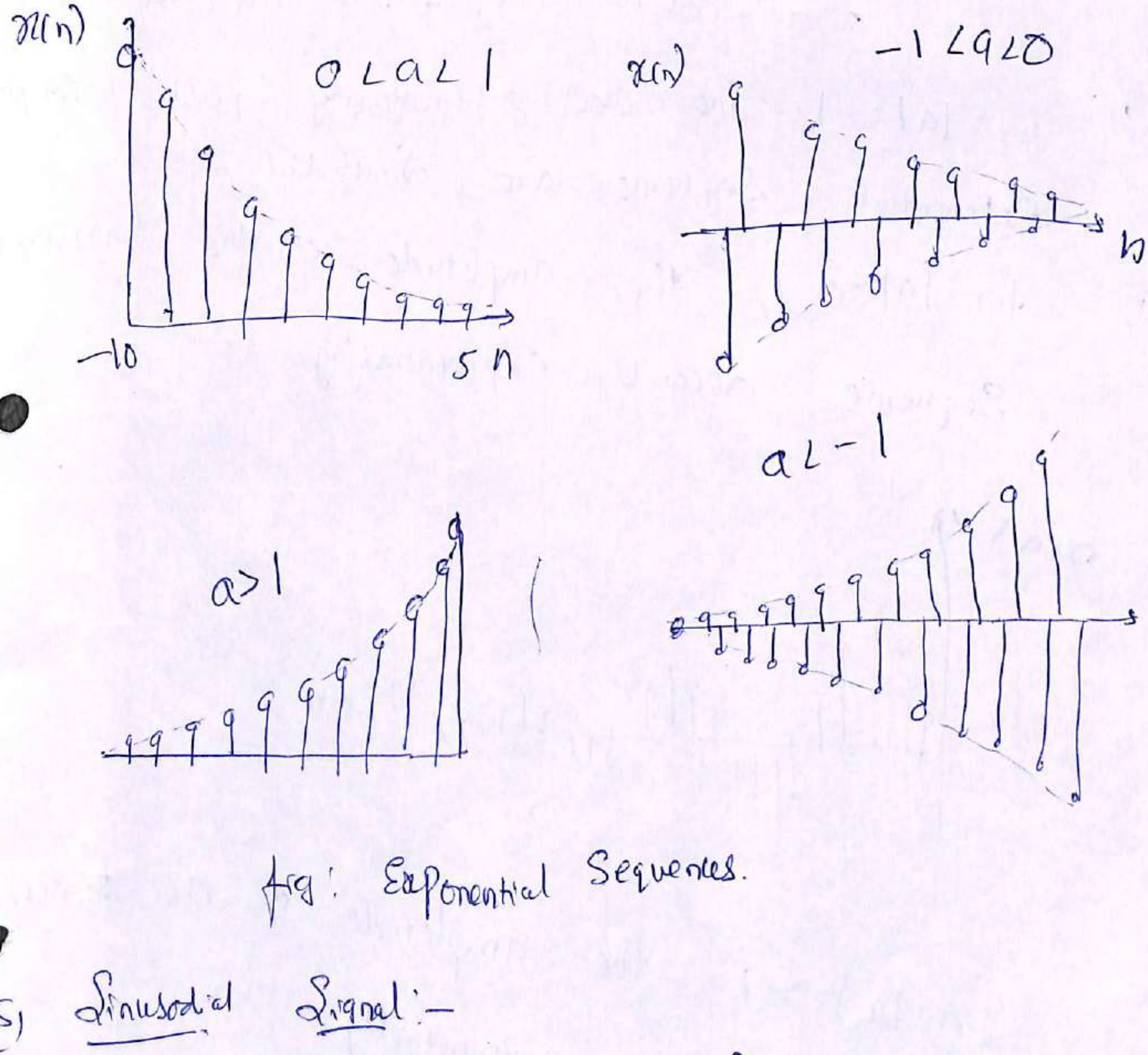


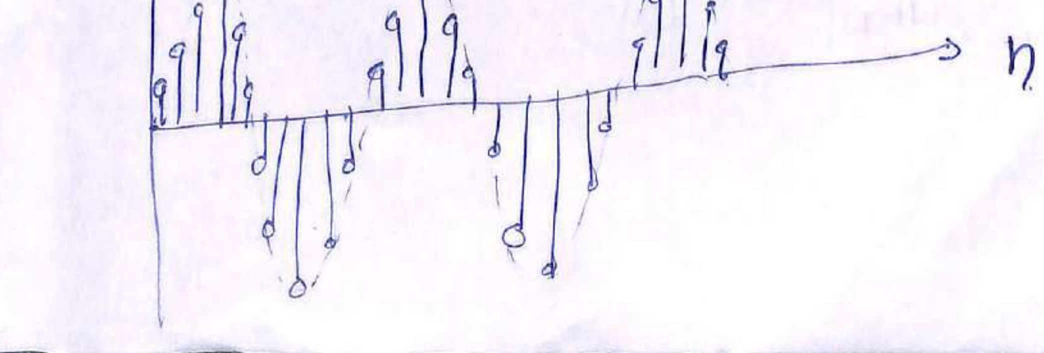
fig: Exponential Sequences.

5) Sinusoidal Signal:-

$$x(n) = A \cdot \cos(\omega_0 n + \phi)$$

$\omega_0 \to$ freq.

$\phi \to$ phase



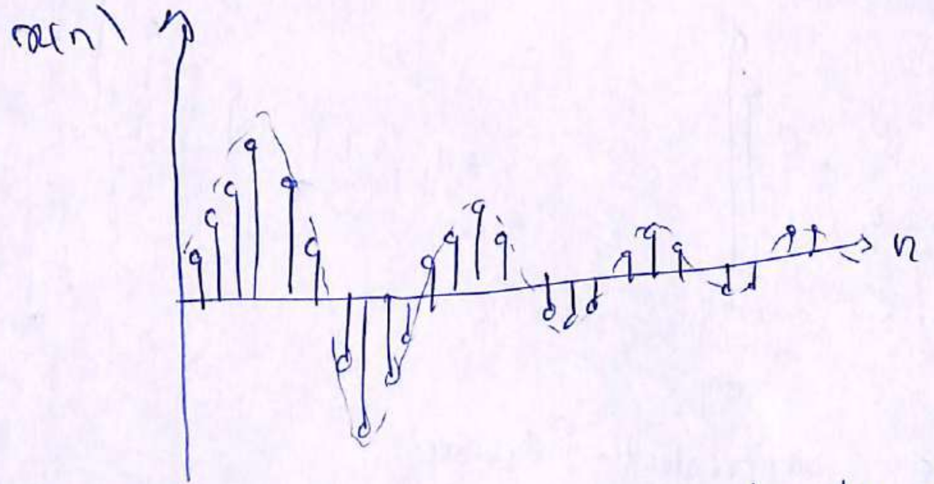
(6) Complex exponential Signal:-

$$x(n) = a^n \cdot e^{j(\omega_0 n + \phi)}$$

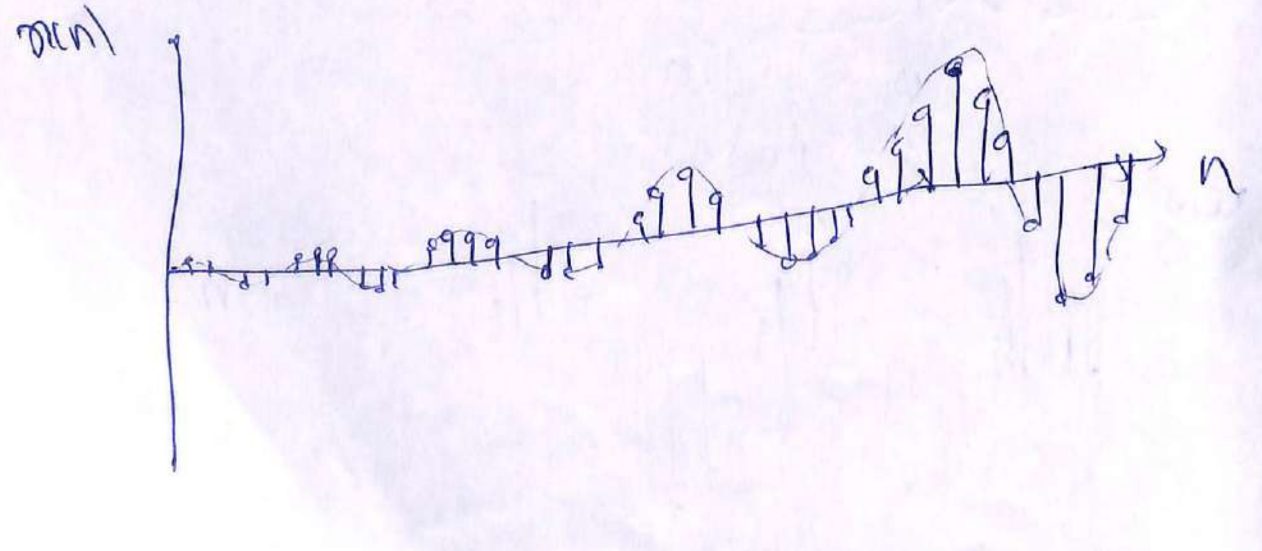
$$= a^n \cdot \cos(\omega_0 n + \phi) + j \cdot a^n \sin(\omega_0 n + \phi)$$

(i) For $|a| = 1$, the real & imaginary parts of complex exponential sequence are sinusodial.

(ii) for $|a| < 1$, the amplitude of the sinusodial sequence decays exponentially.



(iii) for $|a| > 1$, the amplitude of the sinusodi sequence increases exponentially.



Applications of DSP :-

1. Telecommunications :- Echo Cancellation in telephone networks, Telephone dialling application, Modems, line repeaters, channel multiplexing, Data encryption, Video Conferencing, Cellular phone, FAX.

2. Consumer Electronics :- FM stereo applications, Digital Audio / TV, electronic music synthesizer, educational toys etc.

3. Instrumentation and Control :- Spectrum analysis, Digital filters, PLL, function generator etc.

4. Image processing :- Image Compression, Image enhancement, Image analysis & recognition.

5. Medicine :- CT, X-ray Scanning, Spectrum analysis of ECG & EEG, Patient monitoring.

6. Speech Processing :- Automatic Speech recognition, Speaker verification, Speaker Identification, Speech synthesis.

7. Seismology :- DSP techniques are employed in the geophysical exploration for oil & gas etc.

8. Military :- Radar Signal processing, Sonar Signal processing, Navigation, Secure Communications.

Basic Operations on Sequences :-

1. Time Shifting
2. Time reversal
3. Time Scaling
4. Amplitude Scaling
5. Signal addition
6. Signal multiplication.

1. Time Shifting :-

→ The time shifting of a signal may result in time delay or time advance.

$$y(n) = x(n-k)$$

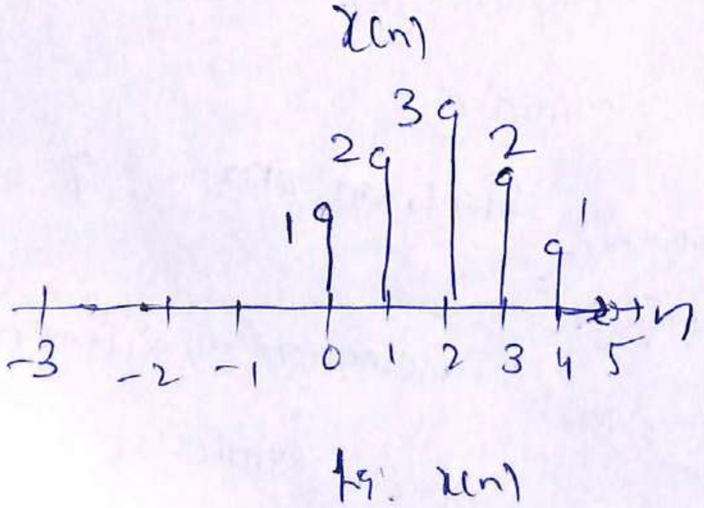
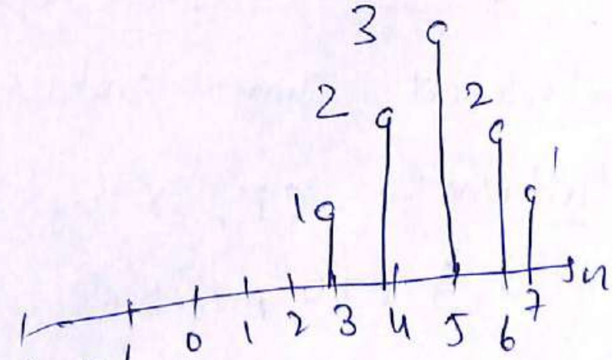
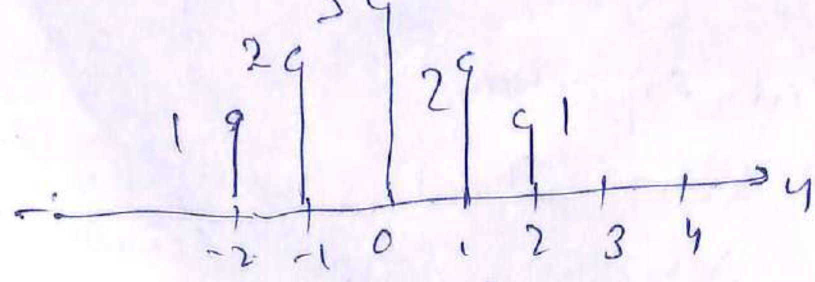


fig: $x(n)$



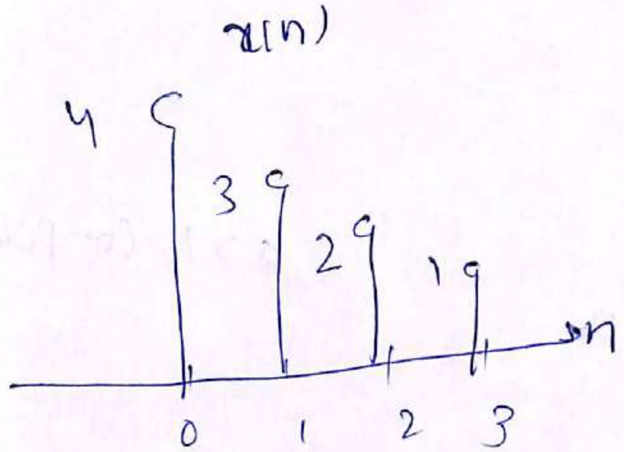
$x(n-3)$

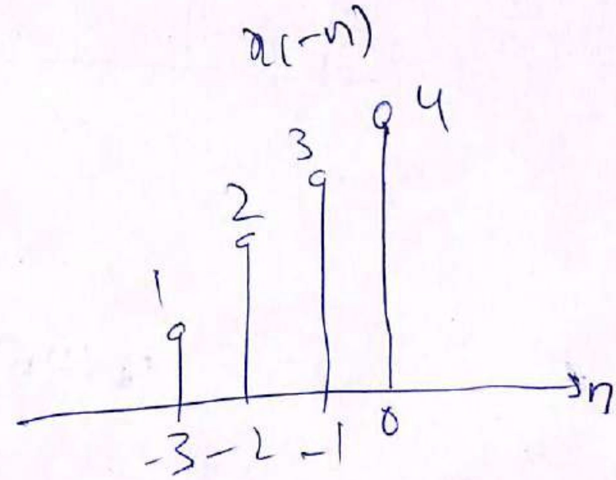


$x(n+2)$

(2) Time Reversal:-

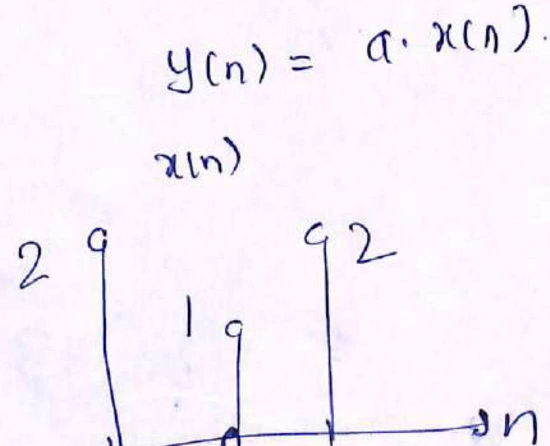
The time reversal also called time folding of a discrete-time signal $x(n)$ can be obtained by folding the sequence about $n=0$.

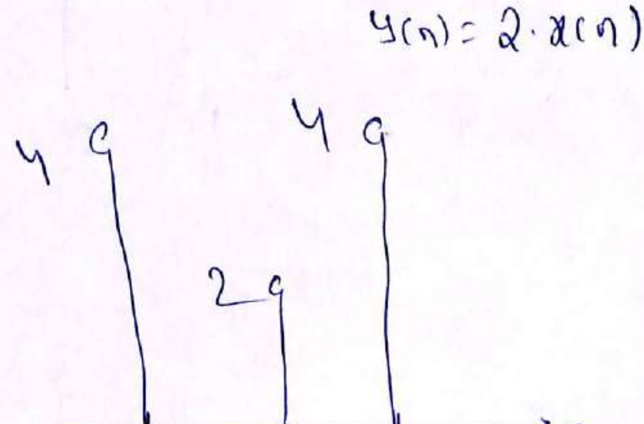




3) Amplitude Scaling:-

$$y(n) = a \cdot x(n).$$





(4) Time Scaling:-

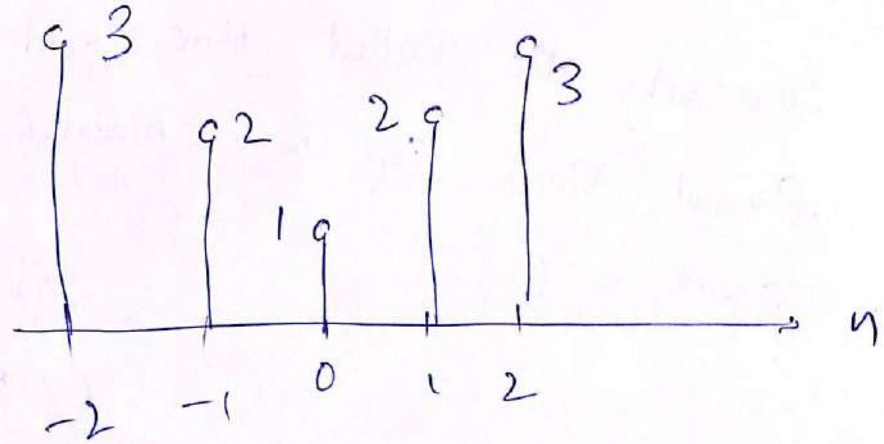
→ Time scaling may be time expansion & time compression.

$$y(n) = x(an).$$

If $a > 1$, time compression.

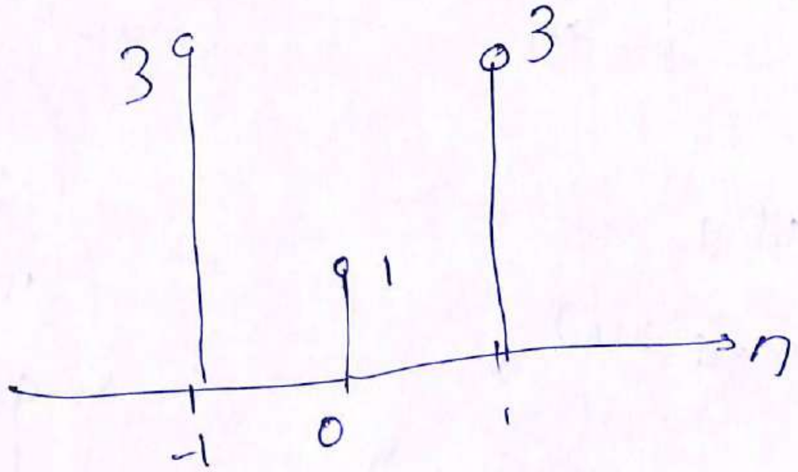
$a < 1$, time expansion.

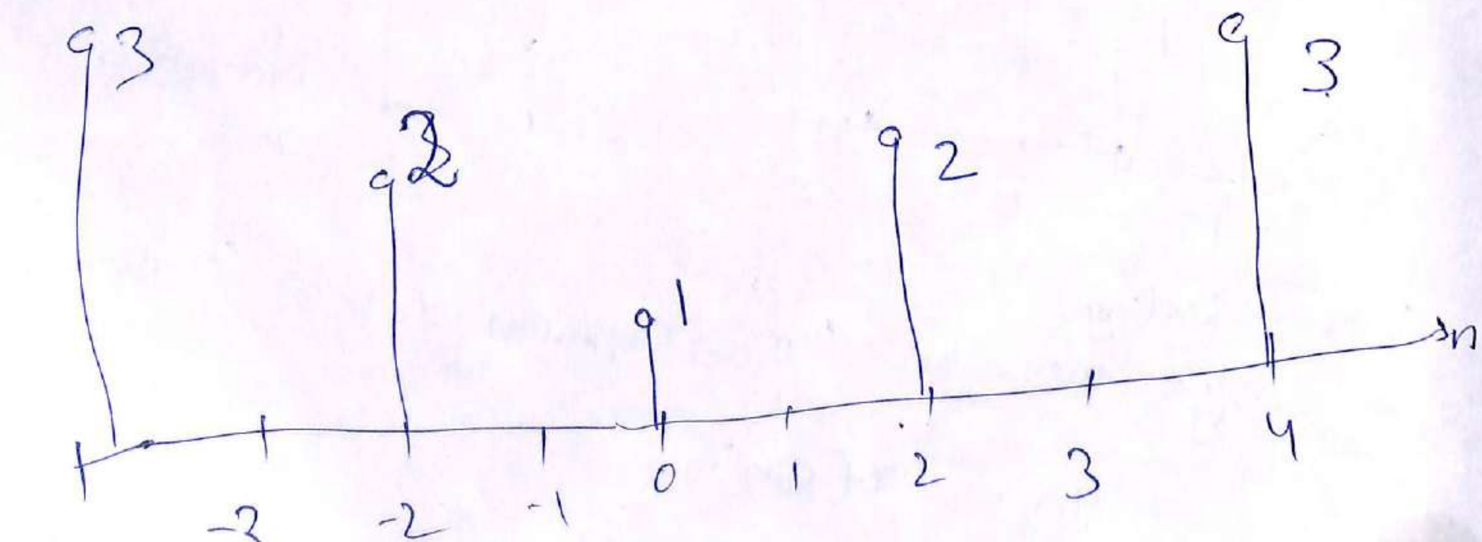
$x(n)$



$x(2n)$

$a>1$, compression





(5) Signal addition:-

$$x(n) = x_1(n) + x_2(n)$$

(1114) $$x(n) = x_1(n) - x_2(n)$$

(6) Signal multiplication:-

$$x(n) = x_1(n) \cdot x_2(n).$$

→ classification of Discrete-time Signals:-

(1) Deterministic & Random Signals
(2) Periodic & non-Periodic Signals
(3) Energy & Power Signals
(4) causual & non-causual Signals
(5) Even & Odd Signals. (Symmetric & Asymmetric)

(1) Deterministic & Random Signals:-

(a) Deterministic:- A signal exhibiting no uncertainity of its magnitude & phase at any given instant of time is called "deterministic Signal". It can be completely represented by mathematical equation. At any time, its nature & amplitude can be completely Predicted.

Eg:- $x(n) = \cos \omega n$,

$x(n) = e^{j\omega n}$ etc.

(b) Nondeterministic:- A signal exhibiting uncertainity of its magnitude & phase at any given instant of time is called "Non deterministic" signal. or "Random Signal".

→ A random signal can't be represented by any mathematical equation. The behaviour of such a signal is probablistic in nature & can be analyzed only stochastically.

→ The pattern of such a signal is quite irregular. Its amplitude & phase at any time instant can't be predicted in advance.

→ Eg: thermal noise.

(2) <u>Periodic</u> & <u>Aperiodic Signals</u>:—

<u>Periodic</u>:— A signal is said to be periodic with period N (N>0) if and only if

$$\boxed{x(n+N) = x(n) \quad \forall\, n.}$$

→ A periodic signal has a definite pattern & repeats itself at regular intervals of time.

→ The smallest value of N for which satisfies the above equation is called fundamental period.

Eg:- $x(n) = A \cdot \sin 2\pi f_0 n$ is periodic when f_0 is a rational number i.e

$\omega_0 = \frac{2\pi \cdot k}{N}$ or $f_0 = \frac{k}{N}$ where k & N are integers.

→ The energy of a periodic signal over $-\infty \le n \le \infty$ is infinite & Avg. Power is finite.

∴ Periodic signals are power signals.

$$N = \frac{2\pi}{\omega_0} \cdot k \quad \text{or} \quad \frac{k}{f_0}$$

Problems.

DSP-1-11

Eg:- (i) Show that the Complex exponential sequence

$x(n) = e^{j\omega_0 n}$ is periodic only if $\frac{\omega_0}{2\pi}$ is a rational no.

Sol

$$x(n) = e^{j\omega_0 n}$$

For Periodic, $x(n) = x(n+N)$

$$e^{j\omega_0 n} = e^{j\omega_0 (n+N)}$$

$$e^{j\omega_0 n} = e^{j\omega_0 n} \cdot e^{j\omega_0 N}$$

it is satisfied only if,

$$e^{j\omega_0 N} = 1.$$

This is true only if,

$$\omega_0 N = 2\pi k.$$

$$f_0 = \frac{\omega_0}{2\pi} = \frac{k}{N} \text{ is a rational no.}$$

(ii) Determine whether the following discrete-time signals are periodic or not. If Periodic, determine the fundamental Period.

(a) $\sin(0.02\pi n)$ (b) $\sin(5\pi n)$

(c) $\cos 4n$ (d) $\sin \frac{2\pi n}{3} + \cos \frac{2\pi n}{5}$

(e) $\cos(\frac{n}{6}) \cdot \cos(\frac{n\pi}{6})$ (f) $\cos(\frac{\pi}{2} + 0.3n)$

(g) $e^{j(\pi/2)n}$ (h) $1 + e^{j\frac{2\pi n}{3}} - e^{j4\pi n/7}$

Sol a) $x(n) = \sin(0.02\pi n)$

compare with $\sin(2\pi f_0 N)$ or $\sin(\omega_0 N)$

$$2\pi f_0 = 0.02\pi$$

$$f_0 = \frac{0.02\pi}{2\pi} = 0.01$$

$$f_0 = \frac{1}{100} = \frac{k}{N}$$

$\because f$ is expressed as ratio of 2 integers $k = 1$ & $N = 100$. It is rational.

$\therefore N = 100$.

$$N = 2\pi\left(\frac{k}{\omega_0}\right)$$

$$= 2\pi\left(\frac{k}{0.02\pi}\right)$$

$\therefore$ Period, $N = 100$.

$$N = 100k$$

for $k = 1$

$\boxed{N = 100}$

b) $x(n) = \sin(5\pi n)$

Compare with $x(n) = \sin(2\pi f_0 n)$

$$2\pi f_0 = 5\pi$$

$$f_0 = \frac{5}{2} = \frac{k}{N} = \frac{1}{2/5}$$

$k = 5$ & $N = 2$ (two integers).

$\therefore N = 2/5$

c) $x(n) = \sin\frac{2\pi n}{3} + \cos\frac{2\pi n}{5}$

Compare with, $\sin 2\pi f_1 n + \cos 2\pi f_2 n$.

$$2\pi f_1 = \frac{2\pi}{3}, \quad 2\pi f_2 = \frac{2\pi}{5}$$

$$f_1 = \frac{1}{3} = \frac{k_1}{N_1} \qquad f_2 = \frac{1}{5} = \frac{k_2}{N_2}.$$

$\frac{N_1}{N_2} = \frac{3}{5}$ is ratio of two integers. the

Sequence is Periodic.

→ The Period of $x(n)$ is LCM of N_1 & N_2 (ie 3 & 5)

$\therefore$ $N = 15$ (LCM of 3 & 5).

e) $$x(n) = \cos\left(\frac{n}{6}\right) \cdot \cos\left(\frac{n\pi}{6}\right)$$

Compare with. $x(n) = \cos(2\pi f_1 n) \cdot \cos(2\pi f_2 n)$

$$2\pi f_1 = \frac{1}{6}, \quad 2\pi f_2 = \frac{\pi}{6}$$

$$f_1 = \frac{1}{12\pi}, \quad f_2 = \frac{1}{12}$$

Not rational, rational.

$\therefore$ $x(n)$ is non-Periodic.

f) $$x(n) = \cos\left(\frac{\pi}{2} + 0.3n\right)$$

Compare with. $x(n) = \cos(2\pi f n + \theta)$

$$2\pi f = 0.3, \quad \theta = \frac{\pi}{2}.$$

$$f = \frac{0.3}{2\pi} = \frac{3}{20\pi} \Rightarrow \text{irrational.}$$

$\therefore$ $x(n)$ is aperiodic.

g) $$x(n) = e^{j(\pi/2)n}$$

Compare with $x(n) = e^{j2\pi f n}$.

$$2\pi f = \frac{\pi}{2} \Rightarrow f = \frac{1}{4} = \frac{k}{N}$$

$\therefore$ $x(n)$ is Periodic with $N = 4$.

h, $x(n) = 1 + e^{j2\pi n/3} - e^{j4\pi n/7}$

$x_1(n) = 1,\quad x_2(n) = e^{j2\pi n/3},\quad x_3(n) = e^{j4\pi n/7}$

→ $x_1(n) = 1$ is a d.c signal with arbitrary Period of '1'.

$\therefore N_1 = 1$

→ $x_2(n) = e^{j2\pi n/3} = e^{j2\pi f_2 n}$

$$\frac{2\pi}{3} = 2\pi f_2$$

$$\Rightarrow f_2 = \frac{1}{3} = \frac{k_2}{N_2}$$

$$N_2 = 3$$

→ $x_3(n) = e^{j4\pi n/7} = e^{j2\pi f_3 n}$

$$\frac{4\pi}{7} = 2\pi f_3$$

$$\Rightarrow f_3 = \frac{2}{7} = \frac{k_3}{N_3} = \frac{1}{7/2}$$

$N_3 = \frac{7}{2} \times k_3$

~~$N_3 = 7$~~. $k_3 = 2,$

$N_3 = 7$. integer

~~$N = 2\pi$~~

$N_1 = 1,\quad N_2 = 3,\quad N_3 = 7$

L.C.M of $N_1, N_2 \& N_3 = 1 \times 3 \times 7 = 21$

$\therefore$ $x(n)$ is Periodic with $N = 21$

2, $x(n) = e^{j6\pi n}$, (ii) $x(n) = e^{j(\frac{3}{5})(n+\frac{1}{2})}$

(iii) $x(n) = \cos(\frac{2\pi}{3})n$ (iv) $\cos\frac{\pi}{3}n + \cos\frac{3\pi}{4}n$.

(3) Energy & Power Signals :-

Energy :- The energy E of a signal $x(n)$ is defined as,

$$\boxed{E = \sum_{n=-\infty}^{\infty} |x(n)|^2}$$

→ The energy of a signal can be finite or infinite.

→ If E is finite, then $x(n)$ is called an energy signal.

→ If we define the energy signal of $x(n)$ over $-N \le n \le N$ as,

$$E_N = \sum_{n=-N}^{N} |x(n)|^2$$

then the signal energy,

$$E = \lim_{N \to \infty} E_N$$

→ Power Signal :- The average power of a discrete-time signal $x(n)$ is defined as,

$$\boxed{P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2}$$

→ If 'P' is finite, the signal is called a power signal.

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \cdot E_N$$

→ If E is finite, $P = 0$.

→ If E is infinite, P may be finite or infinite.

Note:-

1) If E is finite, $P=0$

2) If E is infinite, P may be finite or infinite.

3) Any Periodic signal over $-\infty \leq n \leq \infty$ can't be energy signal. It is Power signal.

4) A signal can't be both Energy & Power signal.

5) A signal can be neither energy nor power signal.

6) Unit Step signal is a power signal. ($\because$ $E=\infty$, $P=$ finite)

7) Complex exponential signal is a Power signal.

8) Unit ramp sequence is neither a Power signal nor an energy signal.

9) For energy signal, E is finite (ie $E<\infty$) & $P=0$

10) For Power signal, P is finite (ie $P<\infty$) & $E=\infty$.

Problems

DSP-1-14

1) Determine the energy & Power of a unit step sequence.

Sol

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} |u(n)|^2$$

$$= \sum_{n=0}^{\infty} (1)^2 = \infty.$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \cdot \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \cdot \sum_{n=-N}^{N} |u(n)|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \cdot \sum_{n=0}^{N} (1)^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \cdot (N+1) = \lim_{N \to \infty} \frac{1+1/N}{2+1/N} = \frac{1}{2} < \infty.$$

$\therefore$ Unit step sequence is a Power signal.

2) Determine the values of energy and power of the following signals. Find whether the signals are power, energy & neighter energy nor power signals.

(i) $x(n) = (\frac{1}{3})^n u(n)$ (ii) $e^{j(\frac{\pi}{2}n + \frac{\pi}{4})}$

(iii) $x(n) = \sin(\frac{\pi}{4}n)$ (iv) $x(n) = e^{2n}u(n)$

Sol) (i) $x(n) = (\frac{1}{3})^n u(n)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} |(\frac{1}{3})^n u(n)|^2$$

$$= \sum_{n=0}^{\infty} |(\frac{1}{3})^n|^2 \quad \left[\because u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}\right]$$

$$= \sum_{n=0}^{\infty} (\frac{1}{9})^n \quad \left[\because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}\right]$$

$$= \frac{1}{1-\frac{1}{9}} = \frac{9}{8}$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |(\frac{1}{3})^n u(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} (\frac{1}{3})^{2n}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1}\left[\frac{1-\left(\frac{1}{9}\right)^{N+1}}{1-\frac{1}{9}}\right] \quad \because \sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}$$

if $a < 1$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} * \left(1-\left(\frac{1}{9}\right)^{N+1}\right) \cdot \frac{8}{9}$$

If $N \to \infty$, we are getting $\frac{\infty}{\infty}$. So, use L-hospitals rule.

$P = 0$

$$\cancel{\lim_{N\to\infty} \frac{-(N+1)\left(\frac{1}{9}\right)^{N}}{2}}$$

$P = -\infty$ ~~Not~~ Power signal.

The signal is an energy signal

(ii) $x(n) = e^{j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)}$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} \left|e^{j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)}\right|^2$$

$$= \sum_{n=-\infty}^{\infty} 1^2 = \infty \quad \left(\because |e^{j\theta}| = |\cos\theta + j\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1.\right)$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \cdot \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \cdot \sum_{n=-N}^{N} \left| e^{j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)} \right|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \cdot \sum_{n=-N}^{N} 1.$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \cdot 2N+1 = \lim_{N \to \infty} 1$$

$$P = 1$$

$\because$ Energy is infinite & power is finite, the above signal is a power signal.

(iii) $x(n) = \sin\left(\frac{\pi}{4} n\right)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} \left| \sin\left(\frac{\pi}{4} n\right) \right|^2$$

$$= \sum_{n=-\infty}^{\infty} \left| \sin^2\left(\frac{\pi}{4} n\right) \right| \qquad \because \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \sum_{n=-\infty}^{\infty} \left| \frac{1-\cos \frac{\pi}{2} n}{2} \right|$$

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$$= \frac{1}{2} \cdot \sum_{n=-\infty}^{\infty} \left| 1 - \cos \frac{\pi}{2} n \right|$$

$$E = \frac{1}{2} [\infty] = \infty.$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \cdot \sum_{n=-N}^{N} \left| \sin^2 \left(\frac{\pi}{4} n\right) \right|$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \cdot \sum_{n=-N}^{N} \left| \frac{1 - \cos \frac{\pi}{2} n}{2} \right|$$

$$= \frac{1}{2} \cdot \lim_{N \to \infty} \frac{1}{2N+1} \cdot \sum_{n=-N}^{N} 1 - \cos \frac{\pi}{2} n$$

$$= \frac{1}{2} \cdot \lim_{N \to \infty} \frac{1}{2N+1} \cdot \sum_{n=-N}^{N} 1.$$

$$= \frac{1}{2} \cdot \lim_{N \to \infty} \frac{1}{\cancel{2N+1}} \cdot \cancel{2N+1}$$

$$P = \frac{1}{2} < \infty$$

$\therefore$ The energy is infinite & the Power is finite.

The signal is a power signal.

(iv) $x(n) = e^{2n} \cdot u(n)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} |e^{2n} u(n)|^2$$

$$= \sum_{n=0}^{\infty} e^{4n} \qquad \left(\because u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}\right)$$

$$E = 1 + e^4 + e^8 + \cdots \infty = \infty$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \cdot \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |e^{2n} u(n)|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \cdot \sum_{n=0}^{N} e^{4n}$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \cdot \left[1 + e^4 + e^8 + \cdots + e^{4N}\right]$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \cdot \left[\frac{e^{4(N+1)} - 1}{e^4 - 1}\right] \qquad \left(\because \sum_{n=0}^{N} a^n = \frac{a^{N+1} - 1}{a - 1} \text{ if } a > 1\right)$$

use L. hospitalls rule

$$= \lim_{N \to \infty} \frac{4 \cdot e^{4(N+1)}}{(e^4 - 1)(2)}$$

$$P = \infty$$

$\because$ Both Energy & Power are infine, the signal is neighther energy nor Power signal

(4) Causual & Non causual Signal:-

→ A signal $x(n)$ is said to be causual if its value is zero for $n<0$. O.W it is non-causual

Eg: $a^n . u(n)$ → causual signal.

$a^n . u(-n+1)$ → Non causual signal.

5) Symmetric (even) & Asymmetric (odd) Signals:-

→ A signal $x(n)$ is said to be symmetric (even) if it satisfies the condition

$$\boxed{x(-n) = x(n)} \quad \forall n. \quad -(1)$$

→ A signal $x(n)$ is said to be asymmetric (odd) if it satisfies the condition

$$\boxed{x(-n) = -x(n)} \quad \forall n \quad -(2)$$

→ Any signal $x(n)$ can be expressed as the sum of even & odd Components

$$\boxed{x(n) = x_e(n) + x_o(n)} \quad -(3)$$

Put $n = -n$

$$x(-n) = x_e(-n) + x_o(-n)$$

$$x(-n) = x_e(n) - x_o(n) \quad \text{(from eq-(1) \& (2))} \quad -(4)$$

(3)+(4) $\quad x(n) + x(-n) = 2 . x_e(n)$

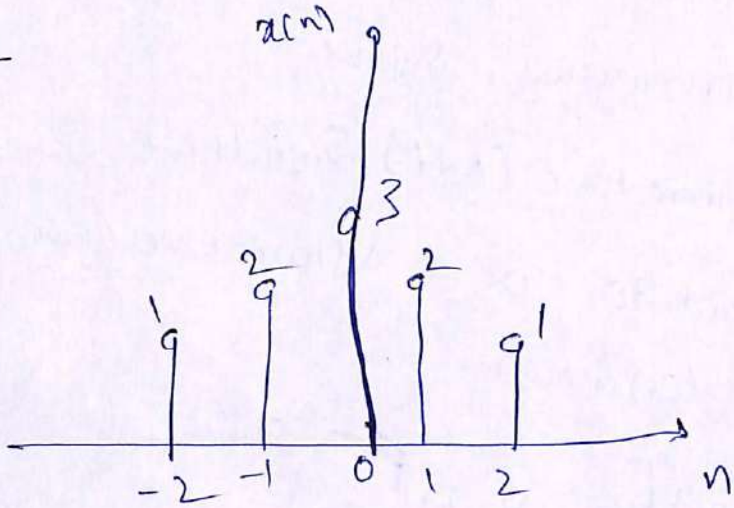
$$\boxed{x_e(n) = \frac{x(n) + x(-n)}{2}}$$

(37)-(4)

$$x(n) - x(-n) = 2 \cdot x_o(n)$$

$$\boxed{x_o(n) = \frac{x(n) - x(-n)}{2}}$$

Eg:



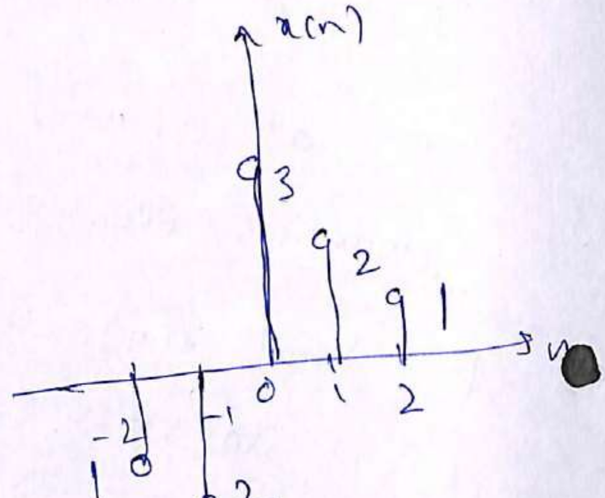


fig: Symmetric (even) signal)

fig: Asymmetric (odd) signal.

→ <u>Classification of Discrete-time Systems</u>:-

1. Static & Dynamic Systems
2. Causual & non causual Systems
3. Linear & non linear Systems
4. Time Variant & Time invariant System
5. FIR & IIR System
6. Stable & Unstable Systems

1. <u>Static & Dynamic Systems</u>:-

→ A discrete time system is called static or memoryless if its O/P at any instant 'n' depends on the i/p samples at the same time but not on past & future samples of i/p. O.W it is said to be dynamic or memory system.

Eg:- $y(n) = ax(n)$ → Static (or memoryless)

$y(n) = x(n-1) + x(n+1)$ → (dynamic)

$y(n) = x(n) \cdot x(n-1)$ → dynamic.

(2) Causual & Noncausual Systems:-

→ A system is said to be causual if the o/p of the system at any time n depends only at present & past i/p's but does not on future i/p's.

→ If the o/p of the system depends on future i/p's, the system is said to be non-causual.

Eg:- $y(n) = x(n) + x(n-1)$ → causual system

$y(n) = a \cdot x(2n)$ → Non-causual system

(3) Linear & Non linear Systems:-

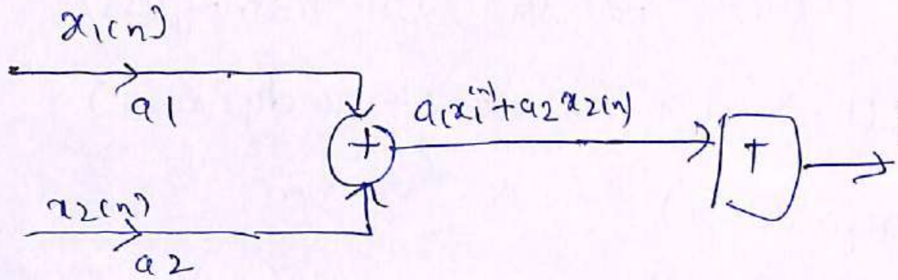
→ A system is called linear system if it satisfies the superposition theorem.

→ The superposition theorem states that the response of the system to a weighted sum of signals be equal to the corresponding weighted sum of responses of the system to each of the individual i/p signals.

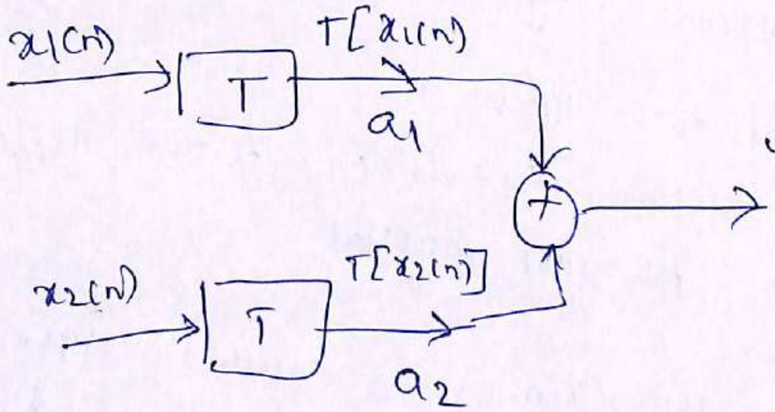
→ A system is linear if

$$\boxed{T[a_1 x_1(n) + a_2 \cdot x_2(n)] = a_1 \, T[x_1(n)] + a_2 \, T[x_2(n)]}$$

for any arbitrary i/p sequences $x_1(n)$ & $x_2(n)$. & any arbitrary constants a_1 & a_2.



$y(n) = T[a_1x_1(n) + a_2x_2(n)]$



$y'(n) = a_1 \cdot T[x_1(n)] + a_2 \cdot T[x_2(n)]$
$= a_1 \cdot y_1(n) + a_2 \cdot y_2(n)$

if $y(n) = y'(n) \Rightarrow$ linear

o.w $\quad y(n) \neq y'(n) \Rightarrow$ Non-linear

Eg:- Determine if the systems described by the following i/p - o/p equations are linear or non-linear.

(a) $y(n) = n \cdot x(n)$ (b) $y(n) = x(n^2)$ (c) $y(n) = x^2(n)$

(d) $y(n) = A \cdot x(n) + B$ (e) $y(n) = e^{x(n)}$

Sol (a) $y(n) = n \cdot x(n)$.

Cond. for linear system. $\boxed{T[a_1x_1(n) + a_2x_2(n)] = a_1 T[x_1(n)] + a_2 \cdot T[x_2(n)]}$

$y(n) = T[a_1x_1(n) + a_2x_2(n)] = n \cdot [a_1x_1(n) + a_2x_2(n)]$

Here ~~Fray~~ $T[\;] \to n$.

$y_1(n) = T[x_1(n)] = n \cdot x_1(n)$

$y_2(n) = T[x_2(n)] = n \cdot x_2(n)$

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$y'(n) = a_1 \cdot y_1(n) + a_2 y_2(n)$

$= a_1 \cdot n \cdot x_1(n) + a_2 \cdot n \cdot x_2(n)$

$y'(n) = n [a_1 x_1(n) + a_2 x_2(n)]$

$\because y(n) = y'(n) \Rightarrow$ System is linear

b) $y(n) = x(n^2)$

$y'(n) = T[a_1 \cdot x_1(n) + a_2 x_2(n)] = a_1 \cdot x_1(n^2) + a_2 \cdot x_2(n^2)$

$y_1(n) = T[x_1(n)] = x_1(n^2)$

$y_2(n) = T[x_2(n)] = x_2(n^2)$

$y''(n) = a_1 \cdot y_1(n) + a_2 y_2(n) = a_1 \cdot x_1(n^2) + a_2 x_2(n^2)$

$\because y'(n) = y''(n) \Rightarrow$ System is linear

c) $y(n) = x^2(n) = [x(n)]^2$

L.H.S $y_3(n) = T[a_1 x_1(n) + a_2 x_2(n)] = [a_1 x_1(n) + a_2 x_2(n)]^2$

$y_1(n) = T[x_1(n)] = x_1^2(n)$

$y_2(n) = T[x_2(n)] = x_2^2(n)$

$y'(n) = a_1 \cdot y_1(n) + a_2 y_2(n) = a_1 \cdot x_1^2(n) + a_2 \cdot x_2^2(n)$

$\because y_3(n) \neq y'(n) \rightarrow$ non-linear

d) $y(n) = A \cdot x(n) + B$

$y_3(n) = T[a_1 x_1(n) + a_2 x_2(n)] = A[a_1 x_1(n) + a_2 x_2(n)] + B$

$y_1(n) = T[x_1(n)] = A x_1(n) + B$

$y_2(n) = T[x_2(n)] = A \cdot x_2(n) + B$

$y'(n) = a_1 \cdot y_1(n) + a_2 \cdot y_2(n) = a_1 [A x_1(n) + B] + a_2 [A \cdot x_2(n) + B]$

$\therefore y_3(n) \neq y'(n) \rightarrow$ Non. linear

(e) $y(n) = e^{x(n)}$

$y_3(n) = T[a_1 x_1(n) + a_2 . x_2(n)] = e^{a_1 x_1(n) + a_2 x_2(n)}$

$y_1(n) = T[x_1(n)] = e^{x_1(n)}$

$y_2(n) = T[x_2(n)] = e^{x_2(n)}$

$y'(n) = a_1 . y_1(n) + a_2 y_2(n) = a_1 . e^{x_1(n)} + a_2 . e^{x_2(n)}$

$\therefore y_3(n) \neq y'(n) \rightarrow$ Non. linear

(4) <u>Time variant & Time - invariant Systems</u> :-

→ A system is said to be time - invariant or shift invariant if the characteristics of the system do not change with time. O.W it is time variant.

→ Condition to test time - invariant :-

$y(n,k) = T[x(n-k)] \rightarrow$ Response of system of i/p is shifted by k.

$y(n-k) \rightarrow$ Response of the system is shifted by k.

if $\boxed{y(n,k) = y(n-k)} \rightarrow$ time invariant

$y(n,k) \neq y(n-k) \rightarrow$ time variant.

Eg:- Determine the systems described by the following i/p - o/p equations are time invariant & variant.

(a) $y(n) = x(n) - x(n-1)$ b) $y(n) = n . x(n)$

c) $y(n) = x(-n)$ d) $y(n) = x(n) . \cos \omega_0 n$.

Sol (a) $y(n) = x(n) - x(n-1)$

$y(n,k) = T[x(n-k)] = x(n-k) - x(n-k-1)$

$y(n-k) = x(n-k) - x(n-k-1)$

$\because y(n,k) = y(n-k) \rightarrow$ time invariant

(b) $y(n) = n \cdot x(n)$

$y(n,k) = T[x(n-k)] = n \cdot x(n-k)$

$y(n-k) = (n-k) \cdot x(n-k)$.

$\because y(n,k) \neq y(n-k) \rightarrow$ time ~~invar~~ Variant

(c) $y(n) = x(-n)$

$y(n,k) = T[x(n-k)] = x(-n-k)$

$y(n-k) = x(-(n-k)) = x(-n+k)$

$\because y(n,k) \neq y(n-k) \rightarrow$ time variant.

(d) $y(n) = x(n) \cdot \cos \omega_0 n$.

$y(n,k) = T[x(n-k)] = x(n-k) \cdot \cos \omega_0 n$

$y(n-k) = x(n-k) \cdot \cos \omega_0 (n-k)$

$\because y(n,k) \neq y(n-k) \rightarrow$ time Variant.

(5) <u>FIR & IIR</u> Systems :-

→ LTI can be classified according to the type of impulse response.

(i) FIR (Finite Impulse Response) System :-

If the Impulse response of the system is of finite duration, then the system is called FIR system

Eg:- $h(n) = \begin{cases} 1 & \text{for } n = -1, 0 \\ 3 & \text{for } n = 1, 2 \\ 0 & \text{o.w} \end{cases}$

(ii) IIR System:-

If the impulse response of the system is of infinite duration, it is called IIR system.

Eg:- $h(n) = a^n . u(n)$.

(6) Stable & Unstable Systems :-

→ An arbitrary relaxed system is said to be bounded i/p - bounded o/p (BIBO) stable if and only if every bounded i/p produced a bounded o/p.

→ The condition that i/p sequences $x(n)$ & the o/p sequence $y(n)$ are bounded ∋ that

$$|x(n)| \leq M_x < \infty, \quad |y(n)| \leq M_y < \infty \quad \forall n$$

where M_x & M_y are some finite nos.

→ For an LTI system,

$x(n) \to$ i/p
$y(n) \to$ o/p
$h(n) \to$ impulse response

$$y(n) = x(n) * h(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) . x(n-k)$$

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k) . x(n-k) \right|$$

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)| . |x(n-k)|$$

$$|y(n)| \leq M . \sum_{k=-\infty}^{\infty} h(k) \quad \text{for bounded i/p}$$

$\therefore$ The above Condition will be satisfied if

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty.$$

$\therefore$ The necessary & Sufficient Condition for stability is

$$\boxed{\sum_{n=-\infty}^{\infty} |h(n)| < \infty}$$

Eg:- Test the stability of the system whose Impulse response

$h(n) = (\frac{1}{2})^n u(n)$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} \left| (\tfrac{1}{2})^n u(n) \right|$$

$$= \sum_{n=0}^{\infty} (\tfrac{1}{2})^n = \frac{1}{1-\frac{1}{2}} = 2 < \infty$$

$\therefore$ System is stable.

Q, check whether the following digital systems are BIBO stable or not.

a) $y(n) = a \cdot x(n+1) + b\,x(n-1)$

b) $y(n) = e^{-x(n)}$

Sol a) $y(n) = a \cdot x(n+1) + b\,x(n-1)$

If $x(n) = \delta(n)$

then $y(n) = h(n)$

$$h(n) = a \cdot \delta(n+1) + b\,\delta(n-1)$$

when $n=0$, $\quad h(0) = a \cdot \delta(1) + b\,\delta(-1) = 0$

$n=1$, $\quad h(1) = a \cdot \delta(2) + b\,\delta(0) = b$

$n=2$, $\quad h(2) = a \cdot \delta(3) + b\,\delta(1) = 0$

In general,
$$h(n) = \begin{cases} b, & n=1 \\ 0, & \text{o.w} \end{cases}$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = b$$

The necessary & sufficient condition for BIBO stability is:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$b < \infty$$

b) $\quad y(n) = e^{-x(n)}$

If $x(n) = \delta(n)$

$y(n) = h(n)$

when $n=0$, $\quad h(0) = e^{-\delta(0)} = e^{-1}$

when $n=1$, $\quad h(1) = e^{-\delta(1)} = e^{0} = 1$

$n=2$, $\quad h(2) = e^{-\delta(2)} = e^{0} = 1$

$$h(n) = \begin{cases} e^{-1} & \text{when } n=0 \\ 1 & \text{when } n \neq 0 \end{cases}$$

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The necessary & sufficient Condition for BIBO stability is:

$$\boxed{\sum_{n=-\infty}^{\infty} |h(n)| < \infty}$$

$= |h(0)| + |h(1)| + \cdots + |h(\infty)|$

$= e^{-1} + 1 + 1 + \cdots \infty = \infty$

∴ The given System is unstable.

(3) Determine whether the following System is linear, stable, causual and time invariant

$$y(n) = n\,x(n) + x(n+2) + y(n-2)$$

Sol 1. Linear:-

$$T[a \cdot x_1(n) + b\,x_2(n)] = a \cdot T[x_1(n)] + b\,T[x_2(n)]$$

$$L.H.S = [a[n\,x_1(n) + x_1(n+2) + y_1(n-2)] + b[n \cdot x_2(n) + x_2(n+2) + y_2(n-2)]$$

$$= n[a\,x_1(n) + b\,x_2(n)] + [a\,x_1(n+2) + b\,x_2(n+2)] + a \cdot y_1(n-2) + b\,y_2(n-2)$$

R.H.S:

$$T[x_1(n)] = n \cdot x_1(n) + x_1(n+2) + y_1(n-2)$$

$$T[x_2(n)] = n \cdot x_2(n) + x_2(n+2) + y_2(n-2)$$

$$R.H.S = a \cdot [n\,x_1(n) + x_1(n+2) + y_1(n-1)] + b[n \cdot x_2(n) + x_2(n+2) - y_2(n-1)]$$

$\because$ L.H.S = R.H.S

System is Linear

c) Stable:-

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

for $x(n) = \delta(n)$, $y(n) = h(n)$

$$h(n) = n.\ \delta(n) + \delta(n+2) + h(n-2)$$

$n = -2$, $h(-2) = -2.\ \delta(-2) + \delta(0) + h(-4)$

$$h(-2) = 1$$

$n = -1$, $h(-1) = -1.\ \delta(-1) + \delta(1) + h(-1) = 0$

$n = 0$, $h(0) = \delta(2) + h(-2) = 1$

$n = 1$, $h(1) = 1.\ \delta(1) + \delta(2) + h(-1) = 0$

$h(2) = 2.\ \delta(2) + \delta(4) + h(0) = 1$

$\vdots$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \ldots 1 + 0 + 1 + 0 + \ldots \infty = \infty$$

$\therefore$ System is Unstable

3) $y(2) = 2.\ x(2) + x(4) + y(0)$

$\because$ the o/p depends on future i/p's, System is non-casual.

e) Time Variant System: $x(n)$ is a fn. of time.

Linear Shift Invariant System (LTI System):-

→ Consider a discrete-time system whose i/p is $x(n)$ & o/p is $y(n)$.

$$x(n) \longrightarrow \boxed{T} \longrightarrow y(n) = T[x(n)]$$

fig: discrete-time system.

→ Consider the system is LTI ie it satisfies the principle of both linearity & time invariance.

$$x(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot \delta(n-k)$$

$$y(n) = T[x(n)]$$

$$y(n) = T\left[\sum_{k=-\infty}^{\infty} x(k) \cdot \delta(n-k)\right]$$

∵ it is a linear system, $[\because T[a_1 x_1(n) + a_2 x_2(n)] = a_1 \cdot T[x_1(n)] + a_2\, T[x_2(n)]]$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot T[\delta(n-k)] \quad -(1)$$

→ If the i/p to the system is unit impulse i/p, ie $x(n) = \delta(n)$, then the o/p of the system is known as impulse response denoted by $h(n)$.

if $x(n) \to \delta(n)$ then $y(n) \to h(n)$

$$h(n) = T[\delta(n)]$$

→ The response due to shifted impulse sequence $\delta(n-k)$ can be denoted as $h(n,k)$ ie

$$h(n,k) = T[\delta(n-k)].$$

Sub in (1)

$$\therefore y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n,k) \quad -(2)$$

→ for Shift-Invariant (time Invariant) System,

$$h(n,k) = h(n-k).$$

Sub in eq-(2)

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)}$$

→ The above is known as convolution sum & is represented as,

$$\boxed{y(n) = x(n) * h(n)}$$

Properties of Convolution:—

1. Commutative law : $x(n) * h(n) = h(n) * x(n)$
2. Associative law : $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$
3. Distributive law: $x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$
4. Shifting property : If $x(n) * h(n) = y(n)$ then

$$x(n-k) * h(n-m) = y(n-k-m)$$

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5. Convolution with an impulse: $x(n) * \delta(n) = x(n)$.

Eg:- (1) Find the convolution of two finite duration sequences:

$$h(n) = a^n u(n) \quad \forall n,$$

$$x(n) = b^n u(n) \quad \forall n.$$

(i) when $a \neq b$ (ii) $a = b$.

Sol

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} a^k u(k) \cdot b^{n-k} u(n-k).$$

$$u(n-k) = \begin{cases} 1 & \text{for } k \le n \\ 0 & \text{for } k > n \end{cases}$$

$$y(n) = \sum_{k=0}^{n} a^k \cdot b^{n-k}$$

$$= b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k \quad -(1)$$

(i)

$$y(n) = b^n \left[\frac{1 - \left(\frac{a}{b}\right)^{n+1}}{1 - \frac{a}{b}} \right] \quad \text{when } a \neq b$$

from (1)

(ii) $$y(n) = b^n \sum_{k=0}^{n} 1^k = b^n [1 + 1 + \cdots + n+1 \text{ terms}]$$

$$= b^n (n+1)$$

(2) find the Convolution of $x(n) = \cos n\pi \cdot u(n)$, $h(n) = (\frac{1}{2})^n u(n)$

Sol $y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$

$$= \sum_{k=-\infty}^{\infty} \cos k\pi \cdot u(k) \cdot (\tfrac{1}{2})^{n-k} \cdot u(n-k)$$

$$= \sum_{k=-\infty}^{\infty} (-1)^k \cdot u(k) \cdot (\tfrac{1}{2})^{n-k} \cdot u(n-k)$$

$$= \sum_{k=0}^{n} (-1)^k \cdot (\tfrac{1}{2})^{n-k}$$

$$= (\tfrac{1}{2})^n \cdot \sum_{k=0}^{n} (-2)^k$$

$\because \sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}$ if $a<1$

$$= (\tfrac{1}{2})^n \cdot \frac{1-(-2)^{n+1}}{1-(-2)}$$

$$= (\tfrac{1}{2})^n \left[\frac{1+2(-2)^n}{3}\right]$$

$$= \frac{1}{3} \cdot (\tfrac{1}{2})^n + \frac{2}{3} (-1)^n \quad \text{for } n \geq 0$$

$$y(n) = \frac{1}{3} (\tfrac{1}{2})^n \cdot u(n) + \frac{2}{3} (-1)^n u(n)$$

→ Unit Step response :- ($S(n)$)

If the i/p to the System unit Step, then the response is called unit Step response.

$$\text{if } x(n) \to u(n) \quad \text{then} \quad y(n) \Rightarrow S(n)$$

$$y(n) = x(n) * h(n)$$

$$S(n) = u(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} u(k) \cdot h(n-k) = \sum_{k=-\infty}^{\infty} h(k) \cdot u(n-k)$$

$$\boxed{S(n) = \sum_{k=0}^{n} h(k).}$$ for a causal System

→ Convolution of finite Sequences :-

→ The Convolution of $y(n)$ of two finite length sequences $x(n)$ & $h(n)$ is also finite.

length of $x(n) \to L_x$

" $h(n) \to L_h$

" $y(n) \to L_y = L_x + L_h - 1$

→ The Starting index of $y(n)$ equals the sum of Starting Indices of $x(n)$ & $h(n)$.

→ The ending index of $y(n)$ equals the sum of the ending indices of $x(n)$ & $h(n)$.

① Determine the Convolution Sum of two Sequences:

$$x(n) = \{4, 2, 1, 3\}, \quad h(n) = \{1, \underset{\uparrow}{2}, 2, 1\}$$

Sol: $x(n)$ starts at $n_1 = 0$, $h(n)$ at $n_2 = -1$

Length of $y(n)$ = length of $x(n)$ + length of $h(n) - 1$

$= 4 + 4 - 1 = 7.$

Starting index of $y(n)$ = St. of $x(n)$ + St. of $h(n)$

$= 0 - 1 = -1.$

Ending index of $y(n)$ = en. of $x(n)$ + en. of $h(n)$

$= 3 + 2 = 5.$

$\therefore$ $y(n)$ is from $n = -1$ to 5.

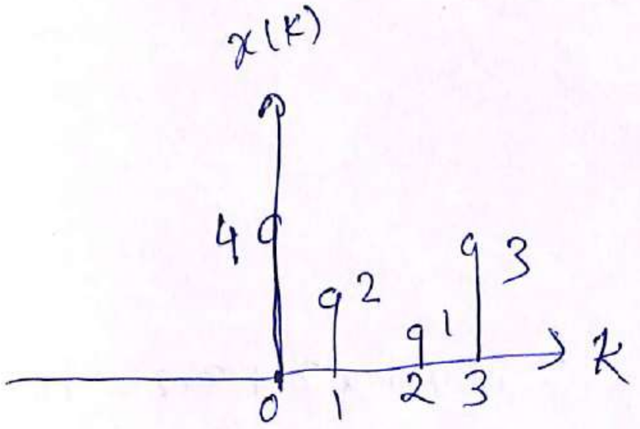
Tabular Method:-

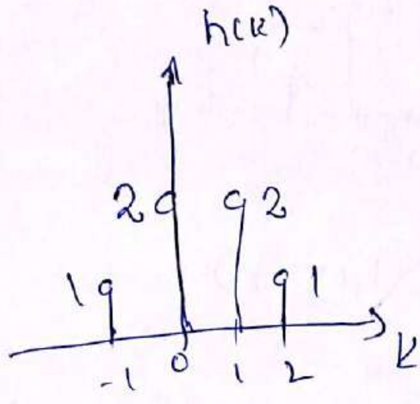
$h(n)$ \ $x(n)$	4	2	1	3
1	4	2	1	3
2	8	4	2	6
2	8	4	2	6
1	4	2	1	3

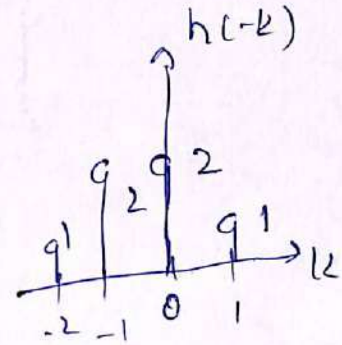
$$y(n) = \{4, \underset{\uparrow}{10}, 13, 13, 10, 7, 3\}$$

Graphical Method :—

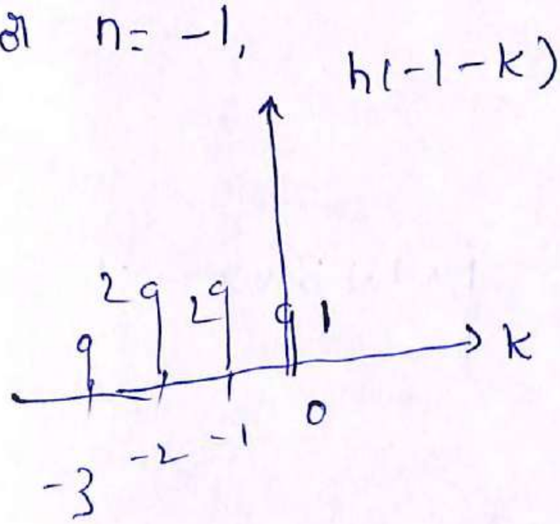
$$y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$$







• for $n = -1$,

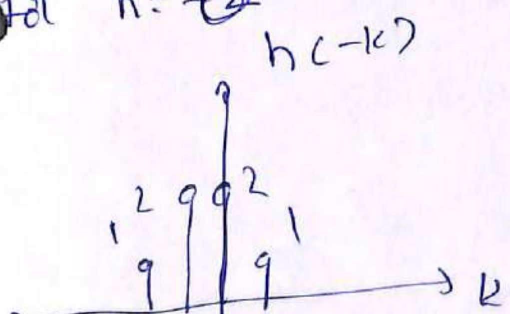


$$y(-1) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(-1-k)$$

Multiply $x(k)$ & $h(-1-k)$ from fig

$$y(-1) = 4 \times 1 = 4$$

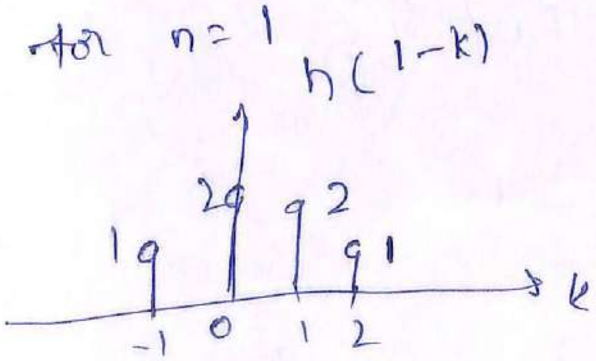
• for $n = 0$



$$y(0) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(-k)$$

$$= 4 \times 2 + 2 \times 1 = 10$$

for $n = 1$

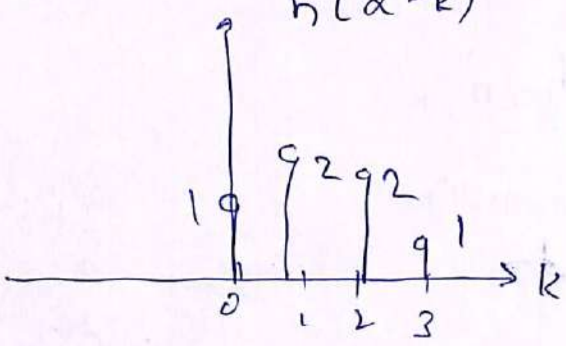


$$y(+1) = 4 \times 2 + 2 \times 2 + 1 \times 1$$

$$= 8 + 4 + 1$$

$$y(1) = 13$$

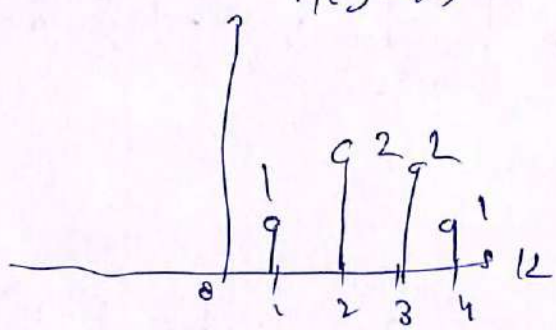
$n = 2$



$y(2) = 4\times1 + 2\times2 + 1\times2 + 3\times1$

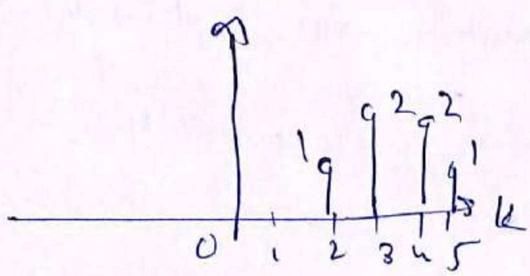
$y(2) = 13$

$n = 3$



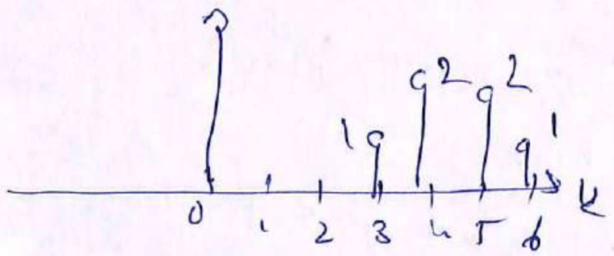
$y(3) = 2\times1 + 1\times2 + 3\times2 = 10$

$n = 4$



$y(4) = 1\times1 + 3\times2 = 7$

$n = 5$



$y(5) = 3\times1 = 3$

$\therefore\ y(n) = \{4, 10, 13, 13, 10, 7, 3\}$
$\uparrow$

→ Evaluate the convolution $y(n) = x(n) * h(n)$ of the sequences $\boxed{\text{DSP-1-27}}$

$$h(n) = \begin{cases} a^n, & 0 \le n < N \\ 0, & \text{else where} \end{cases}$$

$$x(n) = \begin{cases} b^{n-m}, & m \le n \\ 0, & n < m \end{cases}$$

Sol

$$y(n) = x(n) * h(n) = h(n) * x(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k) = \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k)$$

$$h(k) = \begin{cases} a^k, & 0 \le k < N \\ 0, & \text{o.w} \end{cases}$$

$$x(n-k) = \begin{cases} b^{n-k-m}, & k \le n-m \\ 0, & \text{o.w} \end{cases}$$

$$y(n) = \sum_{k=0}^{n-m} a^k \cdot b^{n-k-m} \quad \text{for } m < n < N+m$$

$$= \sum_{k=0}^{N} a^k \cdot b^{n-k-m} \quad \text{for } n > m+N$$

$$= b^{n-m} \cdot \sum_{k=0}^{N} \left(\frac{a}{b}\right)^k$$

$$= b^{n-m}\left[\frac{1-\left(\frac{a}{b}\right)^{n-m+1}}{1-\frac{a}{b}}\right]$$

$$= \frac{b^{n-m+1} - a^{n-m+1}}{b-a} \quad \text{for } n \geq m+N$$

$$y(n) = \sum_{k=0}^{N} a^k . b^{n-k-m} = b^{n-m}\sum_{k=0}^{N}\left(\frac{a}{b}\right)^k$$

$$= b^{n-m+1}\left[\frac{1-\left(\frac{a}{b}\right)^{N+1}}{b-a}\right] \quad \text{for } n > m+N$$

Eg:- Find the Convolution of the Signals

$$x(n) = \begin{cases} 1, & n = -2, 0, 1 \\ 2, & n = -1 \\ 0, & \text{o.w} \end{cases}$$

$$h(n) = \delta(n) - \delta(n-1) + \delta(n-2) - \delta(n-3)$$

$$x(n) = \{1, 2, \underset{\uparrow}{1}, 1\}, \quad h(n) = \{1, -1, 1, -1\}$$

$$y(n) = \{1, 1, \underset{\uparrow}{0}, 1, -2, 0, -1\}$$

Linear Constant - Coefficient Difference Equations:-

→ Already we have seen that, LTI systems response described interms of their impulse responses.

$$y(n) = x(n) * h(n).$$

→ In this section, we will see that LTI systems described by an I/P - O/P relation called a difference equation with constant coefficients.

→ Systems described by a constant coefficient linear difference equations are a sub-class of the recursive & non-recursive systems.

→ Recursive system:- A system whose O/P $y(n)$ at time n depends on any no of past O/P values $y(n-1), y(n-2), \ldots$ is called a recursive equation.

$$y(n) = f\left(y(n-1), y(n-2), \ldots, y(n-N), x(n), x(n-1), \ldots, x(n-M)\right)$$

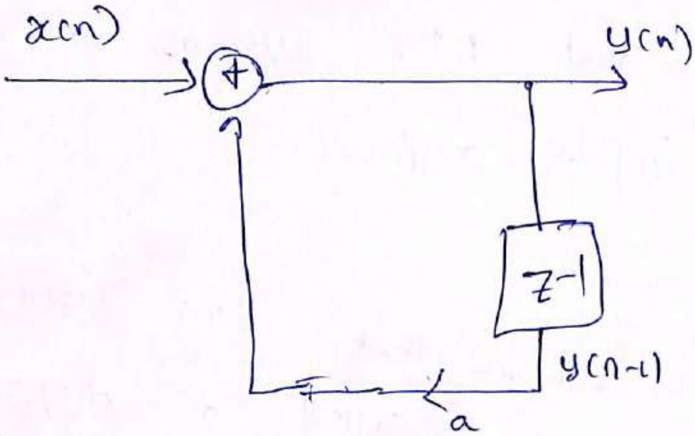
→ Non-recursive System:- A system whose O/P $y(n)$ at time n depends only on the present & past I/P's. is called 'Non-recursive equation'.

$$y(n) = f\left(x(n), x(n-1), \ldots, x(n-M)\right)$$

→ Consider a recursive system with I/P - O/P equation,

$$y(n) = a \cdot y(n-1) + x(n)$$

where a is a constant.



fig! Block diagram realization of a simple recursive system

$$y(n) = a\,y(n-1) + x(n) \quad -(1)$$

$n=0,\quad y(0) = a\,y(-1) + x(0)$

$n=1,\quad y(1) = a\cdot y(0) + x(1)$

But $y(0) = a\cdot y(-1) + x(0)$

$$y(1) = a\,[a\cdot y(-1) + x(0)] + x(1)$$

$$= a^2\,y(-1) + a\cdot x(0) + x(1)$$

$n=2$

$$y(2) = a\cdot y(1) + x(2)$$

but $y(1) = a^2 y(-1) + a\,x(0) + x(1)$

$$y(2) = a\,[a^2 y(-1) + a\,x(0) + x(1)] + x(2)$$

$$y(2) = a^3 y(-1) + a^2 x(0) + a\,x(1) + x(2)$$

$\vdots$

$$y(n) = a^{n+1} y(-1) + a^n x(0) + a^{n-1} x(1) + \cdots + a\,x(n-1) + x(n).$$

$$y(n) = a^{n+1} y(-1) + \sum_{k=0}^{n} a^k\cdot x(n-k), \quad n \geq 0 \quad -(2)$$

→ When the system is at zero state, then its corresponding output (a response) is called zero-state response & is denoted by $y_{zs}(n)$. or "forced response".

In eq-(2), make initial states ie $y(-1) = 0$.

$$y_{zs}(n) = \sum_{k=0}^{n} a^k . x(n-k), \quad n \geq 0$$

→ The O/P of the system with zero i/p for non-relaxed system (i.e $y(-1) \neq 0$), is called "zero-i/p response" or "natural response".

$$y_{zi}(n) = a^{n+1} y(-1), \quad y \geq 0.$$

∴ Forced or zero-state response, y_{zs} → when initial conditions are zero

Natural or zero-i/p response, y_{zi} → when i/p's are zero.

→ ∴ the total response of the system is;

$$y(n) = y_{zs}(n) + y_{zi}(n).$$

→ The system described by eq-(4) is 1st order difference eqn.

→ The Nth order linear constant coefficient difference eqn. is,

$$\boxed{y(n) = -\sum_{k=1}^{N} a_k \cdot y(n-k) + \sum_{k=0}^{M} b_k \cdot x(n-k)}$$

or

$$\cancel{y(n)+} \sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k), \quad a_0 = 1.$$

→ The integer 'N' is called the order of the difference equation.

→ A system which is described by i/p - o/p is said to be linear if it satisfies the following three requirements:

1, The total response is equal to the sum of the zero-i/p and zero-state responses.

2, The principle of superposition applies to the zero-state response (~~re~~ forced or particular solution).

3, The principle of superposition applies to the zero-i/p response. (natural or homogenous)

Solution of Linear Constant-Coefficient Difference Equation:-

→ To find the response of a system which is described by i/p - o/p relation are:

(i) direct method :- $y(n) = y_h(n) + y_p(n)$ by finding homogenous (natural or zero i/p respon) & Particular (forced or zero state) solution.

(ii) Indirect method:- This is based on Z-transform.

(i) Direct method:-

(a) Homogenous Solution:-

$$\sum_{k=0}^{N} a_k\, y(n-k) = 0 \quad , \quad a_0 = 1$$

→ Assume that the solution is in the form of an exponential is:

$$y_h(n) = \lambda^n.$$

$$\sum_{k=0}^{N} a_k \cdot \lambda^{n-k} = 0.$$

$$a_0 \cdot \lambda^n + a_1 \cdot \lambda^{n-1} + a_2 \cdot \lambda^{n-2} + \dots + a_N \lambda^{n-N} = 0$$

$$\lambda^{n-N}\left(\lambda^N + a_1 \cdot \lambda^{N-1} + a_2 \cdot \lambda^{N-2} + \dots + a_N\right) = 0.$$

N^{th} Polynomial eqn. which has N roots.

$\lambda_1, \lambda_2, \dots, \lambda_N$.

$$y_h(n) = c_1 \cdot \lambda_1^n + c_2 \lambda_2^n + \dots + c_N \lambda_N^n.$$

(ii) The Particular solution :- $y_p(n)$ is required to satisfy the difference equation for the specific i/p signal $x(n)$, $n \geq 0$.

General form of the Particular Solution for several i/p signals are:

I/P Signal $x(n)$	Particular Solution $y_p(n)$
1. Unit step, a constant (A)	K
2. $A \cdot M^n$	$K M^n$
3. $A \cdot n^M$	$K_0 n^M + K_1 n^{M-1} + \cdots + K_M$
4. $A^n n^M$	$A^n (K_0 n^M + K_1 \cdot n^{M-1} + \cdots + K_M)$
5. $A \cdot \cos w_0 n$, $A \sin w_0 n$	$K_1 \cdot \cos w_0 n + K_2 \cdot \sin w_0 n$

For homogenous solution :-

(i) if roots are distinct,

$$y_h(n) = c_1 \lambda_1^n + c_2 \lambda_2^n + \cdots + c_N \lambda_N^n.$$

(ii) If roots are repeated,

$$(\lambda - \lambda_1)^m (\lambda - \lambda_2) \cdots (\lambda - \lambda_N) = 0$$

$$y_h(n) = (c_1 + c_2 n + c_3 n^2 + \cdots + c_m n^{m-1}) \cdot (\lambda_1)^n + c_{m+1} \cdot (\lambda_2)^n + \cdots + c_N \cdot \lambda^N$$

(iii) if roots are complex conjugate,

$$\lambda_1 = a + jb, \quad \lambda_2 = a - jb$$

$$y_h(n) = r^n \left[A_1 . \cos n\theta + A_2 . \sin n\theta \right]$$

where $r = \sqrt{a^2+b^2}$

$$\theta = \tan^{-1} \frac{b}{a}.$$

Problems:-

(1) Consider a causual & stable LTI system whose i/p $x(n)$ and output $y(n)$ are related through the second order difference equation:

$$y(n) - \frac{1}{12} . y(n-1) - \frac{1}{12} y(n-2) = x(n)$$

Determine (a) Impulse response (b) Step response

Sol (a) $y(n) = y_h(n) + y_p(n)$

for impulse i/p, $y_p(n) = 0$

$$\therefore y(n) = y_h(n) \qquad -(1)$$

Given $y(n) = \frac{1}{12} . y(n-1) - \frac{1}{12} . y(n-2) = x(n) \quad -(2)$

Consider $y_h(n) = \lambda^n$ form, & for $y_h(n) \rightarrow x(n) = 0$.

$$\lambda^n - \frac{1}{12} . \lambda^{n-1} - \frac{1}{12} \lambda^{n-2} = 0$$

$$\lambda^n (1 - \frac{1}{12} . \lambda^{-1} - \frac{1}{12} . \lambda^{-2}) = 0$$

$$1 - \frac{1}{12} \lambda^{-1} - \frac{1}{12} \lambda^{-2} = 0$$

$$\Rightarrow \lambda^2 - \frac{1}{12} \lambda - \frac{1}{12} = 0$$

→ The roots of above eqn are:

$$\lambda_1 = 1/3, \quad \lambda_2 = -1/4$$

$\therefore$ $y_h(n) = c_1 (\lambda_1)^n + c_2 \lambda_2^n$

$= c_1 (1/3)^n + c_2 (-1/4)^n$ — (2)

from eq-(1)

$\therefore$ $y(n) = y_h(n)$

$y(n) = c_1 (1/3)^n + c_2 (-1/4)^n$.

$n=0$, $y(0) = c_1 + c_2$ — (3)

$n=1$, $y(1) = c_1 \cdot \frac{1}{3} + c_2 \times \frac{-1}{4}$ — (4)

Sub $n=0$ in eq-(2)

$y(0) - \frac{1}{12} \cdot y(-1) - \frac{1}{12} \cdot y(-2) = x(0)$

Here $x(n) = \delta(n)$, $x(0) = \delta(0) = 1$,

& $y(-1) = y(-2) = 0$

→ $y(0) = 1$.

Sub $n=1$ in eq-(2)

$y(1) - \frac{1}{12} y(0) - \frac{1}{12} \cdot y(-1) = x(1)$

$y(1) - \frac{1}{12} - 0 = 0 \Rightarrow y(1) = \frac{1}{12}$.

Sub $y(0) = 1$, $y(1) = \frac{1}{12}$ in ③ & (4)

$$1 = c_1 + c_2$$

$$\frac{1}{12} = \frac{c_1}{3} - \frac{c_2}{4}$$

by Solving above 2 eqns, we get

$$c_1 = \frac{4}{7}, \quad c_2 = \frac{3}{7}.$$

∴ Impulse response $= y(n) = h(n) = \frac{4}{7}\left(\frac{1}{3}\right)^n + \left(\frac{3}{7}\right)\left(-\frac{1}{4}\right)^n$

(b) Step response:–

$$y(n) = y_h(n) + y_p(n)$$

Already, we have cal. $y_h(n) = c_1(1/3)^n + c_2(-1/4)^n$.

$y_p(n)$ for step i/p is $y_p(n) = k$.

For $n \geq 2$, sub $y_p(n) = k$ & $x(n) = 1$ in above eqn,

$$k - \frac{1}{12}k - \frac{1}{12}k = 1$$

$$\Rightarrow k = 6/5$$

$$y_p(n) = 6/5$$

$$\therefore y(n) = y_h(n) + y_p(n)$$

$$y(n) = c_1(1/3)^n + c_2(-1/4)^n + \frac{6}{5} \quad -(5)$$

for $n=0$, $y(0) = c_1 + c_2 + \frac{6}{5}$ — (6)

$y(1) = \frac{1}{3}c_1 - \frac{1}{4}c_2 + \frac{6}{5}$ — (7)

Sub $n=0$ in eq-(1) by assuming $y(-1) = y(-2) = 0$.

$y(0) - \frac{1}{12}y(-1) - \frac{1}{12}y(-2) = x(0)$

$y(0) = 1$ $\quad$ $\cancel{y(0)} = \cancel{\frac{1+6}{5}}$ $\quad$ $\cancel{y(0) = 13}$ $\quad$ $\because$ $x(n) = u(n)$

$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

$n=1$

$y(1) - \frac{1}{12}y(0) - \frac{1}{12}y(-1) = x(1)$

$y(1) - \frac{1}{12} - 0 = 1$

$\Rightarrow y(1) = 1 + \frac{1}{12} = \frac{13}{12}$

Sub $y(0)$ & $y(1)$ in eq-(6) & (7)

$1 = c_1 + c_2 + \frac{6}{5}$ $\quad \Rightarrow c_1 + c_2 = \frac{-1}{5}$

$\frac{13}{12} = \frac{1}{3}c_1 - \frac{1}{4}c_2 + \frac{6}{5}$ $\quad \frac{1}{3}c_1 - \frac{1}{4}c_2 = \frac{13}{12} - \frac{6}{5}$

by solving above two eqns, we get

$c_1 = -\frac{2}{7}$, $c_2 = \frac{3}{35}$.

$\boxed{DSP-1-33}$

2) Determine the solution of the difference equation

$$y(n) = \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + x(n) \text{ for } x(n) = 2^n u(n)$$

Assume initial conditions are zero.

Sol Given

$$y(n) = \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + x(n) \quad -(1)$$

the solution

$$y(n) = y_h(n) + y_p(n) \quad -(2)$$

$y_h(n)$:-

Sub $y_h(n) = \lambda^n$ in eq-(1) & i/p = 0.

$$\lambda^n = \frac{5}{6} \lambda^{n-1} - \frac{1}{6} \lambda^{n-2} + 0$$

$$\Rightarrow \lambda^n \left[1 - \frac{5}{6} \lambda^{-1} + \frac{1}{6} \lambda^{-2}\right] = 0$$

$$\Rightarrow 1 - \frac{5}{6} \lambda^{-1} + \frac{1}{6} \lambda^{-2} = 0$$

$$\lambda^2 - \frac{5}{6} + \frac{1}{6} = 0 = (\lambda - \tfrac{1}{2})(\lambda - \tfrac{1}{3}) = 0$$

By solving, roots are

$$\lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{1}{3}.$$

So, $y_h(n) = C_1 \cdot \lambda_1^n + C_2 \lambda_2^n$

$$y_h(n) = C_1 \cdot (\tfrac{1}{2})^n + C_2 (\tfrac{1}{3})^n \quad -(3)$$

$y_p(n)$:-

$x(n) = 2^n u(n)$

$$x(n) = \begin{cases} 2^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

We K.T for i/p $A \cdot M^n$ is $K \cdot M^n$.

$\therefore y_p(n) = K \cdot 2^n$ (here $A=1$, $M=2$)

Sub. $y_p(n)$ = $K \cdot 2^n$ in eq-(1)

$$K \cdot 2^n = \frac{5}{6} \cdot K \cdot 2^{n-1} - \frac{1}{6} K \cdot 2^{n-2} + 2^n, \quad n \geq 0$$

$$K = \frac{5}{6} \cdot K \cdot 2^{-1} - \frac{1}{6} K \cdot 2^{-2} + 1$$

$$\Rightarrow K = \frac{8}{5}.$$

$$\therefore y_p(n) = \frac{8}{5} \cdot 2^n \quad \text{— (4)}$$

Sub eq-(3) & (4) in (2)

$$y(n) = c_1 \left(\tfrac{1}{2}\right)^n + c_2 \left(\tfrac{1}{3}\right)^n + \frac{8}{5} \cdot 2^n, \quad n \geq 0 \quad \text{— (4A)}$$

$n=0$, $\quad y(0) = c_1 + c_2 + \frac{8}{5} \quad$ — (5)

$n=1$, $\quad y(1) = \frac{c_1}{2} + \frac{c_2}{3} + \frac{16}{5} \quad$ — (6)

Sub $n=0$ & $y(-1) = y(-2) = 0$ in eq-(1)

$n=0$

$$y(0) = \frac{5}{6} y(-1) - \frac{1}{6} y(-2) + 2^0 u(0)$$

$$y(0) = 1.$$

$n=1$

$$y(1) = \frac{5}{6} y(0) - \frac{1}{6} y(-1) + 2^1 u(1)$$

$$y(1) = \frac{5}{6} + 2 = \frac{17}{6}.$$

Sub $y(0)$ & $y(1)$ in eq-(5) & (6)

$$1 = c_1 + c_2 + \frac{8}{5}$$

$$\frac{17}{6} = \frac{c_1}{2} + \frac{c_2}{3} + \frac{16}{5}$$

Solving above two, we get

$$c_1 = -1; \quad c_2 = \frac{2}{5}.$$

Sub c_1 & c_2 in eq-(4A)

$$y(n) = -\left(\frac{1}{2}\right)^n u(n) + \frac{2}{5}\left(\frac{1}{3}\right)^n + \frac{8}{5} 2^n, \quad n \geq 0$$

3) Find the response of the following difference equations

i) $y(n) + y(n-1) = x(n)$ where $x(n) = \cos 2n$

ii) $y(n) - 5y(n-1) + 6y(n-2) = x(n)$ for $x(n) = n$.

Sol (i) $y_h(n) = c(-1)^n$

$y_p(n) = A\cos 2n + B\sin 2n$

$= \frac{1}{2}\cdot\cos 2n - \frac{1}{2}\tan 1\cdot\sin 2n$

$y(n) = y_h(n) + y_p(n)$

$y(n) = c(-1)^n + \frac{1}{2}\cos 2n - \frac{1}{2}\tan 1 \sin 2n$

$n=0,$

$y(0) = c + \frac{1}{2}$

Sub $n=0$ in given eqn.

$y(0) = 1.$

$1 = c + \frac{1}{2} \Rightarrow c = \frac{1}{2}$

$\therefore y(n) = \frac{1}{2}(-1)^n + \frac{1}{2}\cdot\cos 2n - \frac{1}{2}\tan 1\cdot\sin 2n$

$y(n) = \frac{1}{2}(-1)^n + \frac{1}{2}\cdot\frac{\cos(2n+1)}{\cos 1}$

(ii) given $y(n) - 5\cdot y(n-1) + 6y(n-2) = n.$

$y_h(n) = c_1\cdot(3)^n + c_2(2)^n.$

$y_p(n) = k_0\cdot n + k_1.$

Sub in above.

$\cancel{k_0 n - 5}\cdot(k_0 n + k_1) - 5[k_0(n-1) + k_1] + 6[k_0(n-2) + k_1] = n.$

$2k_0 n + 2k_1 - 7k_0 = n$

compare $2k_0 = 1 \Rightarrow k_0 = \frac{1}{2}$

$$2k_1 - 7k_0 = 0 \Rightarrow k_1 = \frac{7}{4}$$

$$y_p(n) = \frac{n}{2} + \frac{7}{4}$$

$\therefore$ $y(n) = y_h(n) + y_p(n)$

$$y(n) = c_1 . 2^n + c_2 3^n + \frac{n}{2} + \frac{7}{4}$$

$n=0$,

$$y(0) = c_1 + c_2 + \frac{7}{4}$$

$n=1$, $y(1) = 2c_1 + 3c_2 + \frac{9}{4}$

Sub $n=0$, in difference eqn.

$$y(0) - 5y(-1) + 6y(-2) = x(0)$$

$x(n) = n$.

$$\Rightarrow y(0) = 0.$$

$n=1$,

$$y(1) - 5y(0) + 6y(-1) = 1 \Rightarrow y(1) = 1$$

Sub in above

$$c_1 + c_2 + \frac{7}{4} = 0$$

$$2c_1 + 3c_2 + \frac{9}{4} = 1$$

Solving, $c_1 = -4$, $c_2 = \frac{9}{4}$.

$\therefore$ $y(n) = \frac{n}{2} + \frac{7}{4} - 4 \cdot (2)^n + \frac{9}{4} 3^n$ for $n \geq 0$.

Frequency domain representation of Discrete-time Signal & Systems:

(i) Frequency domain representation of Discrete - time Signal :-

→ It can be done by using

(i) DTFT (Discrete time FT)

(ii) Z - transform

(iii) DFT (Discrete F.T)

→ For Periodic Signals, we can represent freq. domain by Discrete - time Fourier Series (DTFS).

The DTFS of a periodic discrete - time signal is

$$x(n) = \sum_{k=0}^{N-1} C_k \cdot e^{j2\pi nk/N} \quad n = 0, 1, \ldots, N-1$$

where $$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi nk/N}, \quad k = 0, 1, \ldots, N-1$$

→ For Aperiodic Signals, we can represent by DTFT, Z-transform & DFT. If $N \to \infty$, the periodic sequence becomes aperiodic.

DTFT :-

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega n} \, d\omega$$

→

Z-transform:-

The z-transform of a discrete-time signal $x(n)$ is defined as,

$$\boxed{X(Z) = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n}}$$

when $Z = r \cdot e^{-j\omega}$

Relation b/w DTFT & Z-transform

Inverse Z-transform, $\boxed{x(n) = \frac{1}{2\pi j} \oint_c X(z) \cdot z^{n-1} dz}$

$X(Z) = \cancel{X(}$ Z-transform is & $r = 1$.

Relation b/w DTFT & $X(e^{j\omega}) = X(z) \Big|_{z=e^{j\omega}}$

Discrete F.T:-

DFT of $x(n)$ $\rightarrow$ $X(K) = \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-j2\pi nk}{N}}$, $0 \le k \le N-1$

IDFT of $X(k)$ $\rightarrow$ $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot e^{\frac{+j2\pi nk}{N}}$; $0 \le n \le N-1$

(ii) Frequency Response Analysis of Discrete-time System:-

$\Rightarrow$ The o/p $y(n)$ of any linear time invariant (LTI) system to an i/p $x(n)$ is

$$y(n) = x(n) * h(n)$$

take F.T $\quad Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$

$\because$ Convolution in time domain gives multiplication in freq. domain

$$\Rightarrow \quad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

→ $H(e^{j\omega})$ is called system function or transfer fn. & it can also be called as frequency response of the system.

$$H(z) = \frac{Y(z)}{X(z)} \Big|_{z=e^{j\omega}}.$$

→ Since the freq. response $H(e^{j\omega})$ is a complex-valued function, it can be expressed in polar form is:

$$H(e^{j\omega}) = |H(e^{j\omega})| \, e^{j\theta(\omega)} = H_R(e^{j\omega}) + j H_I(e^{j\omega})$$

Real part + imag. part

where $|H(e^{j\omega})|$ → magnitude response.

$|H(e^{j\omega})| = \sqrt{(H_R(e^{j\omega}))^2 + (H_I(e^{j\omega}))^2}$, $\angle H(e^{j\omega}) = \tan^{-1} \frac{H_I(\omega)}{H_R(\omega)}$.

$\theta(\omega) = \angle H(e^{j\omega})$ → phase response.

→ Because of the even property of $|H(e^{j\omega})|$ & odd property of $\angle H(e^{j\omega})$, we can plot the frequency response only in the frequency range $\omega = 0$ to π.

→ When we calculate B.W, we should not consider negative frequencies because practically they will not exist.

Problems:- $\boxed{DSP-1-37}$

(1) Find the frequency response & plot the magnitude & phase response for the sequence

$$x(n) = \begin{cases} 1 & \text{for } n = -2, -1, 0, 1, 2 \\ 0 & \text{o.w} \end{cases}$$

Sol

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$$

$$= \sum_{n=-2}^{2} x(n) \cdot e^{-j\omega n}$$

$$= x(-2) \cdot e^{2j\omega} + x(-1) \cdot e^{j\omega} + x(0) + x(1) e^{-j\omega} + x(2) \cdot e^{-2j\omega}$$

$$= e^{2j\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-2j\omega}$$

$$X(e^{j\omega}) = 1 + 2 \cdot \cos\omega + 2 \cdot \cos 2\omega \quad \left[\because \frac{e^{j\theta} + e^{-j\theta}}{2} = \cos\theta\right]$$

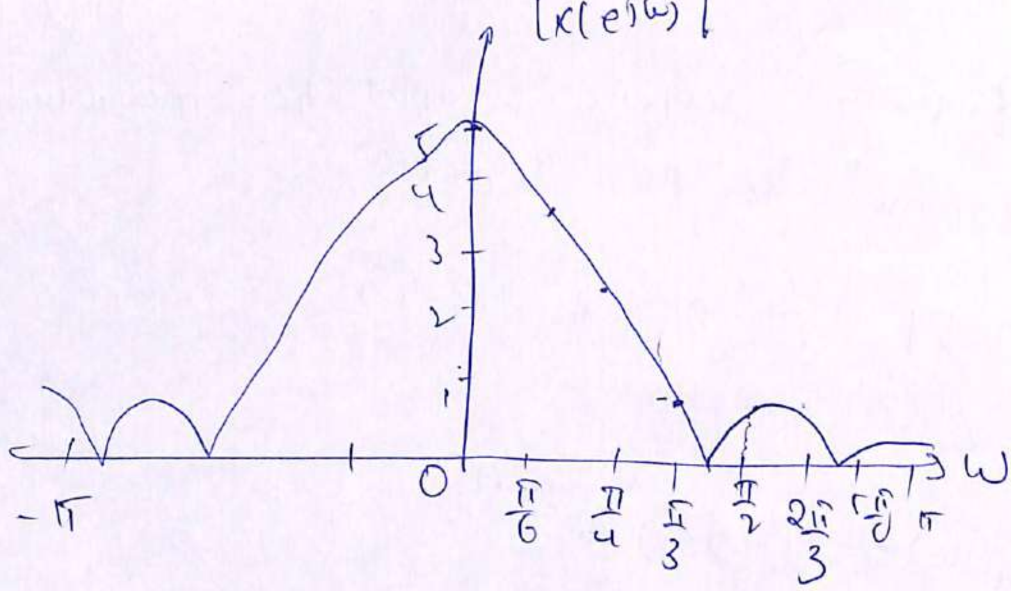
Magnitude response:

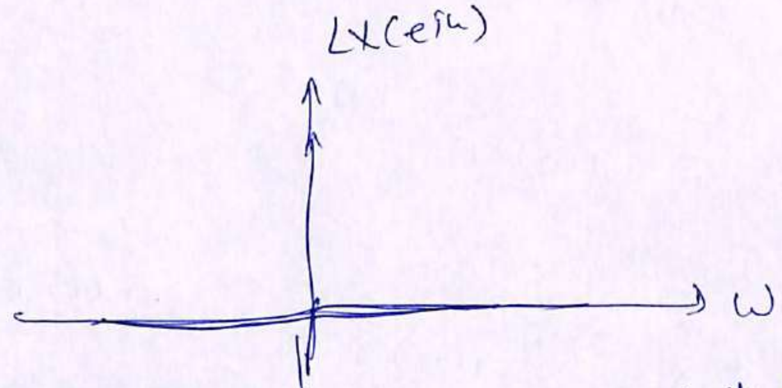
$$|X(e^{j\omega})| = |1 + 2\cos\omega + 2 \cdot \cos 2\omega|$$

phase response

$$\angle X(e^{j\omega}) = 0$$

ω	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{2\pi}{5}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$\lvert X(e^{j\omega})\rvert$	5	3.73	2.414	1	0	1	1	0.268	1
$\angle X(e^{j\omega})$	0	0	0	0	0	0	0	0	0





Q) Determine and sketch the magnitude & phase response of $y(n) = \frac{1}{2}\left[x(n) + x(n-2)\right]$

Sol) $y(n) = \frac{1}{2}\left[x(n) + x(n-2)\right]$

take F.T on B.S

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y(n) \cdot e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2}\left[x(n) + x(n-2)\right] \cdot e^{-j\omega n}$$

$$Y(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\omega}) + e^{-2j\omega} X(e^{j\omega})\right]$$

$$= X(e^{j\omega})\left[\frac{1+e^{-2j\omega}}{2}\right]$$

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{1+e^{-2jw}}{2}$$

$$= \frac{1+\cos 2w - j\sin 2w}{2}$$

$$H(e^{jw}) = \frac{1+\cos 2w}{2} - j\cdot\frac{\sin 2w}{2}$$

(i) Magnitude response

$$|H(e^{jw})| = \left| \frac{1+\cos 2w}{2} - j\frac{\sin 2w}{2} \right|$$

$\because |a+ib| = \sqrt{a^2+b^2}$

$$= \sqrt{\left(\frac{1+\cos 2w}{2}\right)^2 + \left(-\frac{\sin 2w}{2}\right)^2}$$

$\cancel{\cos 2\theta = 2\cos\theta\sin\theta}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cancel{\cos 2\theta}$

$1+\cos 2\theta = 2\cos^2\theta$

$$= \frac{1}{2}\sqrt{1+\cos^2 2w + 2\cdot\cos 2w + \sin^2 2w}$$

$$= \frac{1}{2}\cdot\sqrt{2+2\cdot\cos 2w}$$

$$= \frac{1}{2}\cdot\sqrt{2(1+\cos 2w)} = \frac{1}{2}\sqrt{2\cdot 2\cos^2 w}$$

$$|H(e^{jw})| = \cos w$$

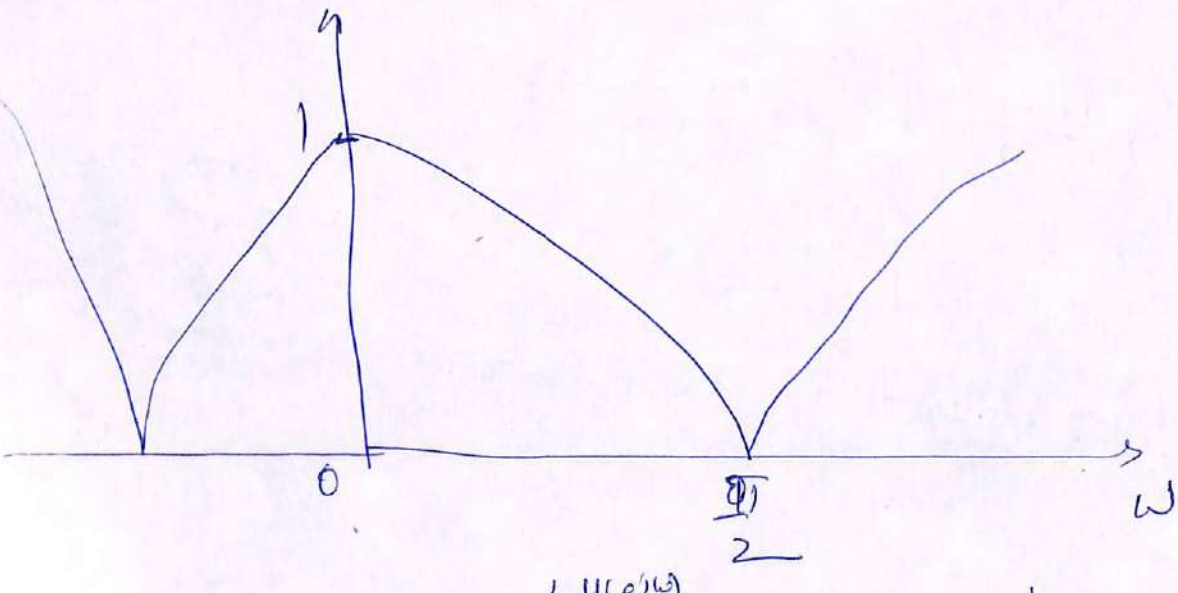
$$\angle H(e^{j\omega}) = \tan^{-1} \frac{-\sin 2\omega}{1+\cos 2\omega}$$

$$= \tan^{-1}\left(- \frac{2\sin\omega \cdot \cos\omega}{2\cos^2\omega} \right)$$

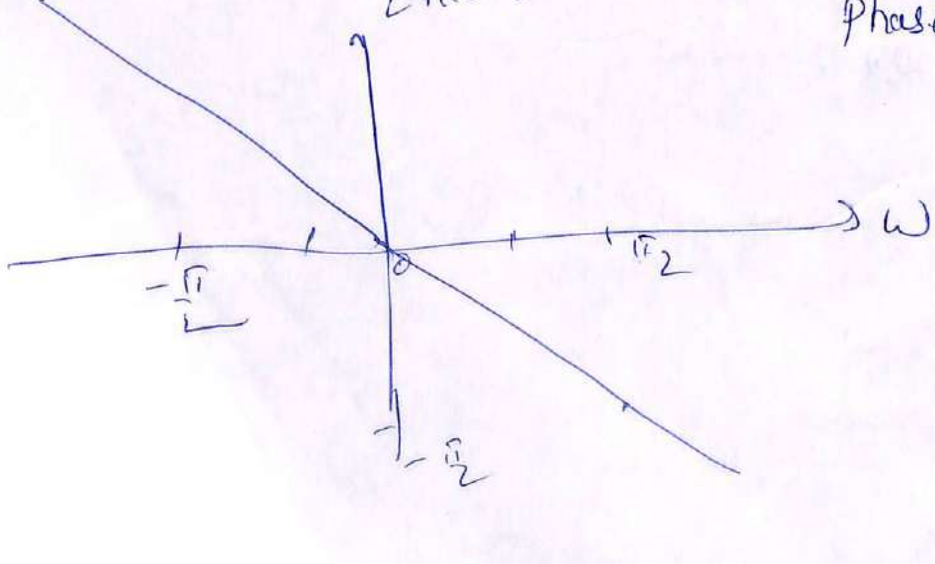
$$= \tan^{-1}(-\tan\omega) = -\omega.$$

ω	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π
$\lvert H(e^{j\omega})\rvert$	1	$0.8\ell 2$	0.707	0.5	0	0.5	0.707	1
$\angle H(e^{j\omega})$	0	$-\frac{\pi}{6}$	$-\frac{\pi}{4}$	$-\frac{\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{2\pi}{3}$	$-\frac{3\pi}{4}$	$-\pi$

Magnitude response



Phase response



Applications of Z-transforms :-

→ Z-transforms plays an important role in the analysis of discrete-time signals & LTI systems.

→ The convolution of two time-domain signals is equivalent to multiplication of their corresponding Z-transforms. This property greatly simplifies the analysis of the response of an LTI system to various signals.

→ Z-transform provides us with a means of characterizing an LTI system, & its response to various signals, by its pole-zero locations.

→ Z-transform :-

The z-transform of a discrete-time signal $x(n)$ is defined as,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

→ The region of convergence (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite values.

→ Determine the Z-transforms of the following signals.

(a) $x(n) = \{1, \underset{\uparrow}{2}, 5, 7, 0, 1\}$

(b) $x(n) = (\frac{1}{2})^n u(n)$

(c) $x(n) = \delta(n)$

Sol (a) $X(Z) = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n}$

$= \sum_{n=-1}^{4} x(n) \cdot Z^{-n}$

$= x(-1) \cdot Z^{1} + x(0) + x(1) Z^{-1} + x(2) Z^{-2} + x(3) Z^{-3} + x(4) Z^{-4}$

$X(Z) = Z + 2 + 5Z^{-1} + 7Z^{-2} + Z^{-4}$

R.O.C. : entire Z-plane except at $Z = 0$ & ∞.

(b) $x(n) = (\frac{1}{2})^n u(n)$

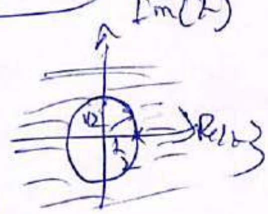
$X(Z) = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n}$

$= \sum_{n=0}^{\infty} (\frac{1}{2})^n Z^{-n}$

$= \sum_{n=0}^{\infty} (\frac{1}{2} Z^{-1})^n = \frac{1}{1 - \frac{1}{2} Z^{-1}}$

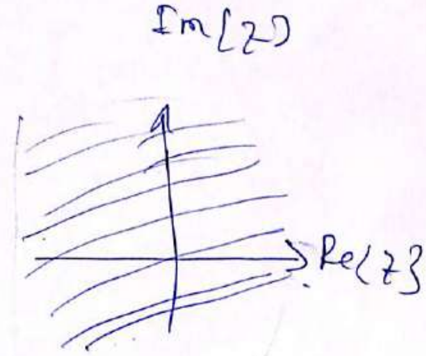
$X(Z) = \frac{1}{1 - \frac{1}{2} Z^{-1}} = \frac{Z}{Z - \frac{1}{2}}$

R.O.C: $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ if $|a|<1$



$$\left|\tfrac{1}{2} z^{-1}\right| < 1 \Rightarrow |z| > \tfrac{1}{2}.$$

(4) $x(n) = \delta(n)$



$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n)\cdot z^{-n} = 1.$$

R.O.C: Entire z-plane.

Q) Determine the z-transform of the signal,

$$x(n) = \alpha^n u(n) + b^n u(-n-1)$$

Sol
$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\alpha^n u(n) + b^n u(-n-1)\right]\cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \alpha^n u(n)\cdot z^{-n} + \sum_{n=-\infty}^{\infty} b^n u(-n-1)\cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \alpha^n\cdot z^{-n} + \sum_{n=-\infty}^{-1} b^n\cdot z^{-n}$$

$= \cancel{1}$

$$= \sum_{n=0}^{\infty} (\alpha z^{-1})^n + \sum_{n=1}^{\infty} b^{-n} z^n$$

$$= \sum_{n=0}^{\infty} (\alpha z^{-1})^n + \sum_{n=1}^{\infty} (b^{-1} z)^n$$

$$= \sum_{n=0}^{\infty} (\alpha z^{-1})^n + \sum_{n=0}^{\infty} (b^{-1} z)^n - 1$$

$$= \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-b^{-1} z} - 1$$

$$= \frac{1}{1-\alpha z^{-1}} + \frac{1-1+b^{-1} z}{1-b^{-1} z}$$

$$= \frac{z}{z-\alpha} + \frac{b^{-1} z}{1-b^{-1} z}$$

$$X(z) = \frac{z}{z-\alpha} + \frac{z}{b-z}$$

R.O.C: $|\alpha z^{-1}| < 1$ & $|b^{-1} z| < 1$

$|z| > \alpha$ & $|z| < b$.

→ Properties of Z-transforms:-

1) Liniarity:-

If $x_1(n) \xleftrightarrow{Z} X_1(Z)$

$x_2(n) \xleftrightarrow{Z} X_2(Z)$

then $a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{Z} a_1 X_1(Z) + a_2 X_2(Z)$

2) Time Shifting:-

If $x(n) \xleftrightarrow{Z} X(Z)$

then $x(n-k) \xleftrightarrow{Z} Z^{-k} X(Z)$

3) Time Reversal:-

If $x(n) \xleftrightarrow{Z} X(Z)$

then $x(-n) \xleftrightarrow{Z} X(Z^{-1})$

4) Scaling:-

If $x(n) \xleftrightarrow{Z} X(Z)$

then $a^n x(n) \xleftrightarrow{Z} X(a^{-1}Z)$

5) Differentiation in Z-domain:-

If $x(n) \xleftrightarrow{Z} X(Z)$

$n \cdot x(n) \longleftrightarrow -Z \dfrac{d\,X(Z)}{dZ}$

6) Conjugation:-

If $x(n) \xleftrightarrow{Z} X(Z)$

then $x^*(n) \xleftrightarrow{Z} X^*(Z^*)$

7, Convolution:-

$$x_1(n) * x_2(n) \xleftrightarrow{z} X_1(z) \cdot X_2(z)$$

8, Initial Value theorem:-

$$\lim_{n \to 0} x(n) = \lim_{|z| \to \infty} X(z)$$

9, Final value theorem:-

$$\lim_{n \to \infty} x(n) = \lim_{|z| \to 1} \left(\frac{z-1}{z}\right) X(z)$$

Some Common z-transform pairs:-

S.No	Signal $x(n)$	$X(z)$	R.O.C.
1.	$\delta(n)$	1	All z.
2	$u(n)$	$\frac{1}{1-z^{-1}} = \frac{z}{z-1}$	$\lvert z\rvert > 1$
3.	$a^n u(n)$	$\frac{1}{1-az^{-1}} = \frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
4.	$n \cdot a^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
5.	$\cos\omega_0 n \cdot u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
6.	$\sin\omega_0 n \cdot u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$

→ **Inverse Z-transform:-**

IZT of $X(z)$ is

→ $$x(n) = \frac{1}{2\pi j} \oint_c X(z) \cdot z^{n-1} \, dz$$

$\oint_c$ → Integration around the circle of radius $|z| = r$ in a counter clock-wise direction.

Using formula, to find Inverse ZT, it is complex.

→ So, it is found using indirect methods:

a) Power series method or long division method

b) Partial fraction expansion method

c) Complex inversion integral or residue method.

d) Convolution integral method.

→ **Partial fraction Expansion Method:-**

→ Consider the partial fraction expansion of $\frac{X(z)}{z}$ instead of $X(z)$.

$$\frac{X(z)}{z} = \frac{b_0 \cdot z^M + b_1 z^{M-1} + b_2 \cdot z^{M-2} + \cdots + b_M}{z^N + a_1 z^{N-1} + \cdots + a_N}$$, when $M \leq N$

If $M \geq N$, it is not a proper function. So write it as:

$$\frac{X(z)}{z} = \underbrace{c_0 \cdot z^{N-M} + c_1 \cdot z^{N-M-1} + \cdots + c_{N-M}}_{\text{Polynomial}} + \underbrace{\frac{N_1(z)}{D(z)}}_{\text{Proper rational fn.}}$$

Case (1) :- $\frac{X(z)}{z}$ has all distinct poles:

$$\frac{X(z)}{z} = \frac{C_1}{z-P_1} + \frac{C_2}{z-P_2} + \cdots + \frac{C_N}{z-P_N}$$

where

$$C_K = (z-P_K) \cdot \frac{X(z)}{z} \Big|_{z=P_K} \quad , k=1,2,\ldots N$$

Case (2) :- $\frac{X(z)}{z}$ has l- repeated poles & the remaining $N-l$ poles are simple.

$$\frac{X(z)}{z} = \underbrace{\frac{C_{1k}}{z-P_1} + \frac{C_{2k}}{z-P_2} + \cdots + \frac{C_{1k}}{z-P_k}}_{\text{distinct Poles. } (N-l)} + \frac{C_{k2}}{(z-P_k)^2} + \cdots + \frac{C_{kl}}{(z-P_k)^l}$$

$$C_{kl} = \frac{1}{(l-i)!} \frac{d^{(l-i)}}{dz^{l-i}} \left((z-P_0)^l \frac{X(z)}{z} \right)$$

$$C_{kl} = \frac{1}{(l-k)!} \frac{d^{l-k}}{dz^{l-k}} \left[(z-P_k)^l \cdot \frac{X(z)}{z} \right]_{z=P_k}$$

where $C_{kl} = (z-P_k)^l \frac{X(z)}{z} \Big|_{z=P_k}$

Case (3) :- $\frac{X(z)}{z}$ has Complex roots.

$$\frac{X(z)}{z} = \frac{C_1}{z-P_1} + \frac{C_1^*}{z-P_1^*}$$

Complex Conjugate poles result in Complex conjugate coefficients

Eg:- Find the Inverse Z-transform of

$$X(z) = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}}, \quad R.O.C: |z| > 1$$

Sol

$$X(z) = \frac{z}{3z^2 - 4z + 1}$$

$$\frac{X(z)}{z} = \frac{1}{3z^2 - 4z + 1} = \frac{1}{(z-1)(z-\frac{1}{3})}$$

$a = 3, \; b = -4, \; c = 1$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 - 12}}{6} = \frac{4 \pm 2}{6}$$

$$= 1, \; \frac{1}{3}.$$

$$\frac{X(z)}{z} = \frac{C_1}{z-1} + \frac{C_2}{z - 1/3}.$$

$$C_1 = (z-1) \cdot \frac{X(z)}{z}\Big|_{z=1} = (\cancel{z-1}) \cdot \frac{1}{(\cancel{z-1})(z-\frac{1}{3})}\Big|_{z=1}$$

$$= \frac{1}{1-\frac{1}{3}} = \frac{1}{2/3} = \frac{3}{2}.$$

$$C_2 = (\cancel{z-\tfrac{1}{3}}) \cdot \frac{1}{(z-1)(\cancel{z-\frac{1}{3}})}\Big|_{z=\frac{1}{3}} = \frac{1}{\frac{1}{3}-1} = -\frac{3}{2}.$$

$$\therefore \frac{X(z)}{z} = \frac{3/2}{z-1} - \frac{3/2}{z-1/3}.$$

$$X(z) = \frac{3}{2} \cdot \frac{z}{z-1} - \frac{3}{2} \cdot \frac{z}{z-1/3}$$

$\because$ IZT

$$x(n) = \frac{3}{2}\left[u(n) - \left(\frac{1}{3}\right)^n u(n)\right];$$

$$(\because a^n x(n) \longleftrightarrow \frac{z}{z-a}.$$

→ <u>Transform Analysis of LTI System:-</u>

(i) <u>System Function (transfer fn) & Impulse response:-</u>

→ Consider an LTI system having an impulse response $h(n)$.

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

then $y(n) = x(n) * h(n)$

take z-transform

$$Y(z) = ZT[x(n) * h(n)]$$
$$Y(z) = X(z) \cdot H(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)}.$$

→ $H(z)$ is called System function or transfer fn.

→ If the I/P $x(n)$ is an impulse sequence, then

$X(z) = 1$. So, $Y(z) = H(z)$.

Relationship between Transfer Function & Difference Equation:-

→ N^{th} order discrete-time LTI system is

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k), \quad a_0 = 1.$$

Expand

$$a_0\, y(n) + a_1 y(n-1) + \cdots + a_N y(n-N) = b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M)$$

take Z-transform on B.S.

$$a_0\, Y(z) + a_1 z^{-1} Y(z) + \cdots + a_N z^{-N} Y(z) = b_0 X(z) + \cdots + b_M z^{-M} X(z)$$

$$Y(z)\left[a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}\right] = X(z)\left[b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

→ To find freq. response, put $z = e^{j\omega}$ in $H(z)$.

→ Stability & Casuality :-

1, Stability :-

→ We know that the necessary & sufficient condition for a causual linear time-invariant discrete-time system to be BIBO stable is :

$$\sum_{n=0}^{\infty} |h(n)| < \infty.$$

i.e an LTI discrete-time system is BIBO stable if its impulse response is absolutely summable.

→ For a causual LTI system to be stable, all the poles of $H(z)$ must lie inside the unit circle in the z-plane. i.e for a causal LTI system to be stable, the ROC of the system function must include the unit circle.

2) Causuality :-

→ For a system to be causal, its impulse response must be equal to zero for $n < 0$.

i.e $[h(n) = 0$ for $n < 0]$.

→ then, the R.O.C for $H(z)$ will be outside the outermost pole.

Problems :-

(1) Consider an LTI System with a System function

$H(z) = \frac{1}{1-(1/2)z^{-1}}$. Find the difference equation.

Determine the Stability.

Sol Given $H(z) = \frac{1}{1-(1/2)z^{-1}} = \frac{z}{z-1/2}$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z-1/2} = \frac{1}{1-(1/2)z^{-1}}$$

$$Y(z)\left(1-\left(\tfrac{1}{2}\right)z^{-1}\right) = X(z)$$

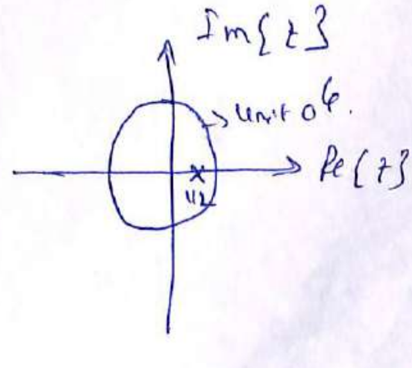
$$Y(z) - \tfrac{1}{2}\cdot z^{-1}Y(z) = X(z)$$

take Inverse Z-transform

$$y(n) - \tfrac{1}{2}\cdot y(n-1) = x(n).$$

Poles of H(z) are values of z at which $H(z) = \infty$.

$\therefore$ pole of H(z) is at $z = 1/2$.



Since all the poles inside the unit circle, the System is stable.

Q) A casual system is represented by,

$$H(z) = \frac{z+2}{2z^2-3z+4}$$

Find the difference equation & freq. response of system.

Multiply with z^{-2} on Nr & Dr

$$H(z) = \frac{z^{-1}+2z^{-2}}{2-3z^{-1}+4z^{-2}}$$

$$\frac{Y(z)}{X(z)} = \frac{z^{-1}+2z^{-2}}{2-3z^{-1}+4z^{-2}}$$

$$Y(z)\left[2-3z^{-1}+4z^{-2}\right] = X(z)\left[z^{-1}+2z^{-2}\right]$$

$$2.Y(z) - 3.z^{-1}Y(z) + 4z^{-2}Y(z) = z^{-1}X(z) + 2.z^{-2}X(z)$$

take Inverse Z-transform.

$$2.y(n) - 3y(n-1) + 4y(n-2) = x(n-1) + 2.x(n-2)$$

→ To find freq. response: $z = e^{j\omega}$ in $H(z)$

$$H(e^{j\omega}) = \frac{e^{j\omega}+2}{2e^{2j\omega}-3e^{j\omega}+4}$$

1-46

(3) Find the impulse response of the system described by difference equation $y(n) - 3\cdot y(n-1) - 4y(n-2) = x(n) + 2\cdot x(n-1)$ using Z-transform.

Sol

$$y(n) - 3y(n-1) - 4\cdot y(n-2) = x(n) + 2\cdot x(n-1)$$

Apply Z-transform on B S.

$$Y(z) - 3\cdot z^{-1}Y(z) - 4\cdot z^{-2}Y(z) = X(z) + 2\cdot z^{-1}X(z)$$

$$Y(z)\left[1 - 3z^{-1} - 4z^{-2}\right] = X(z)\left[1 + 2z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+2z^{-1}}{1-3z^{-1}-4z^{-2}}$$

$$H(z) = \frac{z(z+2)}{z^2-3z-4}$$

$$\frac{H(z)}{z} = \frac{z+2}{z^2-3z-4} = \frac{z+2}{(z-4)(z+1)}$$

$$= \frac{A}{z-4} + \frac{B}{z+1}$$

$$\frac{H(z)}{z} = \frac{6/5}{z-4} + \frac{-1/5}{z+1}$$

$$H(z) = 6/5\cdot\frac{z}{z-4} + -1/5\cdot\frac{z}{z+1}$$

$$h(n) = \frac{6}{5}\cdot(4)^n\cdot u(n) - \frac{1}{5}\cdot(-1)^n u(n)$$

4) Determine the unit step response of the system whose difference equation is $y(n) - 0.7y(n-1) + 0.12y(n-2) = x(n-1) + x(n-2)$ if $y(-1) = y(-2) = 1$.

Sol $y(n) - 0.7y(n-1) + 0.12y(n-2) = x(n-1) + x(n-2)$

Apply Z-transform on B.S

~~$Y(z) - 0.7z^{-1}Y(z) + 0.12z^{-2}Y(z) = z^{-1}X(z) + z^{-2}X(z)$~~

With initial conditions,

$$\boxed{\begin{aligned} Z\{x(n-m)\} &= z^{-m}\left\{X(z) + \sum_{k=1}^{m} x(-k)\cdot z^{k}\right\} \\ Z\{x(n+m)\} &= z^{m}\left\{X(z) - \sum_{k=0}^{m-1} x(k)\cdot z^{-k}\right\} \end{aligned}}$$

$$Y(z) - 0.7[z^{-1}Y(z) + y(-1)] + 0.12[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)] = z^{-1}X(z) + x(-1) + z^{2}X(z) + z^{-1}x(-1) + x(-2)$$

$y(-1) = y(-2) = 1, \quad x(-1) = x(-2) = 0$

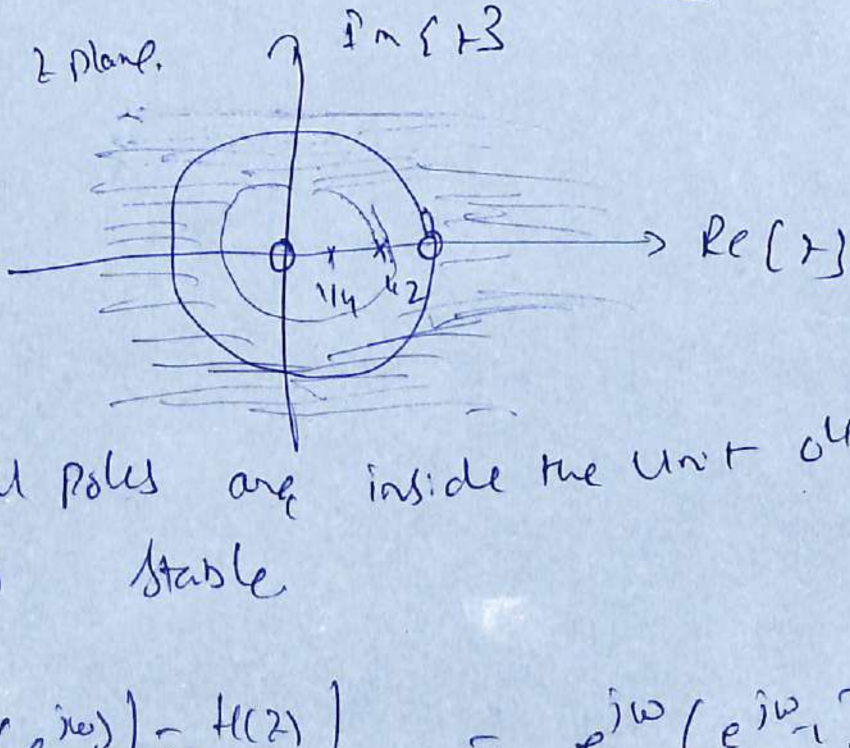
Determine the pole-zero plot for the system described by the difference eqn. $y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n) - x(n-1)$ & also stability & find freq. response [L-4]

$$H(z) = \frac{1 - z^{-1}}{1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}}$$

$$= \frac{z(z-1)}{(z-\frac{1}{4})(z-\frac{1}{2})}$$

R.O.C $= |z| > 1/2$

Zeros $= 0, 1$, Poles $= \frac{1}{4}, \frac{1}{2}$



Since all poles are inside the unit cir, the system is stable

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \frac{e^{j\omega}(e^{j\omega}-1)}{(e^{j\omega}-\frac{1}{4})(e^{j\omega}-\frac{1}{2})}$$

Mag response $= |H(e^{j\omega})|$

Phase response $= \angle H(e^{j\omega})$

→ Evaluate the freq. response of the system

$$y(n) - 0.5\,y(n-1) = x(n)\,.$$

Sol

$$Y(z) - 0.5\,z^{-1}\,Y(z) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.5z^{-1}} = \frac{z}{z-0.5}\,,$$

ROC: $|z| > \frac{1}{2}$

As ROC includes unit circle, $H(e^{j\omega})$ exists.

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \frac{e^{j\omega}}{e^{j\omega}-0.5}$$

$$= \frac{\cos\omega + j\sin\omega}{\cos\omega + j\sin\omega - 0.5}$$

$$|H(e^{j\omega})| = \frac{|\cos\omega + j\sin\omega|}{|\cos\omega - 0.5 + j\sin\omega|}$$

$$= \frac{1}{\sqrt{(\cos\omega - 0.5)^2 + \sin^2\omega}}$$

$$|H(e^{j\omega})| = \frac{1}{\sqrt{1.25 - \cos\omega}}$$

$$\angle H(e^{j\omega}) = \tan^{-1}\frac{b}{a} = \frac{(\cos\omega + j\sin\omega)(\cos\omega - 0.5 - j\sin\omega)}{(\cos\omega - 0.5)^2 + \sin 2\omega}$$

$$= \omega - \tan^{-1}\frac{\sin\omega}{\cos\omega - 0.5}$$

(1-48)

→ A Casual system has i/p $x(n)$ & o/p $y(n)$. find the system fn. freq response & impulse response of the system if

$$x(n) = \delta(n) + \frac{1}{6}\delta(n-1) - \frac{1}{6}\delta(n-2)$$

$$y(n) = \delta(n) - \frac{2}{3}\delta(n+1)$$

Sg

$$X(z) = 1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}$$

$$Y(z) = 1 - \frac{2}{3}\cdot z^{-1}$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1-\frac{2}{3}z^{-1}}{1+\frac{1}{6}z^{-1}-\frac{2}{6}z^{-2}}$$

$$H(z) = \frac{z\left(z-\frac{2}{3}\right)}{\left(z-\frac{1}{3}\right)\left(z+\frac{1}{2}\right)}$$

freq resp. $H(e^{j\omega}) = \dfrac{e^{j\omega}\left(e^{j\omega}-\frac{2}{3}\right)}{\left(e^{j\omega}-\frac{1}{3}\right)\left(e^{j\omega}+\frac{1}{2}\right)}$

By partl fr. exp

$$\frac{H(z)}{z} = \frac{z-\frac{2}{3}}{\left(z-\frac{1}{3}\right)\left(z+\frac{1}{2}\right)}$$

$$\frac{H(z)}{z} = \frac{-2/5}{z-\frac{1}{3}} + \frac{7/5}{z+1/2}$$

$$h(n) = -\frac{2}{5}\left(\frac{1}{3}\right)^n u(n) + \frac{7}{5}\cdot\left(-\frac{1}{2}\right)^n u(n)$$

2) $x(n) = (\frac{1}{3})^n u(n) - \frac{1}{5}(\frac{1}{3})^{n-1} u(n-1)$, $y(n) = (\frac{1}{2})^n u(n)$

de $H(z)$, impulse resp. $h(n)$ & freq response $H(\omega)$ of the system that satisfies this condition. Also answer the stability

$$X(z) = \frac{z}{z-\frac{1}{3}} - \frac{1}{5} \cdot z^{-1} \cdot \frac{z}{z-1/3}$$

$$X(z) = \frac{z-1/5}{z-1/3}, \quad Y(z) = \frac{z}{z-1/2}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z(z-1/3)}{(z-1/2)(z-1/5)}$$

$$\frac{H(z)}{z} = \frac{5/9}{z-1/2} + \frac{4/9}{z-1/5}$$

$$H(z) = \frac{5}{9} \cdot \frac{z}{z-1/2} + \frac{4}{9} \cdot \frac{z}{z-1/5}$$

$$h(n) = \frac{5}{9}(\frac{1}{2})^n u(n) + \frac{4}{9}(\frac{1}{5})^n u(n)$$

$$H(e^{j\omega}) = \frac{e^{j\omega}(e^{j\omega}-1/3)}{(e^{j\omega}-1/2)(e^{j\omega}-1/5)}$$

Realization of Digital filters:-

→ A digital filter transfer (system) function can be realized in two ways:

1) for recursive realization, the current o/p $y(n)$ is a function of past o/p's, past & present i/p's. This form corresponds to an IIR digital filter.

2) for non-recursive realization current o/p sample $y(n)$ is a function of only past & present i/p's. This form corresponds to FIR ~~A~~ digital filter.

→ IIR & FIR filter can be realized in many forms. they are:

1) Direct form { Direct form-I, Direct form-II (Canonic) }

2) Cascade form

3) Parallel form

4) Canonic

1) Direct form-I Realization:-

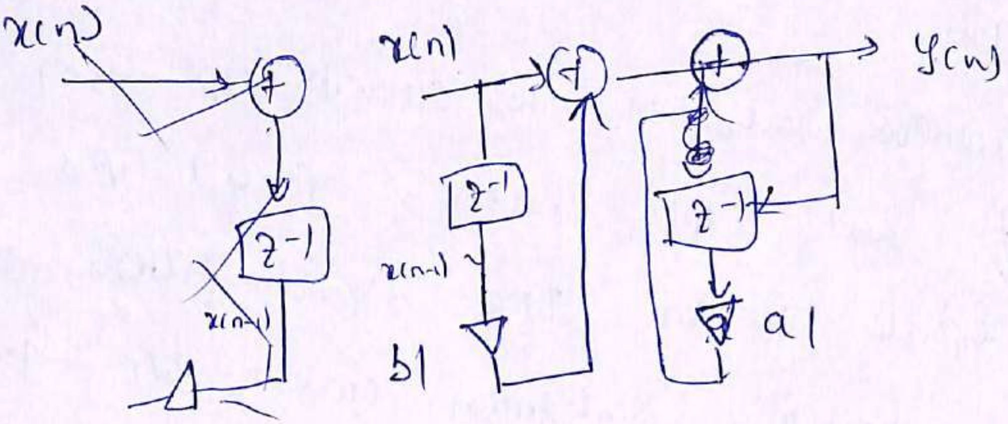
→ The basic building blocks for realization of filters are

1) Addition → $x_1(n) \rightarrow (+) \leftarrow x_2(n)$, $\rightarrow x_1(n) + x_2(n)$

2) multiplier → $x(n) \xrightarrow{a} a \cdot x(n)$

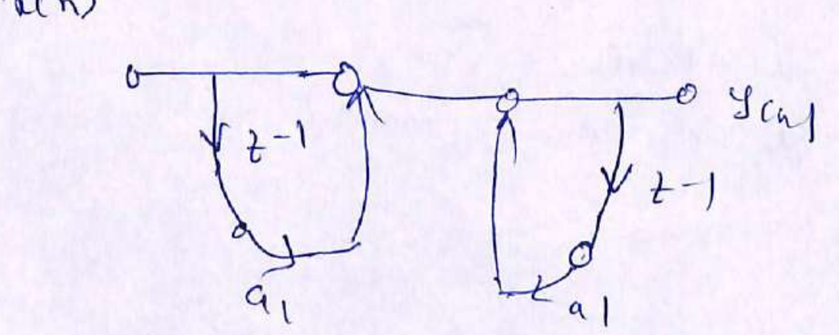
3) delay → $x(n) \rightarrow \boxed{z^{-1}} \rightarrow x(n-1)$

For Eg: $y(n) = a_1 \cdot y(n-1) + x(n) + b_1 x(n-1)$



~~Direct~~

→ In signal flow graph, the nodes represent both branch points & adders in the block diagram



Direct form-I Realization:—

→ ~~we~~ Let us consider an LTI recursive system described by difference equation.

$$y(n) = -\sum_{k=1}^{N} a_k \, y(n-k) + \sum_{k=0}^{M} b_k \, x(n-k)$$

Let $$w(n) = \sum_{k=0}^{M} b_k \, x(n-k)$$

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + w(n).$$

$$\therefore y(n) = -a_1 y(n-1) - a_2 y(n-2) + \cdots - a_N y(n-N) + w(n) \quad \text{—(1)}$$

where $w(n) = \sum_{k=0}^{M} b_k x(n-k)$

$$w(n) = b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M) \quad \text{—(2)}$$

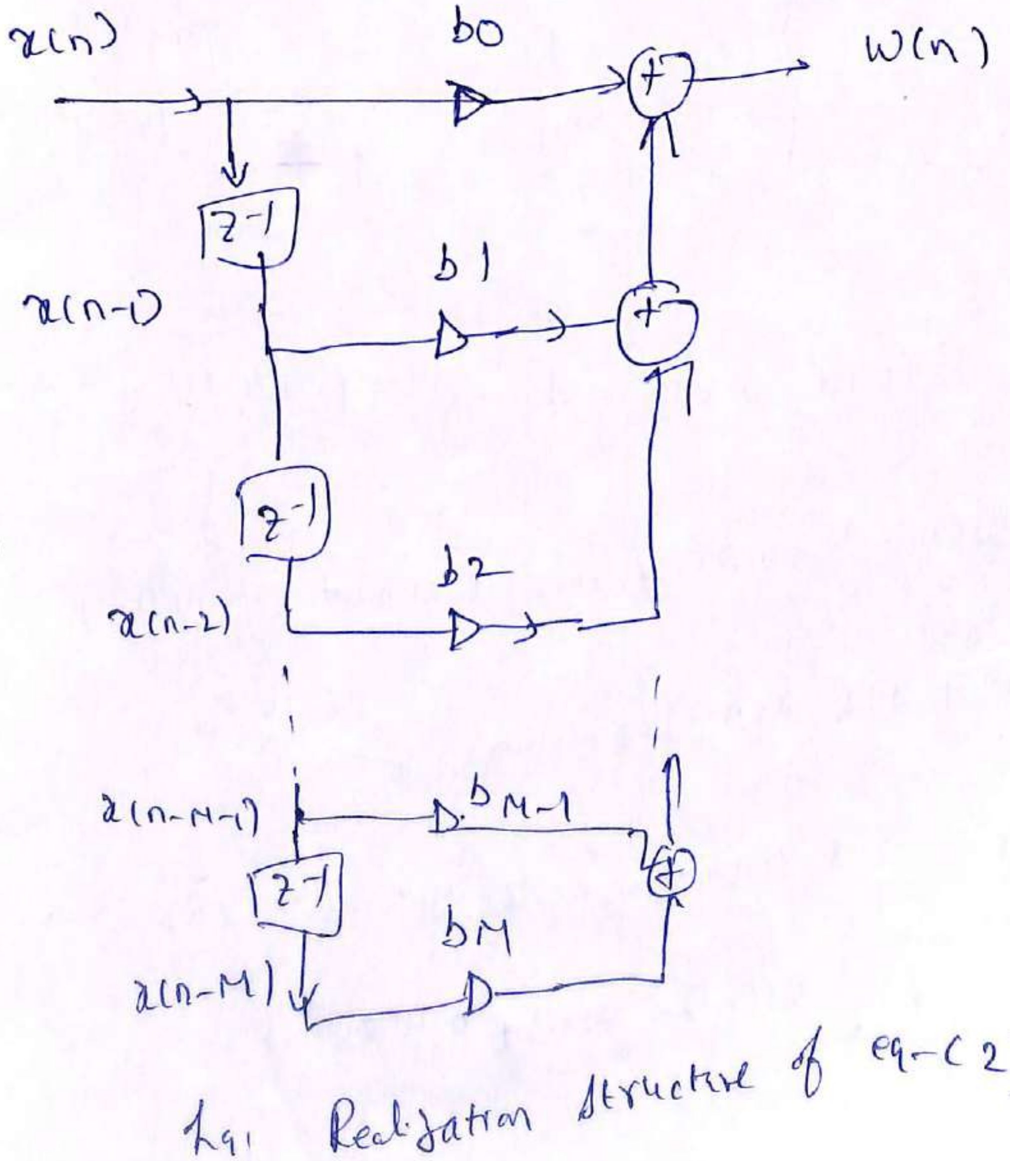
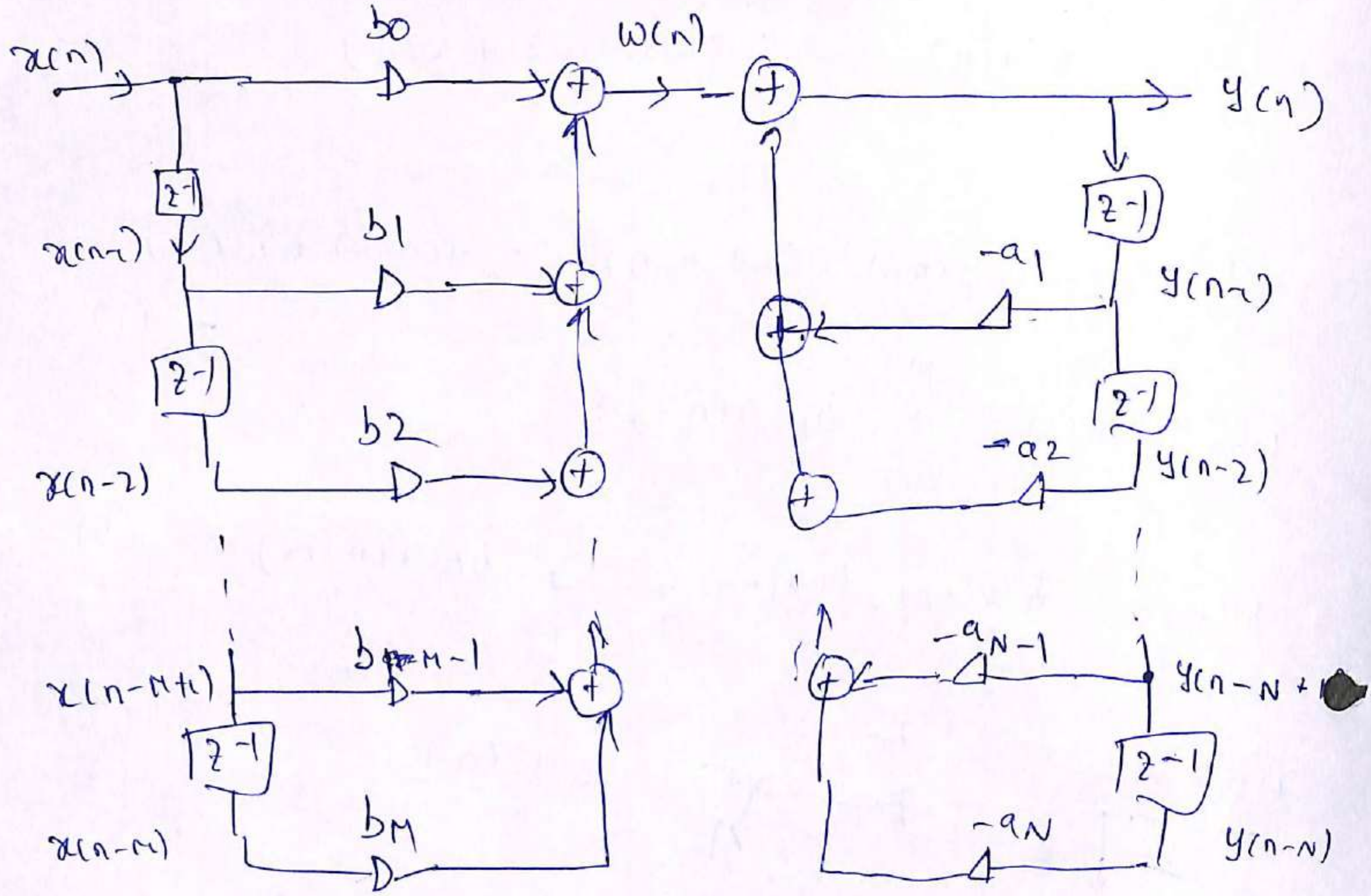


fig: Realization structure of eq-(2)



Eg:-

(i) Realize the second order digital filter using direct form - I

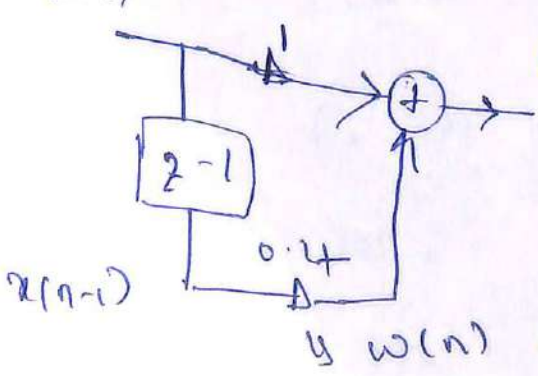
$y(n) = 0.5\, y(n-1) - 0.25\, y(n-2) + x(n) + 0.4\, x(n-1)$

Sol

let

$$x(n) + 0.4\, x(n-1) = w(n) \quad -(1)$$

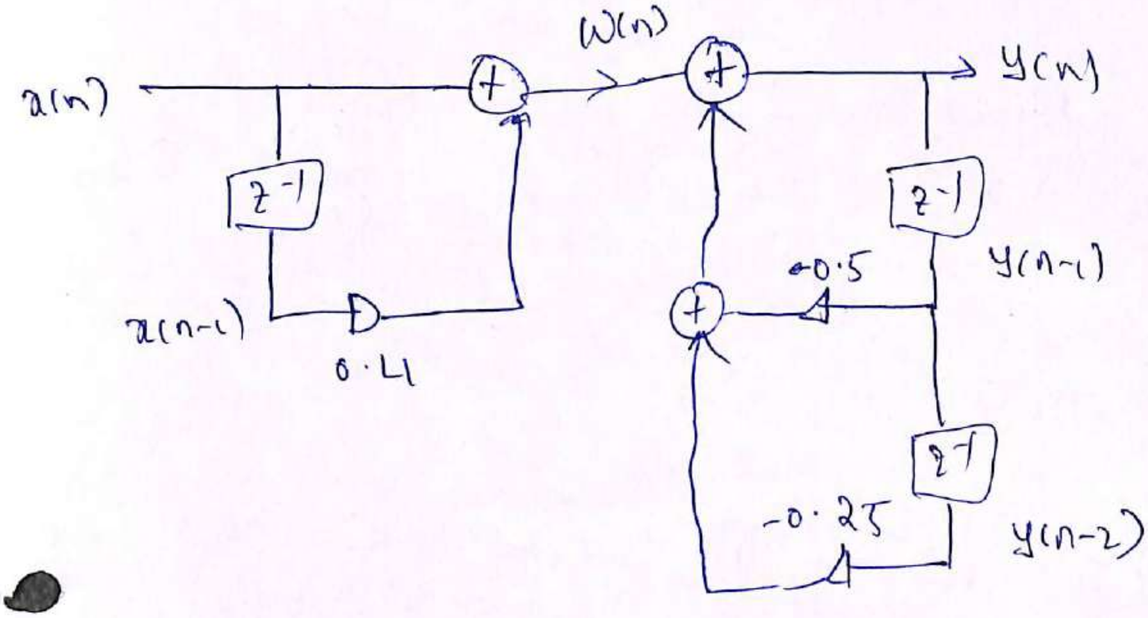
$$y(n) = 0.5\, y(n-1) - 0.25\, y(n-2) + w(n) \quad -(2)$$



$w(n) = x(n) + 0.4\, x(n-1)$

y w(n) realization

∴ Direct form realization of eq-(2) is



2) $y(n) = 2y(n-1) + 3y(n-2) + x(n) + 2 \cdot x(n-1) + 3 \cdot x(n-2)$

3) $y(n) = 0.5\,y(n-1) + 0.06\,y(n-2) + 0.3\,x(n) + 0.5\,x(n-1)$

Direct form-II realization:- (or canonic form).

→ Consider the difference eqn.

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k)$$

The System function,

$$\cancel{H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M} b_k \cdot x(n-k)}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k \cdot z^{-k}}$$

Let $\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$

where $\frac{W(z)}{X(z)} = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}$, $\frac{Y(z)}{W(z)} = \sum_{k=0}^{M} b_k z^{-k}$

$$W(z)\left(1+\sum_{k=1}^{N} a_k z^{-k}\right) = X(z) \quad -$$

$$W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) - \cdots - a_N z^{-N} W(z)$$

taking IZT

$$W(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) - \cdots - a_N w(n-N) \quad -(1)$$

"

$$\frac{Y(z)}{W(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

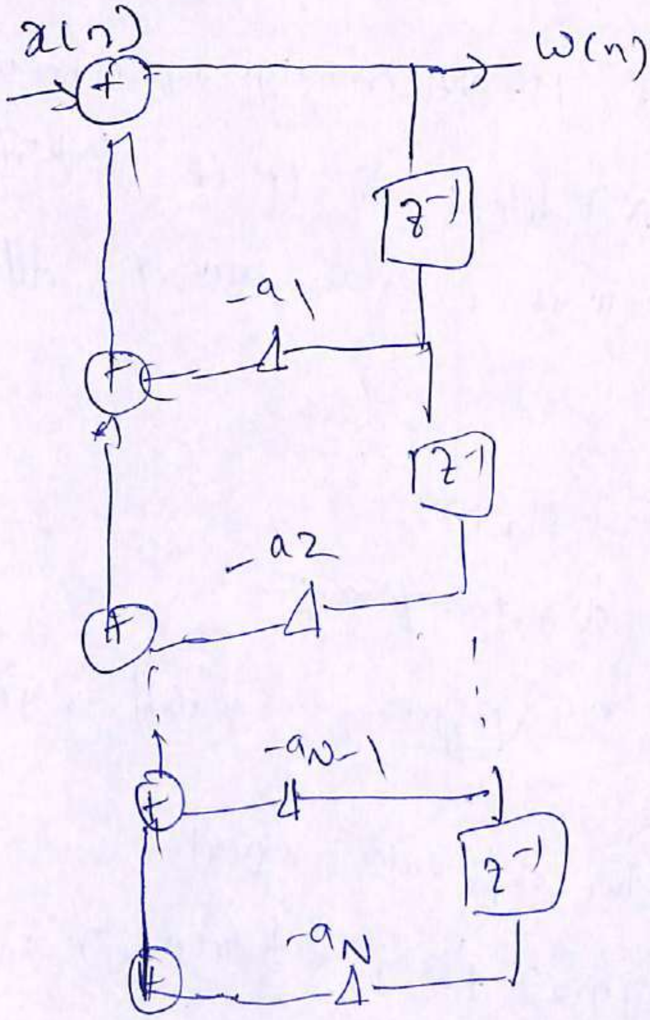
$$Y(z) = W(z)\left[b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}\right]$$

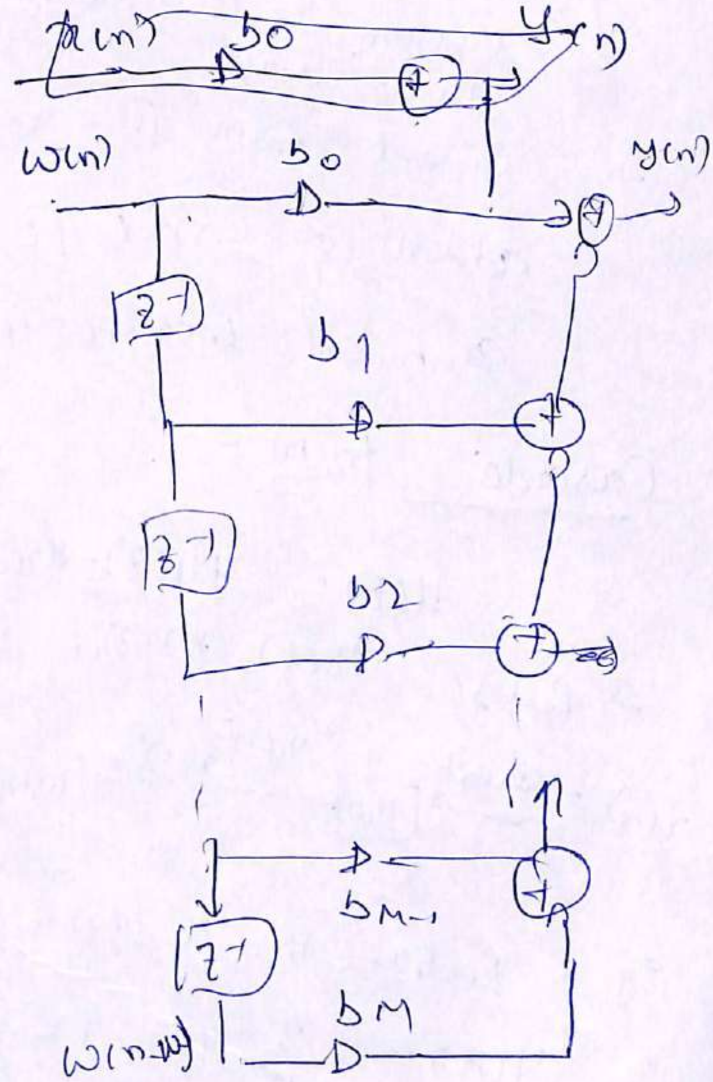
taking IZT

$$y(n) = b_0 \cdot w(n) + b_1 w(n-1) + \cdots + b_M w(n-M) \quad -(2)$$

drawing eq-(1)

eq-(2)





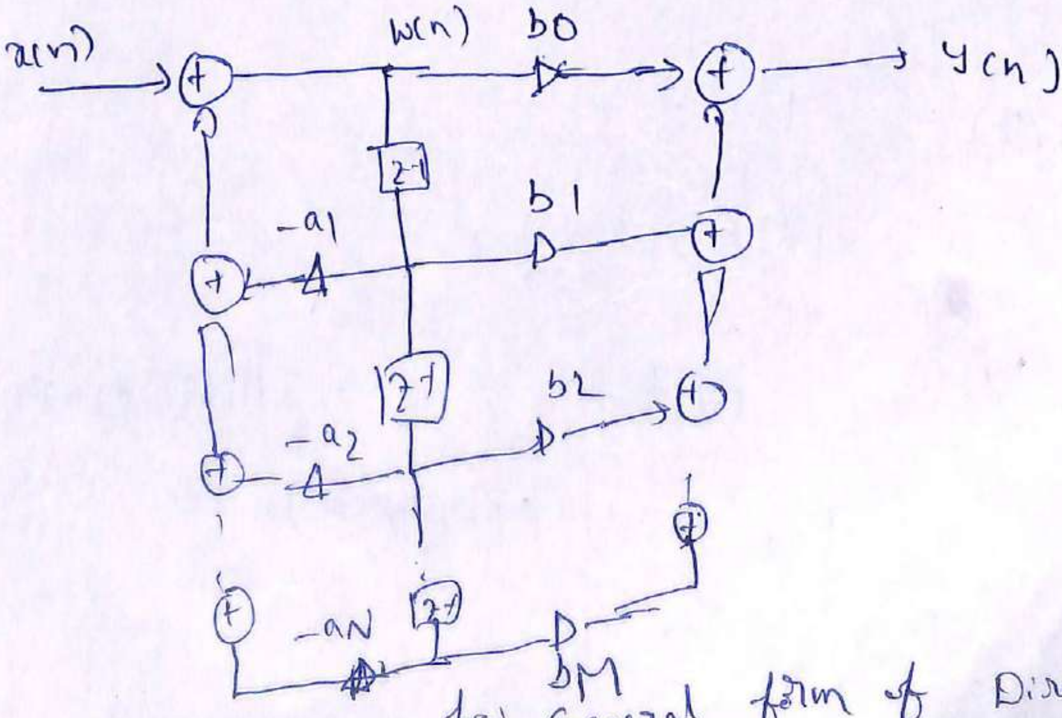


fig: General form of Direct - form -II

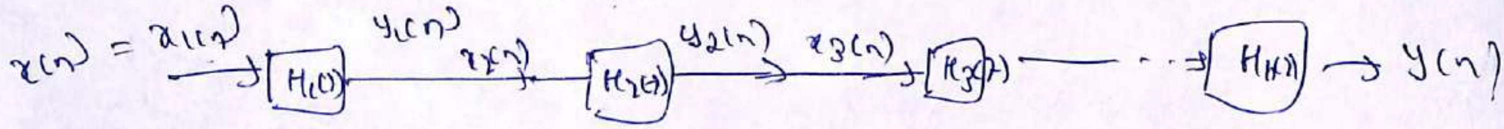
→ Direct form-I requires $M+N+1$ multipliers, $M+N$ adders & $M+N+1$ delay for M^{th} & N^{th} order IIR system. & It is called non-canonical. b'coz it requires more no of delays.

→ Direct form-II requires $M+N+1$ multipliers, $M+N$ adders & max. [(M, N] delays. & It is called canonical, because it requires a min. no of delays.

<u>Cascade form</u> :-

$$H(z) = H_1(z) \cdot H_2(z) \cdots H_k(z).$$

→ Realize $H_1(z)$, $H_2(z)$, ... in direct form-II.



<u>Eg</u>:- Realize the system with difference equation,

$$y(n) = \frac{3}{4} y(n-1) - \frac{1}{8} y(n-2) + x(n) + \frac{1}{3} x(n-1)$$ in cascade form.

$$H(z) = \frac{1 + \frac{1}{3} z^{-1}}{1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}}$$

$$= \frac{1 + \frac{1}{3} z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 - \frac{1}{4} z^{-1})} = H_1(z) \cdot H_2(z)$$

$$H_1(z) = \frac{1 + \frac{1}{3} z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

Parallel- form realization:-

$$H(z) = \frac{Y(z)}{X(z)} = C + \sum_{i=1}^{k} H_i(z).$$

$$= C + \frac{C_1}{1-P_1 z^{-1}} + \frac{C_2}{1-P_2 z^{-2}} + \cdots + \frac{C_N}{1-P_N z^{-1}}$$

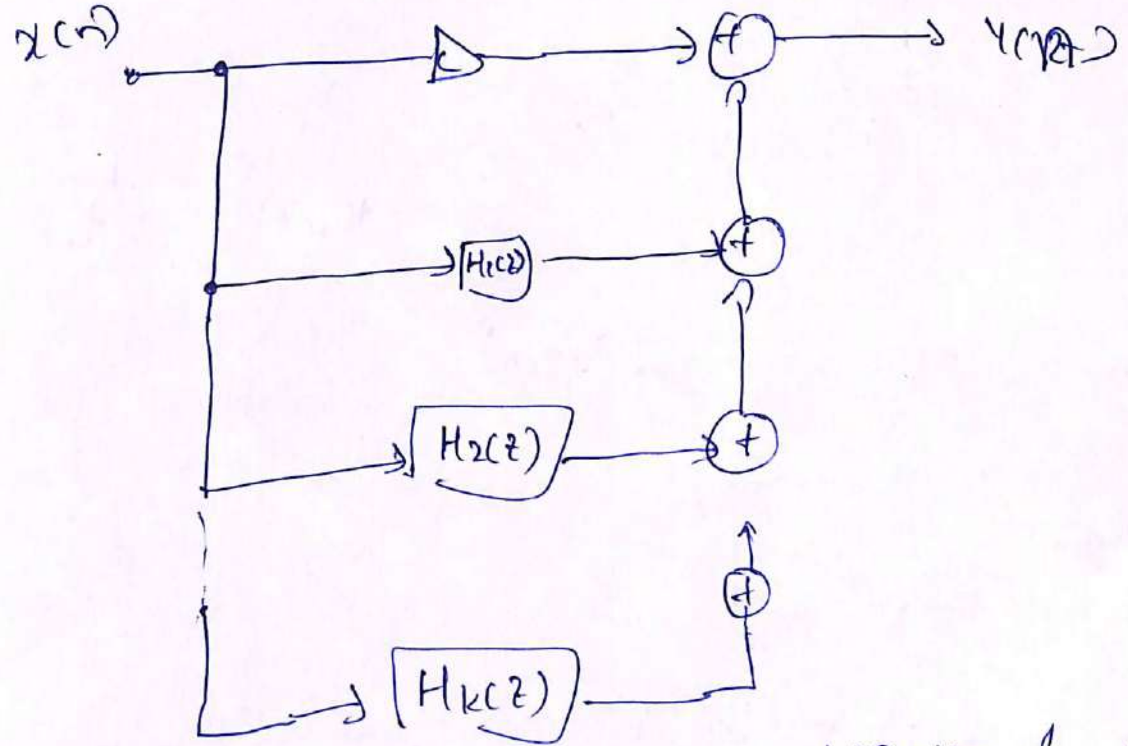
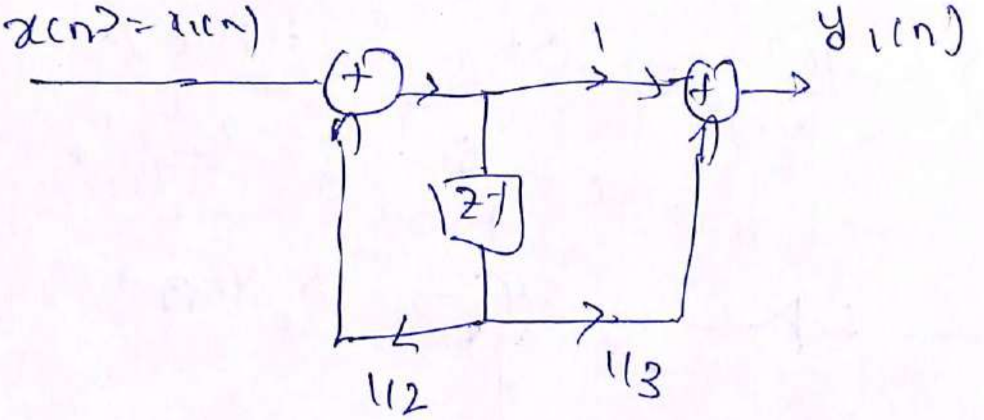


fig: Parallel form realization of IIR system.

→ As the filter operation is performed in parallel i.e the processing is performed simultaneously, the parallel form structure is used for high-speed filtering action.

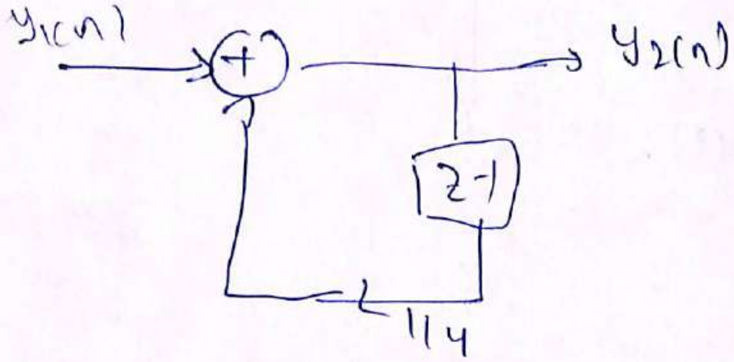
→ The drawback is expressing the transfer function in partial form is not easy for high order systems.

$H_1(z)$ in direct form-II as

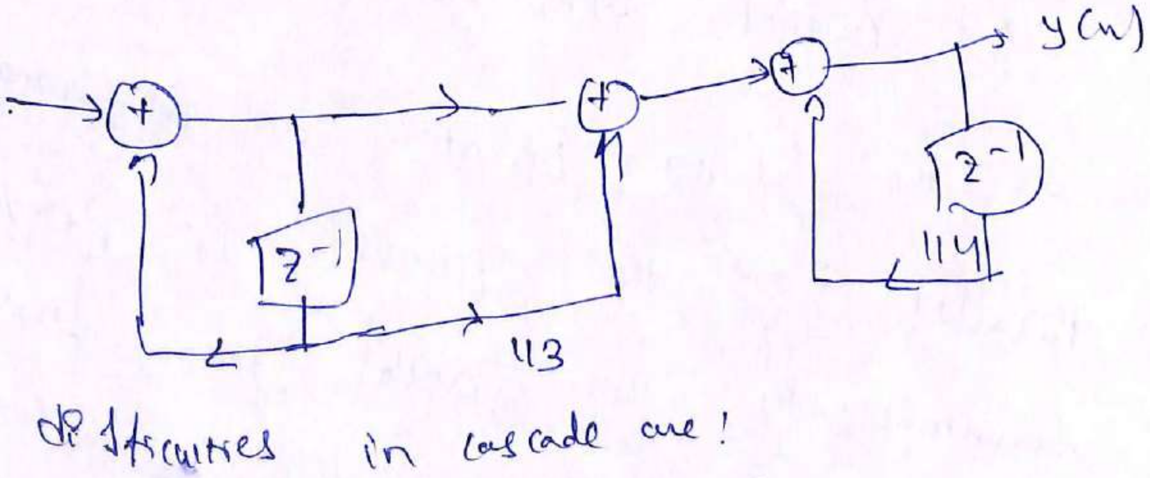


$H_2(z)$

$H_2(z) = \frac{1}{1-\frac{1}{4}z^{-1}}$



Cascading $H_1(z)$ & $H_2(z)$,



The difficulties in cascade are!

1) Decision of pairing poles & zeros

2) Deciding the order of cascading the 1st & 2nd order sections.

realize eqn (1) & (2).

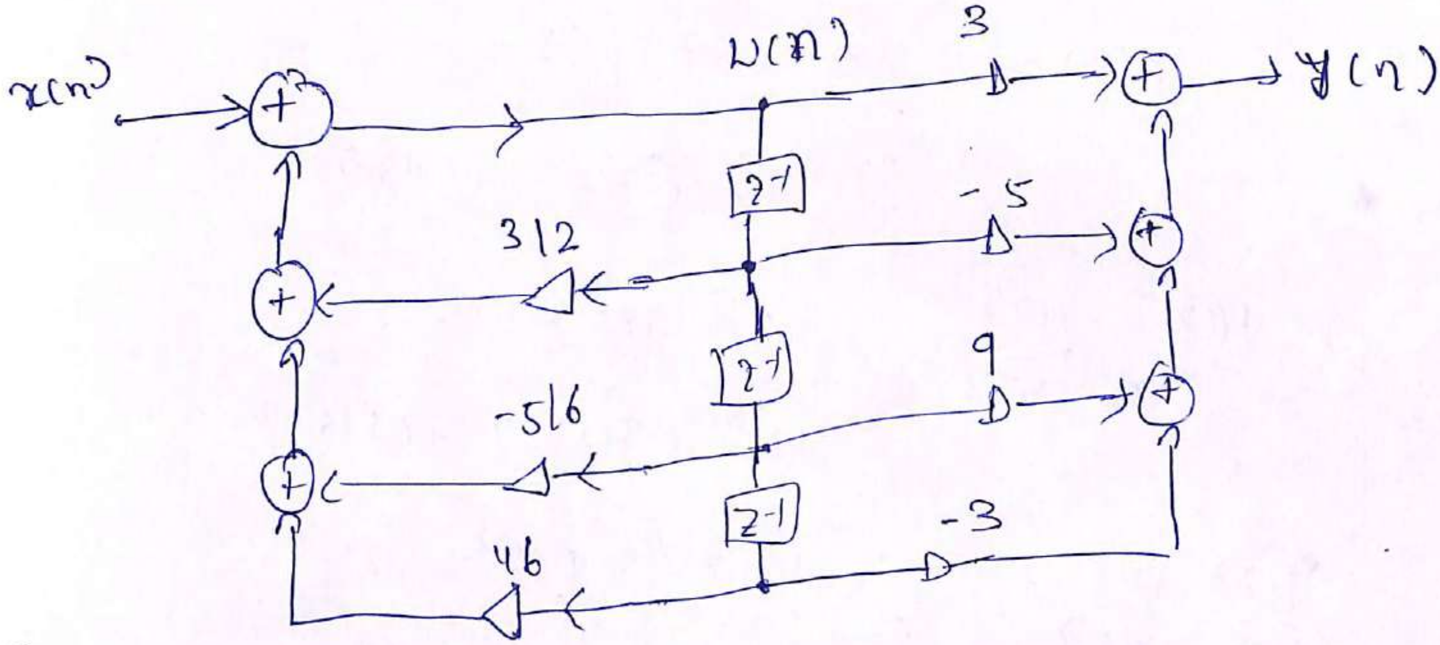


fig: Direct - form - II.

Eg:- Find the direct form-II realizations of a discrete-time system represented by the transfer function.

$$H(z) = \frac{3z^3 - 5z^2 + 9z - 3}{(z-\frac{1}{2})(z^2 - z + (1/3))}$$

Sol

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3z^3 - 5z^2 + 9z - 3}{z^3 - (3/2)\cdot z^2 + (5/6)z - (1/6)}$$

$$\frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)} = \frac{3 - 5z^{-1} + 9z^{-2} - 3z^{-3}}{1 - \frac{3}{2}\cdot z^{-1} + \frac{5}{6}\cdot z^{-2} - 1/6 \cdot z^{-3}}$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{5}{6}\cdot z^{-2} - \frac{1}{6} z^{-3}} \quad -(1)$$

$$\frac{Y(z)}{X(z)} = 3 - 5z^{-1} + 9z^{-2} - 3z^{-3} \quad -(2)$$

from (1)

$$X(z) = W(z) - \frac{3}{2}\cdot z^{-1}W(z) + \frac{5}{6}\cdot z^{-2}W(z) - \frac{1}{6}z^{-3}W(z)$$

Apply Inverse Z.T

$$x(n) = w(n) - \frac{3}{2}\cdot w(n-1) + \frac{5}{6}w(n-2) - \frac{1}{6}w(n-3)$$

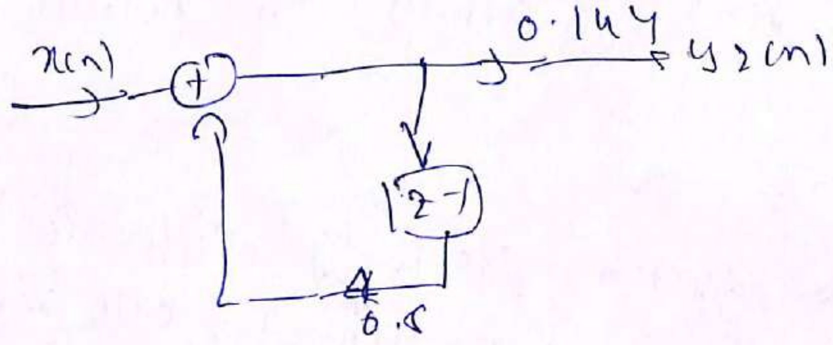
from (2)

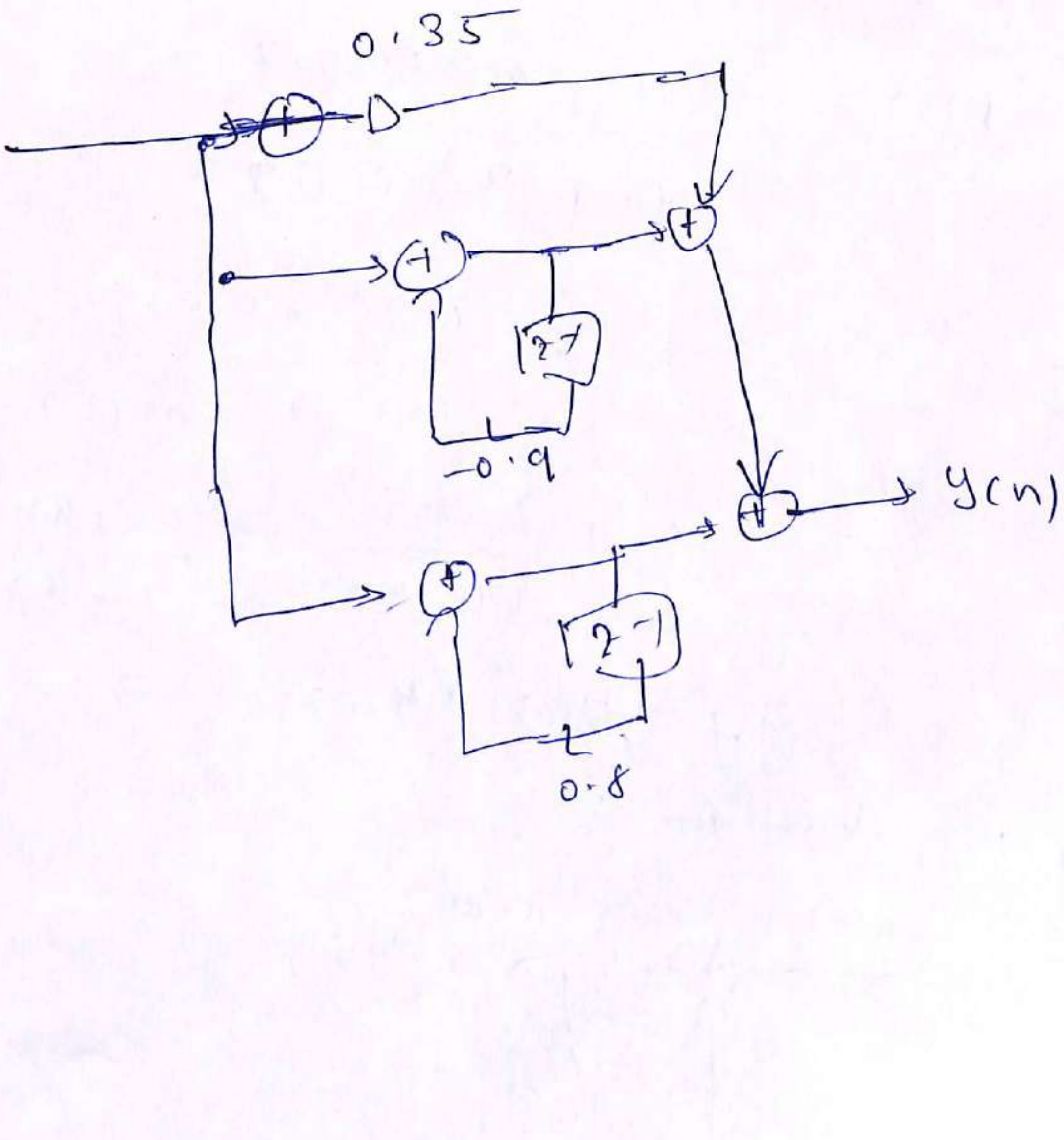
$$Y(z) = 3X(z) - 5z^{-1}W(z) + 9z^{-2}W(z) - 3z^{-3}W(z)$$

Apply Inverse Z.T

$$y(n) = 3\cdot x(n) - 5w(n-1) + 9w(n-2) - 3\cdot w(n-3)$$

$H_2(z)$.





Eg:-1. Realize the system with difference eqn. $y(n) = \frac{3}{4} y(n-1) - \frac{1}{4} y(n-2) + x(n) + \frac{1}{3} x(n-1)$ in parallel form.

Sol

~~$H(z) =$~~

2. Realize the system given by difference equation $y(n) = -0.1\, y(n-1) + 0.72\, y(n-2) + 0.7\, x(n) - 0.252\, x(n-2)$ in parallel form

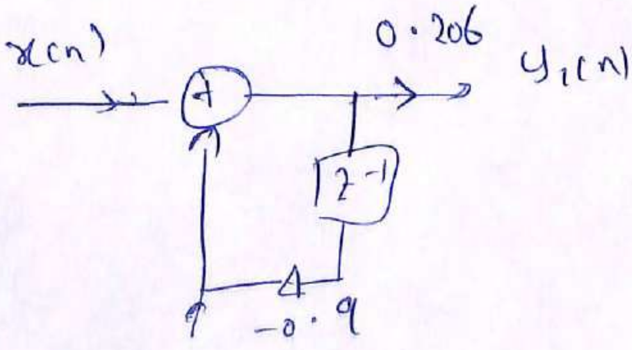
$$H(z) = \frac{0.7 - 0.252 z^{-2}}{1 + 0.1 z^{-1} - 0.72 z^{-2}}$$

$$= 0.35 + \frac{0.35 - 0.035 z^{-1}}{1 + 0.1 z^{-1} - 0.72 z^{-2}}$$

$$= 0.35 + \frac{0.206}{1 + 0.9 z^{-1}} + \frac{0.144}{1 - 0.8 z^{-1}}$$

$$= c + H_1(z) + H_2(z)$$

$H_1(z)$ in direct form-II.



→ Obtain the direct form I, direct form II, Cascade & Parallel form realization for the system $y(n) = -0.1\,y(n-1) + 0.2\,y(n-2) + 3x(n) + 3.6\,x(n-1) + 0.6\,x(n-2)$

Sol

Direct form I :-

let $w(n) = 3.x(n) + 3.6\,x(n-1) + 0.6\,x(n-2)$

$$y(n) = -0.1\,y(n-1) + 0.2\,y(n-2) + w(n)$$

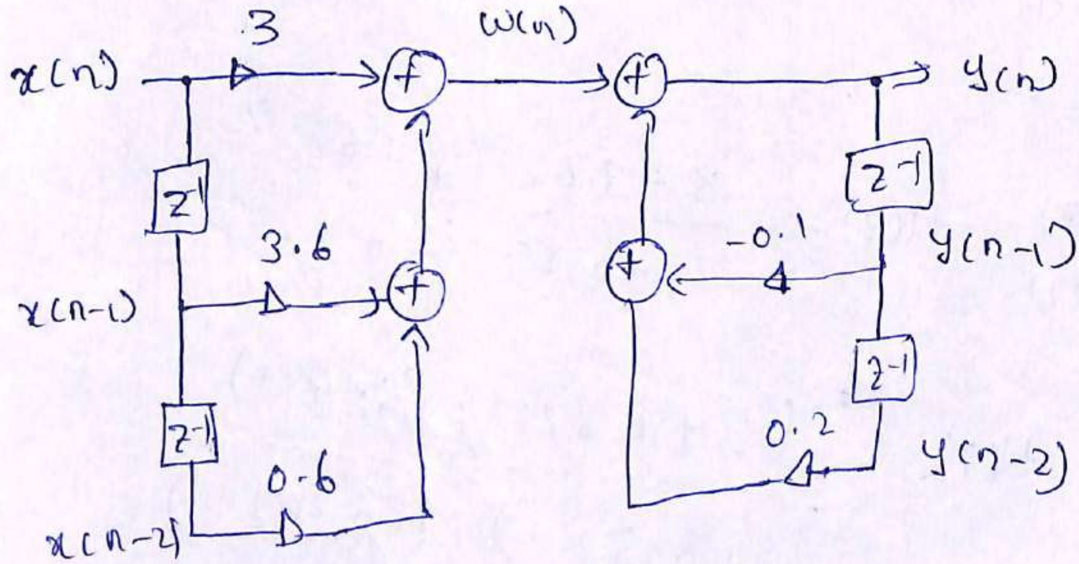


fig: Direct form I realization

Direct form II :-

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

$\therefore$ $b_0 = 3,\ b_1 = 3.6,\ b_2 = 0.6$

$a_0 = 1,\ a_1 = 0.1,\ a_2 = -0.2$

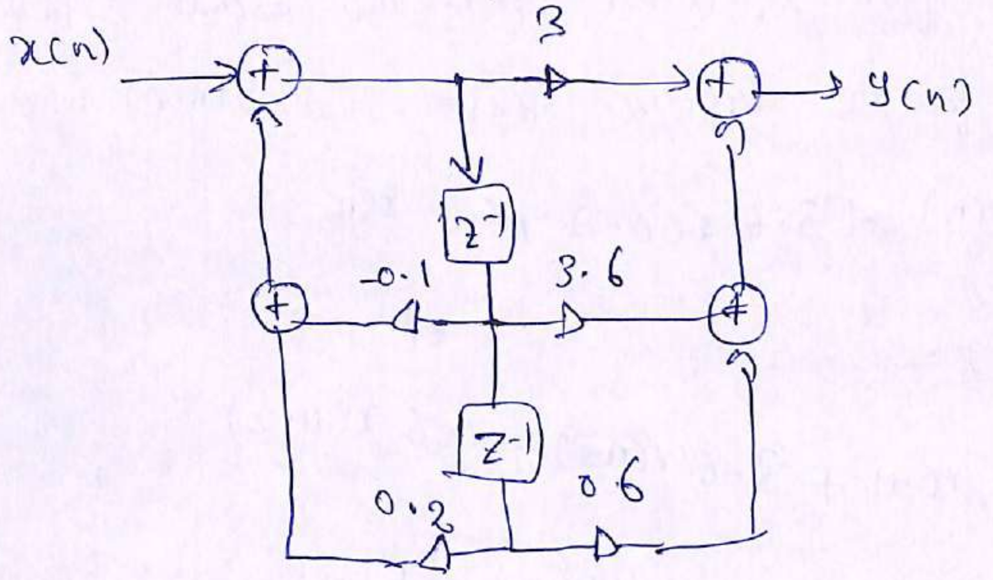


fig: Direct form II realization

Cascade form:-

$$\frac{Y(z)}{X(z)} = H(z) = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

$$H(z) = \frac{(3 + 0.6z^{-1})(1 + z^{-1})}{(1 + 0.5z^{-1})(1 - 0.4z^{-1})}$$

$$H(z) = H_1(z) \cdot H_2(z)$$

$$H_1(z) = \frac{3 + 0.6z^{-1}}{1 + 0.5z^{-1}}, \quad H_2(z) = \frac{1 + z^{-1}}{1 - 0.4z^{-1}}$$

Realize $H_1(z)$ & $H_2(z)$ using direct form-II

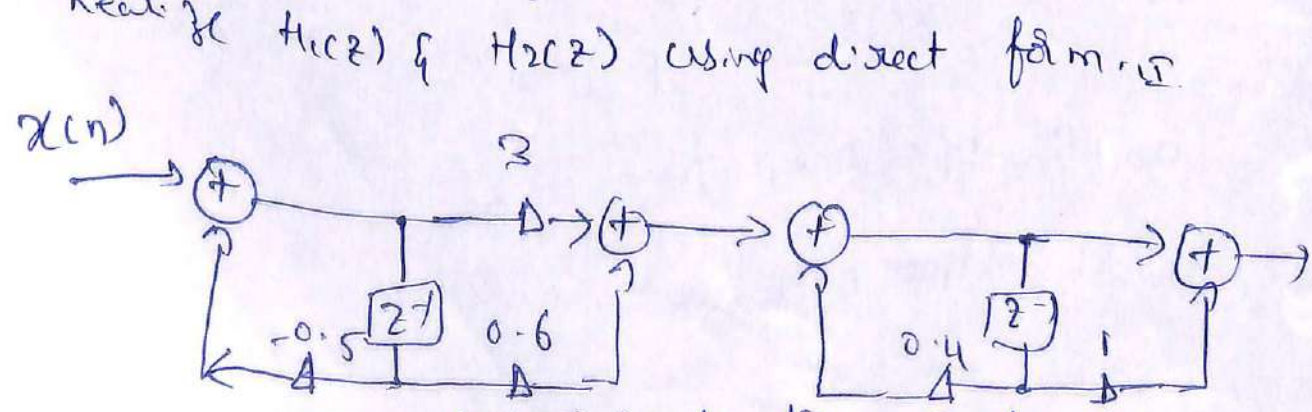


fig: Cascade form realization

Parallel form :-

$$H(z) = \frac{3 + 3.6z^{-1} + 0.6z^{-2}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

$$\begin{array}{r|l} & \quad -3 \\ \hline -0.2z^{-2} + 0.1z^{-1} + 1 & 0.6z^{-2} + 3.6z^{-1} + 3 \\ & 0.6z^{-2} - 0.3z^{-1} - 3 \\ & \quad\quad - \quad\quad + \\ \hline & 3.9z^{-1} + 6 \end{array}$$

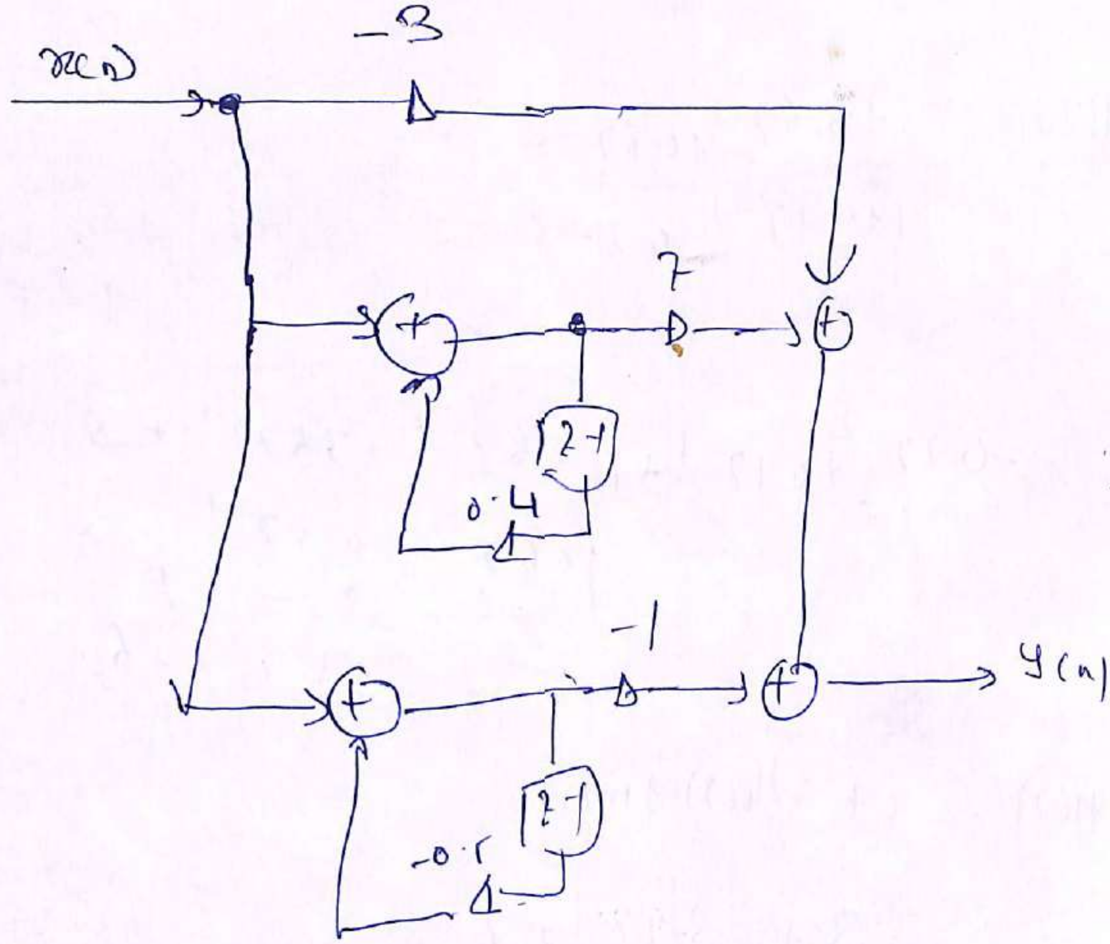
$$H(z) = C + H_1(z) + H_2(z)$$

$$= -3 + \frac{3.9z^{-1} + 6}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

$$= -3 + \frac{A}{1 - 0.4z^{-1}} + \frac{B}{1 + 0.5z^{-1}}$$

where $A = 7, B = -1$

$$= -3 + \frac{7}{1 - 0.4z^{-1}} + \frac{-1}{1 + 0.5z^{-1}}$$



x(n)
-3
7
z-1
0.4
-1
z-1
-0.5
y(n)

UNIT-II

Discrete Fourier Series & Fast Fourier Transforms

TOPICS:- DFS representation of Periodic Sequences, Properties of Discrete Fourier Series, Discrete Fourier Transforms, Properties of DFT, linear Convolution of Sequences using DFT, Computation of DFT: over-lap Add method, Over-lap Save method, Relation between DTFT, DFS, DFT, Z-transform.

FFT:- Radix 2 decimation in time & decimation in frequency FFT algorithms, Inverse FFT & FFT with general radix-N.

DFS representation of Periodic Sequences:-

→ The Fourier Series representation of a periodic discrete-time sequence is called discrete fourier series (DFS).

→ Consider a Sequence $x_p(n)$ with a period of N samples so that $x_p(n) = x_p(n+lN)$ for any integer value of l.

→ Since $x_p(n)$ is Periodic, it can be represented as a weighted sum of complex exponentials whose frequencies are integer multiples of the fundamental frequency $\frac{2\pi}{N}$.

→ These periodic complex exponentials are of the form

$$e^{j2\pi kn/N} = e^{j2\pi k(n+lN)/N}$$

→ Now any Periodic Sequence $x_p(n)$ can be expressed as,

$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_p(k) \cdot e^{j2\pi kn/N} \quad , \quad n=0,1,\dots N-1$$

$X_p(k) \rightarrow$ are called discrete Fourier Series Coefficients

→ Multiply B.S. with $e^{-j\frac{2\pi}{N}mn}$ & sum the Product from $n=0$ to $N-1$,

$$\sum_{n=0}^{N-1} x_p(n) \cdot e^{-j\frac{2\pi}{N}mn} = \frac{1}{N}\sum_{k=0}^{N-1} X_p(k) \cdot \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-m)n}$$

We K.T

$$\sum_{n=0}^{N-1} e^{j\left(\frac{2\pi}{N}\right)\cdot(k-m)\cdot n} \begin{cases} = N & \text{if } k-m = 0, \pm N, \dots \\ = 0 & \text{o.w} \end{cases}$$

$$\sum_{n=0}^{N-1} x_p(n) \cdot e^{-\frac{j2\pi}{N}m \cdot n} = X_p(m).$$

$$\therefore X_p(k) = \sum_{n=0}^{N-1} x_p(n) \cdot e^{-\frac{j2\pi}{N}nk}$$

**

$$\boxed{\begin{aligned} DFS\,[x_p(n)] &= X_p(k) = \sum_{n=0}^{N-1} x_p(n) \cdot e^{-\frac{j2\pi nk}{N}} \\ IDFS\,[X_p(k)] &= x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_p(k) \cdot e^{\frac{j2\pi nk}{N}} \end{aligned}}$$

Properties of DFS:-

(1) Linearity:-

Consider two periodic sequences $x_{1p}(n)$ & $x_{2p}(n)$ with period N, such that

$$DFS[x_1(n)] = X_1(k) \quad \&$$

$$DFS[x_2(n)] = X_2(k)$$

then

$$DFS[a x_1(n) + b x_2(n)] = a X_1(k) + b X_2(k)$$

Proof:- We KT

$$DFS[x(n)] = X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-j2\pi nk}{N}}$$

$$L.H.S \quad DFS[a x_1(n) + b x_2(n)] = \sum_{n=0}^{N-1} [a x_1(n) + b x_2(n)] \cdot e^{\frac{-j2\pi nk}{N}}$$

$$= \sum_{n=0}^{N-1} a x_1(n) \cdot e^{\frac{-j2\pi nk}{N}} + \sum_{n=0}^{N-1} b \cdot x_2(n) \cdot e^{\frac{-j2\pi nk}{N}}$$

$$= a \cdot \sum_{n=0}^{N-1} x_1(n) \cdot e^{\frac{-j2\pi nk}{N}} + b \cdot \sum_{n=0}^{N-1} x_2(n) \cdot e^{\frac{-j2\pi nk}{N}}$$

$$= a \cdot X_1(k) + b X_2(k) = R.H.S$$

$$\therefore DFS[a x_1(n) + b x_2(n)] = a X_1(k) + b X_2(k)$$

2, Time Shifting :-

If $x(n)$ is Periodic sequence with Period N $\ni$

$$DFS[x(n)] = X(k)$$

then

$$DFS[x(n-m)] = e^{-j\frac{2\pi}{N}mk} X(k)$$

Proof:-

We k.T

$$DFS[x(n)] = X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi}{N}nk}$$

$$DFS[x(n-m)] = \sum_{n=0}^{N-1} x(n-m) \cdot e^{-j\frac{2\pi}{N}nk}$$

let $n-m = l \Rightarrow n = l+m$ $\quad n=0, \; l=-m$ $\quad n=N-1, \; l=N-1-m$

$$R.H.S. = \sum_{l=-m}^{N-1-m} x(l) \cdot e^{-j\frac{2\pi}{N}(l+m)k}$$

$$= \sum_{l=-m}^{N-1-m} x(l) \cdot e^{-j\frac{2\pi}{N}lk} \cdot e^{-j\frac{2\pi}{N}mk}$$

$$= e^{-j\frac{2\pi}{N}mk} \sum_{l=-m}^{N-1-m} x(l) \cdot e^{-j\frac{2\pi}{N}lk}$$

Replace l by n

$$= e^{-j\frac{2\pi}{N}mk} \cdot \sum_{n=-m}^{N-1-m} x(n) \cdot e^{-j\frac{2\pi}{N}nk}$$

$\therefore$ DFS $[x(n-m)] = e^{-\frac{j2\pi}{N}mk} \cdot X(k)$ — R.H.S

3) ~~Symmetry~~ Frequency Property :-

If $x(n)$ is periodic with period 'N' &

$$DFS[x(n)] = X(k)$$

then $$DFS\left[e^{\frac{j2\pi l n}{N}} \cdot x(n)\right] = X(k-l)$$

Proof :- we k.T $$DFS[x(n)] = \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-j2\pi nk}{N}} = X(k)$$

$$L.H.S = DFS\left[e^{\frac{j2\pi l n}{N}} x(n)\right] = \sum_{n=0}^{N-1} e^{\frac{j2\pi l n}{N}} x(n) \cdot e^{\frac{-j2\pi nk}{N}}$$

$$= \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi}{N}(k-l)n}$$

$$= X(k-l) = R.H.S$$

(4) Periodic Convolution Property :-

If $x_1(n)$ & $x_2(n)$ be two periodic sequences with period N,

with $DFS[x_1(n)] = X_1(k)$ &

$DFS[x_2(n)] = X_2(k)$

then $DFS[x_1(n) * x_2(n)] = X_1(k) \cdot X_2(k)$.

Proof:-

$$x_1(n) * x_2(n) = \sum_{m=0}^{N-1} x_1(m) \cdot x_2(n-m).$$

We K.T $\quad DFS[x(n)] = \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-j2\pi nk}{N}}$

L.H.S

$$DFS[x_1(n) * x_2(n)] = \sum_{n=0}^{N-1} x_1(n) * x_2(n) \cdot e^{\frac{-j2\pi nk}{N}}$$

$$= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x_1(m) \cdot x_2(n-m) \cdot e^{\frac{-j2\pi nk}{N}}.$$

let $n-m = l \quad \Rightarrow \quad n = l+m$

$n=0, \quad l = -m.$

$n = N-1, \quad l = N-1-m.$

$$= \sum_{m=0}^{N-1} x_1(m) \cdot \sum_{l=-m}^{N-1-m} x_2(l) \cdot e^{\frac{-j2\pi}{N}(l+m) \cdot k}$$

$$= \sum_{m=0}^{N-1} x_1(m) \cdot e^{\frac{-j2\pi mk}{N}} \cdot \sum_{l=-m}^{N-1-m} \cdot x_2(l) \cdot e^{\frac{-j2\pi lk}{N}}$$

$DFS[x_1(n) * x_2(n)] = X_1(k) \cdot X_2(k)$

(5) Multiplication Property:—

If $x_1(n)$ & $x_2(n)$ are two Periodic Sequences with Period 'N' $\ni$

$$DFS[x_1(n)] = X_1(k) \;\&$$
$$DFS[x_2(n)] = X_2(k)$$

then $$DFS[x_1(n) \cdot x_2(n)] = \frac{1}{N} \sum_{l=0}^{N-1} X_1(l) \cdot X_2(k-l)$$
$$= \frac{1}{N} X_1(k) * X_2(k).$$

Proof:—

LH. We K.T

$$DFS[x(n)] = \sum_{n=0}^{N-1} x_1(n) \cdot e^{\frac{-j2\pi nk}{N}}$$

$$IDFS[X(k)] = x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot e^{\frac{j2\pi nk}{N}}$$

Replace k by l.

$$IDFS[X(l)] = x_1(n) = \frac{1}{N} \sum_{l=0}^{N-1} X_1(l) \cdot e^{\frac{j2\pi nl}{N}}.$$

$$DFS[x_1(n) \cdot x_2(n)] = \sum_{n=0}^{N-1} x_1(n) \cdot x_2(n) \cdot e^{\frac{-j2\pi nk}{N}}$$

Sub. $$x_1(n) = \frac{1}{N} \sum_{l=0}^{N-1} X_1(l) \cdot e^{\frac{j2\pi nl}{N}}$$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{l=0}^{N-1} X_1(l) \cdot e^{\frac{j2\pi nl}{N}} \right] \cdot x_2(n) \cdot e^{\frac{-j2\pi nk}{N}}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} X_1(l) \cdot \sum_{n=0}^{N-1} x_2(n) \cdot e^{\frac{-j2\pi}{N}(k-l)n}$$

$$\therefore DFS[x_1(n) \cdot x_2(n)] = \frac{1}{N} \sum_{l=0}^{N-1} X_1(l) \cdot X_2(k-l)$$

$$= \frac{1}{N} \left[X_1(k) * X_2(k) \right]$$

(6) <u>Symmetry</u> <u>property</u>:-

If $DFS[x_p(n)] = X_p(k)$ then

$DFS[x^*_p(n)] = X^*_p(-k)$ &

$DFS[x^*_p(-n)] = X_p^*(k)$

& $DFS[Re\{x_p(n)\}] = X_{pe}(k)$,

$DFS[j\, imag\{x_p(n)\}] = X_{po}(k)$.

<u>Proof</u>:-

we k.T $X_p[k] = \sum_{n=0}^{N-1} x_p(n) \cdot e^{\frac{-j2\pi nk}{N}}$

Replace k by -k

$$X_p[-k] = \sum_{n=0}^{N-1} x_p(n) \cdot e^{\frac{j2\pi nk}{N}}$$

take complex conjugate on B.S

$$X_p^*(-k) = \left[\sum_{n=0}^{N-1} x_p(n) \cdot e^{\frac{j2\pi nk}{N}} \right]^*$$

$$= \sum_{n=0}^{N-1} x_p^*(n) \cdot e^{\frac{-j2\pi nk}{N}} = DFS[x_p^*(n)]$$

we k.T $X_p(k) = \sum_{n=0}^{N-1} x_p(n) \cdot e^{-j\frac{2\pi nk}{N}}$

Replace n by $-n$

$$X_p(k) = \sum_{n=0}^{-(N-1)} x_p(-n) \cdot e^{j\frac{2\pi nk}{N}}$$

take Complex Conjugate on B.S

$$X_p^*(k) = \sum_{n=0}^{-(N-1)} x_p^*(-n)\, e^{-j\frac{2\pi nk}{N}}$$

$$X_p^*(k) = DFS\{x_p^*(-n)\} = L.H.S$$

we k.T $x_p(n) = x_{pe}(n) + x_{po}(n)$

where $x_{pe}(n) = \frac{1}{2}[x_p(n) + x_p^*(-n)]$

& $x_{po}(n) = \frac{1}{2}[x_p(n) - x_p^*(-n)]$

$\rightarrow$ $DFS[x_{pe}(n)] = DFS\left[\frac{1}{2}(x_p(n) + x_p^*(-n))\right]$

$DFS[x_{pe}(n)] = \frac{1}{2}\cdot[X_p(k) + X_p^*(k)] = X_{pe}(k)$

$|||^{ly}$ $DFS[x_{po}(n)] = DFS\left[\frac{1}{2}(x_p(n) - x_p^*(-n))\right]$

$= \frac{1}{2}[X_p(k) - X_p^*(k)] = j\, Imag\{X_p(k)\}$

$= X_{po}(k).$

→ Frequency - analysis of Signals :-

→ For frequency analysis of signals, different methods are:

(1) The Fourier Series for Continuous - time Periodic Signals

2) " " discrete - time "

3) The Fourier transform of continuous - time aperiodic Signals

4) " " discrete - time "

→ Continuous - time Signals have aperiodic Spectra :- A close inspection of the Fourier Series & Fourier transform analysis formulas for continuous - time Signals does not reveal any kind of Periodicity in the Spectral domain. This lack of Periodicity is a Consequence of the fact that the complex exponential $e^{j2\pi f t}$ is a function ~~Consequence~~ of the continuous variable t & hence it is not Periodic in F. Thus the freq. range of the continuous - time Signals extends from $f = 0$ to ∞.

→ Discrete - time Signal have Periodic Spectra :- Both the Fourier Series & the Fourier transform for discrete - time Signals are Periodic with Period $\omega = 2\pi$. As a result of this Periodicity, the freq. range of discrete - time Signals is finite & extends from $\omega = -\pi$ to π rad or $\omega = (0$ to $2\pi)$.

[DSP-2-6]

→ **Fourier Transform of Discrete-Time Aperiodic Signals (DTFT):-**

→ The fourier transform of a finite-energy discrete-time signal $x(n)$ is defined as,

$$DTFT[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}.$$

· $X(\omega)$ represents the frequency content of the signal $x(n)$. i.e decomposition of $x(n)$ into its frequency components.

→ for Continuous-time signals, the Fourier transform, i.e the spectrum of the signal, have a frequency range of $(-\infty, \infty)$. Spectrum is aperiodic

→ for discrete-time signals, the frequency range is unique over the frequency interval of $(-\pi, \pi)$ or over $(0, 2\pi)$. Spectrum is Periodic.

$$X(\omega) = X(\omega + 2\pi k).$$

$X(\omega)$ is periodic with period 2π.

$$X(\omega+2\pi k) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j(\omega+2\pi k)n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n} \cdot e^{-j2\pi k n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n} = X(\omega).$$

Freq. Analysis of Discrete-time Aperiodic Signals:

Analysis eqn. (DTFT) $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$

Synthesis eqn (IDTFT). $x(n) = \frac{1}{2\pi} \int_{2\pi} X(\omega) \cdot e^{j\omega n} \, d\omega$

Eg:- Determine the F.T of the following Signals.

a) $x(n) = u(n)$

b) $x(n) = (-1)^n u(n)$

c) $x(n) = \cos\omega_0 n \cdot u(n)$

Sol a) $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$

$= \sum_{n=-\infty}^{\infty} u(n) \cdot e^{-j\omega n} = \sum_{n=0}^{\infty} 1 \cdot e^{-j\omega n} = \frac{1}{1-e^{-j\omega}}$

b) $X(\omega) = \sum_{n=-\infty}^{\infty} (-1)^n \cdot u(n) \cdot e^{-j\omega n}$

$= \sum_{n=0}^{\infty} (-1)^n \cdot e^{-j\omega n} = \sum_{n=0}^{\infty} (-e^{-j\omega})^n$

$= \frac{1}{1+e^{-j\omega}}$.

Discrete Fourier transform (DFT):-

→ We K.T Discrete - time Fourier transform of a sequence is periodic. & freq. range from 0 to 2π or $-\pi$ to π. There are infinitely many ω in this range. If we use a digital computer to N equally spaced points over the interval $0 \le \omega \le 2\pi$, then N points should be located at

$$\omega_K = \frac{2\pi}{N} \cdot K, \qquad K = 0, 1, \ldots N-1.$$

→ These N equally spaced frequency samples of DTFT are known as DFT & denoted by $X(K)$.

$$X(K) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N} \cdot K} \qquad 0 \le K \le N-1$$

→ Let $x(n)$ is a causal, finite duration sequence containing L samples, then its DTFT is

$$X(\omega) = X(e^{j\omega}) = \sum_{n=0}^{L-1} x(n) \cdot e^{-j\omega n}.$$

→ Sample $X(e^{j\omega})$ at N equally spaced points over $0 \le \omega \le 2\pi$, then

$$X(K) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} = \sum_{n=0}^{L-1} x(n) \cdot e^{\frac{-j2\pi kn}{N}}$$

→ To evaluate $X(k)$ using DFT formula, it requires N^2 multiplications & $N(N-1)$ additions.

→ If $N < L$, then time domain aliasing occurs.

∴ recovering of signal is difficult. So, to prevent it, increase the duration of $x(n)$ from L to N samples by appending appropriate number of zeros, which is known as "zero padding".

$$\boxed{\begin{aligned} DFT[x(n)] = X(k) &= \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-j2\pi nk}{N}}, \quad 0 \le k \le N-1 \\ IDFT[X(k)] = x(n) &= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot e^{\frac{+j2\pi nk}{N}}, \quad 0 \le n \le N-1 \end{aligned}}$$

$N \ge L$.

→ Let $W_N = e^{\frac{-j2\pi}{N}}$ & W_N is known as twiddle factor.

N-point DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{nk},$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot W_N^{-nk}$$

Eg:- Find the DFT of a sequence $x(n) = \{1, 1, 0, 0\}$ & find also IDFT.

Sol: Let $N = L = 4$

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-j2\pi nk}{N}}, \quad k = 0, 1, \ldots N-1$$

$$X(k) = \sum_{n=0}^{3} x(n) \cdot e^{\frac{-j2\pi nk}{4}}$$

DSP-2-8

$$X(0) = \sum_{n=0}^{3} x(n) = x(0) + x(1) + x(2) + x(3) = 2$$

$$X(1) = \sum_{n=0}^{3} x(n) \cdot e^{-j\pi n/2} = 1 - j$$

$$X(2) = \sum_{n=0}^{3} x(n) \cdot e^{-j\pi n} = 0$$

$$X(3) = \sum_{n=0}^{3} x(n) \cdot e^{-j\frac{3n\pi}{2}} = 1 + j$$

$$\therefore X(k) = \{2,\ 1-j,\ 0,\ 1+j\}$$

$$IDFT\,[X(k)] = x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot e^{\frac{j2\pi nk}{N}}$$

$$x(n) = \frac{1}{4} \sum_{k=0}^{3} X(k) \cdot e^{\frac{j2\pi nk}{4}}$$

$$x(0) = \frac{1}{4} \sum_{k=0}^{3} X(k) = 1$$

$$x(1) = \frac{1}{4} \sum_{k=0}^{3} X(k) \cdot e^{\frac{j2\pi k}{4}} = 1$$

$$x(2) = \frac{1}{4} \sum_{k=0}^{3} X(k) \cdot e^{j\pi k} = 0$$

$$x(3) = \frac{1}{4} \sum_{k=0}^{3} X(k) \cdot e^{\frac{j3k\pi}{2}} = 0 \quad \therefore x(n) = \{1, 1, 0, 0\}$$

2) Find the DFT of the sequence $x(n)$ of length 'N'.

i) $x(n) = \delta(n)$ ii) $x(n) = \delta(n-n_0)$ iii) $x(n) = a^n, 0 \le n \le N-1$

Sol i) $$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-j2\pi nk}{N}} = \sum_{n=0}^{N-1} \delta(n) \cdot e^{\frac{-j2\pi nk}{N}} = 1.$$

$\therefore X(k) = 1, \quad 0 \le k \le N-1$

ii) $$X(k) = \sum_{n=0}^{N-1} \delta(n-n_0) \cdot e^{\frac{-j2\pi nk}{N}} = e^{\frac{-j2\pi n_0 k}{N}}, \quad 0 \le k \le N-1$$

iii) $$X(k) = \sum_{n=0}^{N-1} a^n \cdot e^{\frac{-j2\pi nk}{N}} = \sum_{n=0}^{N-1} \left(a e^{\frac{-j2\pi k}{N}}\right)^n$$

$$= \frac{1 - a^N e^{\frac{-j2\pi k}{N}}}{1 - a e^{\frac{-j2\pi k}{N}}}$$

3) Find the 8-DFT of a sequence & plot mag & phase spectrum

$x(n) = 1$ for $0 \le n \le 2$

$= 0$ O.W

$L = \text{length}(x(n)) = 3$

Sol $x(n) = \{ \underset{\uparrow n=0}{1}, 1, 1 \}$

$N = 8$. So add $N-L$ zeros to $x(n)$

$x(n) = \{1, 1, 1, 0, 0, 0, 0, 0\}$

1, 6, 7, 9, 11, 12, 15, 17, 19, 20,
2, 3, 6, 30, 2, 3, 4, 40, 4, 6, 8,
9, be-4, 5

DSP-29

$$X(k) = \sum_{n=0}^{7} x(n) \cdot e^{\frac{-j2\pi nk}{8}}, \quad k = 0, 1, \ldots, 7$$

$X(0) = 3$, $|X(0)| = 3$, $\angle X(0) = 0$

$X(1) = 1.707 - j1.707$, $|X(1)| = 2.414$, $\angle X(1) = -\frac{\pi}{4}$

$X(2) = -j$, $|X(k)| = 1$, $\angle X(2) = -\frac{\pi}{2}$

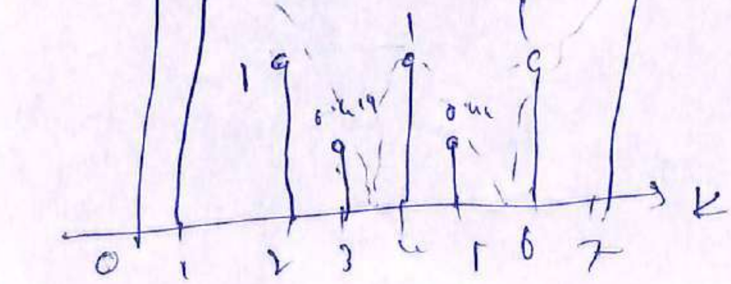
$X(3) = 0.293 + j0.293$, $|X(3)| = 0.414$, $\angle X(3) = \frac{\pi}{4}$

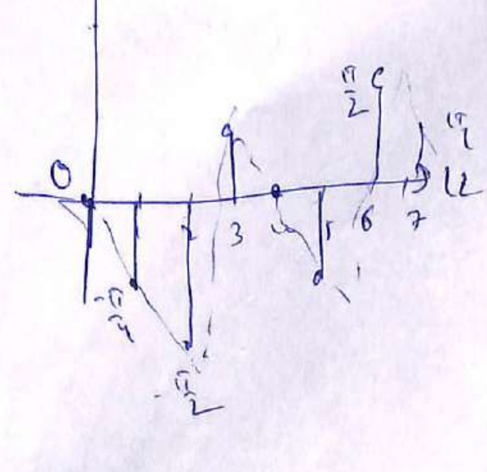
$X(4) = 1$, $|X(4)| = 1$, $\angle X(4) = 0$

$X(5) = 0.293 - j0.293$, $|X(5)| = 0.414$, $\angle X(5) = -\frac{\pi}{4}$

$X(6) = j$, $|X(6)| = 1$, $\angle X(6) = +\frac{\pi}{2}$

$X(7) = 1.707 + j1.707$, $|X(7)| = 2.414$, $\angle X(7) = \frac{\pi}{4}$





Properties of DFT :-

1) Periodicity :-

If a sequence $x(n)$ is periodic with periodicity of 'N' samples, then N-Point DFT of the sequence, $X(K)$ is also periodic with periodicity of N samples,

$$x(n) \xrightarrow{DFT} X(K)$$

If $x(n+N) = x(n) \quad \forall n$

then $X(K+N) = X(K) \quad \forall K.$

Proof:-

$$X(K) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi nK/N}$$

$$X(K+N) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi n(K+N)/N}$$

$$= \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-j2\pi nK}{N}} \cdot e^{-j2\pi n}$$

$$X(K+N) = \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-j2\pi nK}{N}} = X(K)$$

$\because e^{-j2\pi n} = 1$ for n is integer value.

$\therefore X(K+N) = X(K)$

a) Liniarity:-

If two finite duration sequences $x_1(n)$ & $x_2(n)$ are ~~periodic~~ linearly combined as, $x_3(n) = a x_1(n) + b x_2(n)$ then DFT of $x_3(n)$ is $X_3(k) = a X_1(k) + b X_2(k)$.

If $x_1(n) \xleftrightarrow{N\text{-DFT}} X_1(k)$

$x_2(n) \xleftrightarrow{N\text{-DFT}} X_2(k)$

then $a x_1(n) + b x_2(n) \xleftrightarrow{N\text{-DFT}} a X_1(k) + b X_2(k)$

Proof:- $N\text{-DFT}[x_1(n)] = X_1(k) = \sum_{n=0}^{N-1} x_1(n) \cdot e^{\frac{-j2\pi nk}{N}}$.

$N\text{-DFT}[x_2(n)] = X_2(k) = \sum_{n=0}^{N-1} x_2(n) \cdot e^{\frac{-j2\pi nk}{N}}$

L.H.S $N\text{-DFT}[a x_1(n) + b x_2(n)] = \sum_{n=0}^{N-1} (a x_1(n) + b x_2(n)) \cdot e^{\frac{-j2\pi nk}{N}}$.

$= a \sum_{n=0}^{N-1} x_1(n) \cdot e^{\frac{-j2\pi nk}{N}} + b \sum_{n=0}^{N-1} x_2(n) \cdot e^{\frac{-j2\pi nk}{N}}$

$= a. X_1(k) + b X_2(k) =$ R.H.S

circular time shift:-

If $x(n) \xleftrightarrow{N\text{-DFT}} X(k)$

then $x((n-m))_N \xleftrightarrow{DFT} e^{-j2\pi km/N} \cdot X(k)$.

Proof:- $DFT[x((n-m))_N] = \sum_{n=0}^{N-1} x((n-m))_N \cdot e^{-j2\pi nk/N}$

$$= \sum_{n=0}^{m-1} x((n-m))_N \cdot e^{-j2\pi nk/N} + \sum_{n=m}^{N-1} x((n-m))_N \cdot e^{-j2\pi nk/N}$$

but $\boxed{x((n-m))_N = x(N-m+n)}$

$$= \underbrace{\sum_{n=0}^{m-1} x(N-m+n) \cdot e^{-j2\pi nk/N}}_{1^{st}\text{ Part}} + \underbrace{\sum_{n=m}^{N-1} x(N-m+n) \cdot e^{-j2\pi nk/N}}_{2^{nd}\text{ Part}}$$

1^{st} Part

$$\sum_{n=0}^{m-1} x(N-m+n) \cdot e^{-j2\pi nk/N} = \sum_{l=N-m}^{N-1} x(l) \cdot e^{-j2\pi k(-N+m+l)/N}$$

let $l = N-m+n$. $\Rightarrow$ $n = l-N+m$.

$n \to 0,\ l = N-m$

$n \to m-1,\ l = N-1$.

$$= \sum_{l=N-m}^{N-1} x(l) \cdot e^{+j2\pi k} \cdot e^{-j2\pi km/N} \cdot e^{-j2\pi kl/N}$$

$$\therefore \ DFT[x((n-m))_N] = e^{\frac{-j2\pi km}{N}} \cdot \left[\sum_{l=N-m}^{N-1} x(l)\cdot e^{\frac{-j2\pi lk}{N}} + \sum_{l=N}^{2N-m-1} x(l)\cdot e^{\frac{-j2\pi lk}{N}} \right]$$

$$= e^{\frac{-j2\pi km}{N}} \sum_{l=N-m}^{2N-m-1} x(l)\cdot e^{\frac{-j2\pi lk}{N}}$$

$$= e^{\frac{-j2\pi km}{N}} \cdot X(k).$$

(4) Time Reversal of the Sequence :-

If $x(n) \xleftrightarrow{DFT} X(k)$

then $x((-n))_N \xleftrightarrow{DFT} X((-k))_N$

ie $x(N-n) \xleftrightarrow{DFT} X(N-k)$.

Proof:

$$x((-n))_N = x(N-n).$$

$$X((-k))_N = X(N-k).$$

change the index from n to m.

$$DFT[x((-n))_N] = \sum_{n=0}^{N-1} x((-n))_N \cdot e^{\frac{-j2\pi nk}{N}}$$

$$= \sum_{n=0}^{N-1} x(N-n)\cdot e^{\frac{-j2\pi nk}{N}}$$

$$= e^{\frac{-j2\pi km}{N}} \sum_{l=N-m}^{N-1} x(l) \cdot e^{\frac{-j2\pi kl}{N}} \quad (\because e^{\frac{j2\pi k}{N}} = 1,$$

$k = 0, 1, \ldots N-1)$

$$= \boxed{e^{\frac{-j2\pi km}{N}} \cdot \sum_{l}}$$

Q.1174 2nd part,

$$\sum_{n=m}^{N-1} x(N-m+n) \cdot e^{\frac{-j2\pi nk}{N}} = e^{\frac{-j2\pi km}{N}} X(k)$$

$$\boxed{\text{L.H.S.}, \ \sum_{n=0}^{N-1} x((n-m))_N \cdot e^{\frac{j2\pi nk}{N}} = e^{\frac{-j2\pi km}{N}} X(k)}$$

let $\lambda = N-m+n$, $n = \lambda + m - N$

$n = m, \rightarrow \lambda = N,$

$n = N-1, \quad \lambda = 2N-m-1.$

$$= \sum_{l=N}^{2N-m-1} x(l) \cdot e^{-j2\pi k(l+m-N)/N}$$

$$= \sum_{l=N}^{2N-m-1} x(l) \cdot e^{\frac{-j2\pi lk}{N}} \cdot e^{\frac{-j2\pi mk}{N}} \cdot e^{-j2\pi k}$$

$$= e^{\frac{-j2\pi mk}{N}} \cdot \sum_{l=N}^{2N-m-1} x(l) \cdot e^{\frac{-j2\pi lk}{N}}.$$

(5) <u>Circular frequency shift :-</u>

If $\quad DFT[x(n)] = X(k)$

then $\quad DFT\left[x(n) \cdot e^{\frac{j2\pi ln}{N}}\right] = X((k-l))_N = X(N+k-l)$

<u>Proof:-</u> $\quad DFT\left[x(n) \cdot e^{\frac{j2\pi ln}{N}}\right] = \sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi ln}{N}} \cdot e^{\frac{-j2\pi nk}{N}}$

$$= \sum_{n=0}^{N-1} x(n) \cdot e^{-\frac{j2\pi}{N}(k-l)n}$$

$$= \sum_{n=0}^{N-1} x(n) \cdot e^{-\frac{j2\pi}{N}(k-l)} \cdot e^{-\frac{j2\pi}{N} \cdot Nn}$$

$$= \sum_{n=0}^{N-1} x(n) \cdot e^{-\frac{j2\pi}{N}(N+k-l)n} \qquad (\because e^{-j2\pi n} = 1)$$

$$= X(N+k-l)$$

$$= X((k-l))_N = R.H.S$$

let $m = N - n, \Rightarrow n = N - m$

$n = 0, \quad m = N$

$n = N-1, \quad m = N-(N-1) = 1.$

$$\therefore = \sum_{m=N}^{1} x(m) \cdot e^{-j\frac{2\pi}{N}(N-m)k}$$

$$= \sum_{m=1}^{N} x(m) \cdot e^{\frac{j2\pi mk}{N}}$$

$$= \sum_{m=0}^{N-1} x(m) \cdot e^{-j\frac{2\pi}{N}m(N-k)}$$

$= X(N-k) = X((-k))_N = R.H.S$

$\therefore$ DFT $[x((-n))_N] = X((-k))_N$.

(6) Circular Convolution:-

If ~~DFT~~ $x_1(n) \xleftrightarrow{N\text{-DFT}} X_1(k)$

$$x_2(n) \xleftrightarrow{N\text{-DFT}} X_2(k)$$

then $x_1(n) \textcircled{N} x_2(n) \xleftrightarrow{DFT} X_1(k) \cdot X_2(k)$

Proof:

$$x_1(n) \textcircled{N} x_2(n) = \sum_{m=0}^{N-1} x_1((m))_N \, x_2((n-m))_N .$$

$$x_1(n) \textcircled{N} x_2(n) = \sum_{m=0}^{N-1} x_1(N+m) \cdot x_2(N+n-m) = \sum_{m=0}^{N-1} x_1(m) \cdot x_2(N+n-m)$$

$$N\text{-DFT}\left[x_1(n) \textcircled{N} x_2(n)\right] = \sum_{n=0}^{N-1} \left[\sum_{m=0}^{N-1} x_1(m) \cdot x_2(N+n-m)\right] e^{\frac{-j2\pi nk}{N}}$$

$$= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x_1(m) \cdot x_2(N+n-m) \cdot e^{\frac{-j2\pi nk}{N}}$$

$$= \sum_{m=0}^{N-1} x_1(m) \cdot \underbrace{\sum_{n=0}^{N-1} x_2((n-m))_N \cdot e^{\frac{-j2\pi nk}{N}}}$$

$$= \sum_{m=0}^{N-1} x_1(m) \cdot X_2(k) \cdot e^{\frac{-j2\pi mk}{N}}$$

$$= X_2(k) \cdot \sum_{m=0}^{N-1} x_1(m) \cdot e^{\frac{-j2\pi mk}{N}} = X_1(k) \cdot X_2(k)$$

(7) Multiplication of two Sequences:-

If $x_1(n) \xleftrightarrow{N\text{-DFT}} X_1(k)$ &

$x_2(n) \xleftrightarrow{N\text{-DFT}} X_2(k)$

then $x_1(n) \cdot x_2(n) \xleftrightarrow{N\text{-DFT}} \frac{1}{N} X_1(k) \circledN X_2(k)$

Proof:-

$$DFT[x_1(n) \cdot x_2(n)] = \sum_{n=0}^{N-1} x_1(n) x_2(n) \cdot e^{\frac{-j2\pi nk}{N}} \quad -(1)$$

$$IDFT[X_1(l)] = x_1(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(l) \cdot e^{\frac{j2\pi nl}{N}}$$

Sub in (1)

$$= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X_1(l) \cdot e^{\frac{j2\pi nl}{N}}\right] \cdot x_2(n) \cdot e^{\frac{-j2\pi nk}{N}}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X_1(l) \cdot \sum_{n=0}^{N-1} x_2(n) \cdot e^{\frac{-j2\pi}{N}(l-k)n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X_1((l))_N \cdot \sum_{n=0}^{N-1} x_2(n) \cdot e^{\frac{j2\pi}{N}(l-k)n} \cdot e^{-\frac{j2\pi}{N}(N)n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X_1((l))_N \cdot \sum_{n=0}^{N-1} x_2(n) \cdot e^{-\frac{j2\pi}{N}(k-l+N)}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} x_1((l))_N \cdot X_2(N+k-l)$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} X_1((l))_N \cdot X_2((k-l))_N$$

$$= \frac{1}{N} \left[X_1(k) \, (N) \, X_2(k) \right] = R.H.S$$

(8) Complex conjugate Property :-

If $\quad x(n) \xleftrightarrow{DFT} X(k)$

then i) $x^*(n) \xleftrightarrow{DFT} X^*(N-k) = X^*((-k))_N$ &

ii) $x^*((-n))_N = x^*(N-n) \xleftrightarrow{DFT} X^*(k)$.

Proof :- we K.T $\quad DFT[x(n)] = \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-j2\pi nk}{N}}$

$$DFT[x^*(n)] = \sum_{n=0}^{N-1} x^*(n) \cdot e^{\frac{-j2\pi nk}{N}}$$

$$= \left[\sum_{n=0}^{N-1} x(n) \cdot e^{\frac{j2\pi nk}{N}} \right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-j2\pi}{N} n(N-k)} \right]^*$$

$$\therefore DFT[x^*(n)] = \cancel{X^*(k)} \cdot X^*(N-k) = X^*((-k))_N$$

(ii) let $IDFT[X^*(k)] = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k).e^{\frac{j2\pi kn}{N}}$

$$= \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k).e^{\frac{-j2\pi kn}{N}}\right]^*$$

$$= \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k).e^{j2\pi k(N-n)/N}\right]^*$$

$$= x^*(N-n)$$

$\therefore IDFT[X^*(k)] = x^*((-n))_N$

$\Rightarrow \quad DFT[x^*((-n))_N] = X^*(k).$

(9) <u>Circular correlation</u> :-

For complex-valued sequences $x(n)$ & $y(n)$, if

$DFT[x(n)] = X(k)$,

$DFT[y(n)] = Y(k)$

then $DFT[\tilde{r}_{xy}(l)] = DFT\left[\sum_{n=0}^{N-1} x(n).y^*((n-l))_N\right] = X(k).Y^*(k)$

(10) <u>Parseval's</u> theorem :-

If $DFT[x(n)] = X(k)$ &

$DFT[y(n)] = Y(k)$

then $\sum_{n=0}^{N-1} x(n).y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k).Y^*(k).$

Comparision b/w circular convolution & linear convolution:-

→ linear convolution	circular convolution
1. $y(n) = x(n) * h(n)$ $y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$	1. $y(n) = x(n) \; (N) \; h(n)$ $y(n) = \sum_{k=0}^{N-1} x(k) \cdot h((n-k))_N$
2. If length of $x(n) = L$ & length of $h(n) = M$ then length of $y(n) = L+M-1$	2. length of $y(n) = Max(L, M)$.
3. It can be used to find the filter response.	3. It can't be used to find the response of a filter w/o zero padding.
4. Methods to find linear convolution are: (1) Graphical method (2) Tabular column method (3) Matrix method etc	4. Methods to find circular convolution are: (1) Concentric circle method (2) Matrix method etc

Methods to evaluate circular convolution of 2 sequences:-

(1) Concentric ○le method:-

1. Graph N samples of $x_1(n)$ as equally spaced points around the outer ○le in counterclockwise direction.
2. Start at the same point as $x_1(n)$ graph. 'N' samples of $x_2(n)$ as equally spaced points

around an inner circle in clockwise direction

3. Multiply corresponding samples on the two circles & sum the products to produce O/P.

4. Rotate the inner circle one sample at a time in counterclockwise direction & goto step 3 to obtain the next value of O/P.

5. Repeat step No. 4 until the inner circle first sample lines up with the first sample of the exterior circle once again.

Eg:- Find the circular convolution of two finite duration sequences $x_1(n) = \{1, -1, -2, 3, -1\}$; $x_2(n) = \{1, 2, 3\}$.

Sol: Length of $x_1(n) = L = 5$

Length of $x_2(n) = M = 3$.

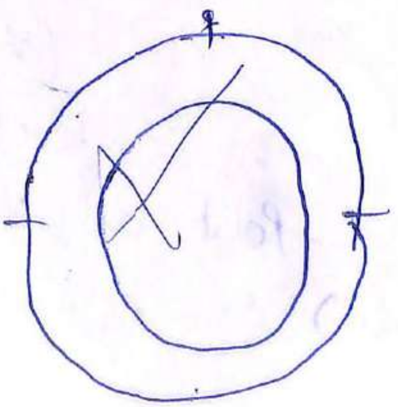
Length of $y(n) = \text{Max}(L, M) = \text{Max}(5, 3) = 5$.

$L > M$, add $L - M$ zeros to $x_2(n)$.

$5 - 3 = 2$ zeros to $x_2(n)$

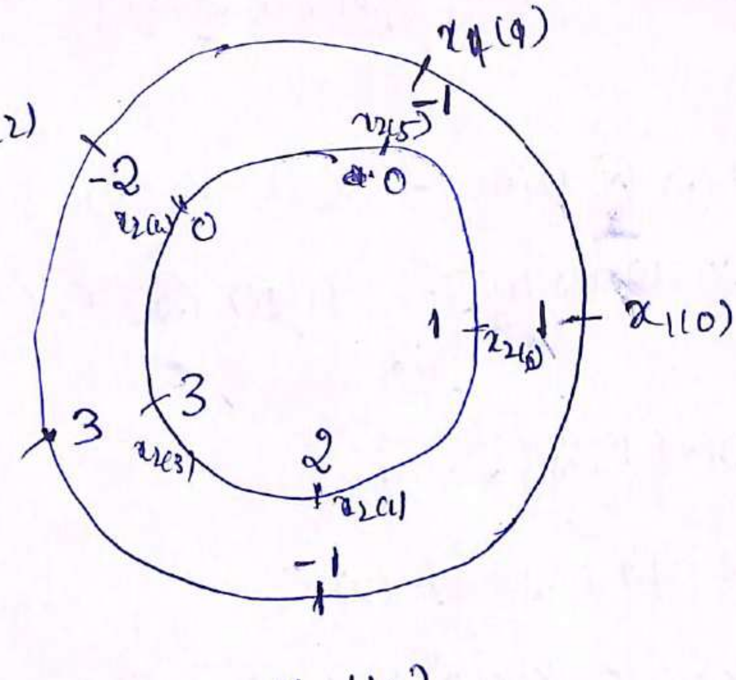
$x_2(n) = \{1, 2, 3, 0, 0\}$.

(i) $y(0)$:



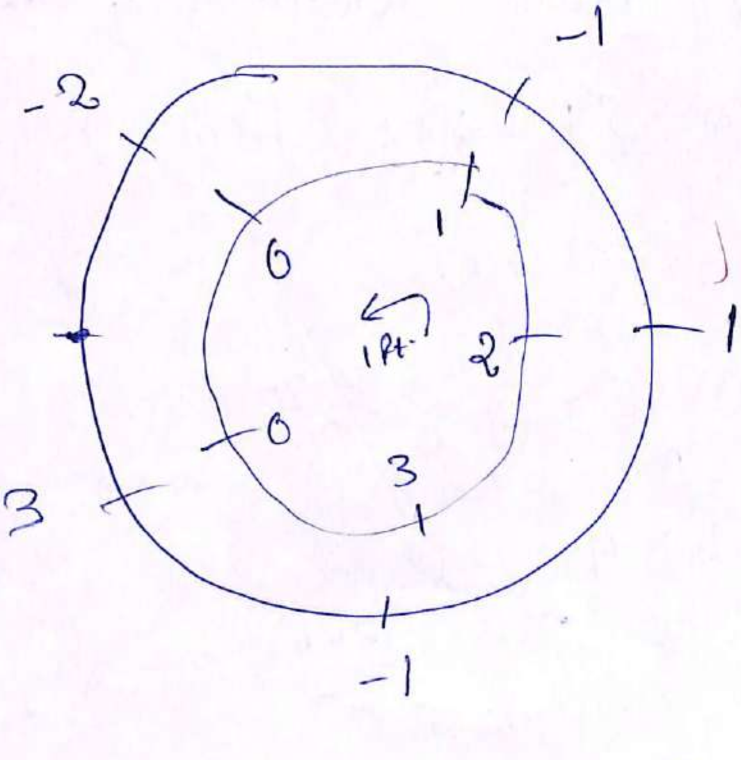
DSP-2-16

$y(0)$



$y(0) = 1(1) + 0(-1) + 0(-2) + 3(3) + 2(-1) = 8.$

$y(1)$



$y(1) = 1\times 2 + 1\times -1 + -2\times 0 + 3\times 0 + 3\times -1 = -2$

lly

$y(2) = 3(1) + 2(-1) + 1(-2) + 0(3) + (-1)(0) = -1$

$y(3) = 0(1) + 3(-1) + 2(-2) + 1(3) + (-1)(0) = -4$

$y(4) = 0(1) + 0(-1) + 3(-2) + 3(2) + 1(-1) = -1$

$\therefore y(n) = \{8, -2, -1, -4, -1\}$

→ Finding Circular Convolution using DFT & IDFT method!-

We K.T

$$DFT[x_1(n) \, (N) \, x_2(n)] = X_1(k) \cdot X_2(k)$$

or $$IDFT[X_1(k) \cdot X_2(k)] = x_1(n) \, (N) \, x_2(n).$$

Steps:-

(1) find $DFT[x_1(n)] = X_1(k)$

(2) " $DFT[x_2(n)] = X_2(k)$

(3) Multiply $X_1(k)$ & $X_2(k)$. ie $X_3(k) = X_1(k) \cdot X_2(k)$

(4) find IDFT of $X_3(k)$.

Eg:- Perform the circular convolution of the following sequences. $x(n) = \{1, 1, 2, 1\}$, $h(n) = \{1, 2, 3, 4\}$

Sol

$$X_3(k) = X_1(k) \cdot X_2(k)$$

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) \cdot e^{\frac{-j2\pi nk}{N}} \qquad k = 0, 1, \dots N-1$$

$$N = L = 4.$$

$$X_1(0) = \sum_{n=0}^{3} x_1(n) \cdot e^{\frac{-j2\pi n \times 0}{4}}$$

$$= x_1(0) + x_1(1) + x_1(2) + x_1(3)$$

$$X_1(0) = 1 + 1 + 2 + 1 = 5$$

$$X_1(1) = \sum_{n=0}^{3} x_1(n) \cdot e^{-j\frac{\pi n}{2}} = 1-j-2+j = -1$$

$$X_1(2) = \sum_{n=0}^{3} x_1(n) \cdot e^{-j\pi n} = 1-1+2-1 = 1$$

$$X_1(3) = \sum_{n=0}^{3} x_1(n) \cdot e^{-j3\pi n/2} = -1$$

$$\therefore X_1(k) = \{5, -1, 1, -1\}$$

$$X_2(k) = \sum_{n=0}^{3} x_2(n) \cdot e^{-j\frac{2\pi nk}{N}} , \quad k = 0, 1, \dots N-1$$

$N = 4 = L$

$$X_2(0) = \sum_{n=0}^{3} x_2(n) = 10$$

$$X_2(1) = \sum_{n=0}^{3} x_2(n) \cdot e^{-j\pi n/2} = -2+j2$$

$$X_2(2) = \sum_{n=0}^{3} x_2(n) \cdot e^{-j\pi n} = -2$$

$$X_2(3) = \sum_{n=0}^{3} x_2(n) \cdot e^{-j3\pi n/2} = -2-2j$$

$$\therefore X_2(k) = \{10, -2+j2, -2, -2-2j\}$$

$$X_3(k) = X_1(k) \cdot X_2(k) = \{50, 2-j2, -2, 2+j2\}$$

$$x_3(n) = IDFT[X_3(k)] = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k) \cdot e^{\frac{j2\pi nk}{N}}, \quad n=0,1,\ldots N-1$$

$$x_3(0) = \frac{1}{4}\sum_{k=0}^{3} X_3(k) = 13$$

$$x_3(1) = \frac{1}{4}\left[\sum_{k=0}^{3} X_3(k) \cdot e^{\frac{j\pi k}{2}}\right] = 14$$

$$x_3(2) = \frac{1}{4} \cdot \sum_{k=0}^{3} X_3(k) \cdot e^{j\pi k} = \cancel{12}\ 11$$

$$x_3(3) = \frac{1}{4}\sum_{k=0}^{3} X_3(k) \cdot e^{-j3\pi k/2} = 12$$

$$\therefore x_3(n) = \{13, 14, 11, 12\}$$

Linear Convolution from circular convolution :-

→ To find linear convolution from Circular Convolution, we have to increase the length of the sequences $x(n)$ & $h(n)$ to $L+M-1$ points by appending zeros & then find circular convolution to find linear convolution

Eg:- Determine the O/P response $y(n)$ of $h(n) = \{1,1,1\}$, $x(n) = \{1,2,3,1\}$ using
(i) Linear Convolution (ii) Circular Convolution
(iii) Linear Conv. from circular conv. (or) Circular Conv. with zero Padding

Sol (i) Linear Convolution.

$$
\begin{array}{c|cccc}
 & \multicolumn{4}{c}{x(n)} \\
 & 1 & 2 & 3 & 1 \\
\hline
1 & 1 & 2 & 3 & 1 \\
h(n) \quad 1 & 1 & 2 & 3 & 1 \\
1 & 1 & 2 & 3 & 1
\end{array}
$$

$$y(n) = x(n) * h(n) = \{1, 3, 6, 6, 4, 1\}$$

(ii) Circular Convolution.

$$
\begin{bmatrix}
x_2(0) & x_2(N-1) & x_2(N-2) & \cdots & x_2(1) \\
x_2(1) & x_2(0) & x_2(N-1) & \cdots & x_2(2) \\
\vdots & \vdots & x_2(0) & \cdots & x_2(3) \\
 & & \vdots & & \vdots \\
x_2(N-1) & x_2(N-2) & x_2(N-3) & \cdots & x_2(0)
\end{bmatrix}
\begin{bmatrix} x_1(0) \\ x_1(1) \\ \vdots \\ x_1(N-1) \end{bmatrix}
=
\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix}
$$

$$x_1(n) = \{1, 2, 3, 1\}, \quad x_2(n) = \{\underset{x_2(0)}{1}, \underset{x_2(1)}{1}, \underset{x_2(2)}{1}, \underset{x_2(3)}{0}\}$$

$$
\begin{bmatrix}
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}
=
\begin{bmatrix} 5 \\ 4 \\ 6 \\ 6 \end{bmatrix}
$$

$$\therefore \; y(n) = x(n) \; (N) \; h(n) = \{5, 4, 6, 6\}$$

$L+M-1 = 4+3-1 = 6.$

(ii) $x(n) = \{1, 2, 3, 1, 0, 0\}$, $h(n) = \{1, 1, 1, 0, 0, 0\}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 6 \\ 4 \\ 1 \end{bmatrix}$$

1, 6, 13, 18, 38, 44, 6, 1e-5
14, 34, 39, 44, 6, 1e-3,

$\therefore y(n) = \{1, 3, 6, 6, 4, 1\}$

Filtering long duration Sequences:-

Over lap - Save Method :-

⇒ Suppose an i/p sequence $x(n)$ of long duration is to be processed with a system having impulse response of finite duration by convolving the two sequences.

→ B'coz of the length of the i/p sequence long, it would not be practical to store it before performing linear convolution.

→ ∴ the i/p sequence must be divided into blocks. The successive blocks are processed seperately one at a time & the results are combined later to yield the desired o/p sequence which is identical to the sequence obtained by linear convolution

→ Two methods that are commonly used for filtering the sectioned data: DSP-2-19

(i) Overlap - Save method

(ii) Overlap - add method.

→ Overlap - Save method: & add method

Problem: Find the O/P $y(n)$ of a filter whose impulse response is $h(n) = \{1,1,1\}$ & I/P signal $x(n) = \{3,-1,0,1,3,2,0,1,2,1\}$ using (i) overlap - Save method

(ii) Over-lap add method.

Sol (i) Overlap-Save method:-

The I/P sequence can be divided into blocks of data as follows.

∵ length of $\underset{M=}{h(n)} = 3$, take 3 data $\underset{Ls=}{\text{points}}$ of $x(n)$

$M=3$, $Ls = 3$. & length of each block is

~~for 1st seq, add~~ $3+3-1$ (i.e $Ls+M-1$) $= 5$.

1st block $x_1(n) = \{\underbrace{0,0}_{M-1\text{ zeros}}, \underbrace{3,-1,0}_{3\text{ data points}}\}$ ∵ for 1st block, add M-1 zeros

$x_2(n) = \{\underbrace{-1,0}_{\text{Two data points from previous block}}, \underbrace{1,3,2}_{3\text{ new data points}}\}$

$x_3(n) = \{3,2,0,1,2\}$

$x_4(n) = \{1,2,1,0,0\}$

→ Increase the length of sequence to $L+M-1 = 5$ by adding 2 zeros.

$h(n) = \{1, 1, 1, 0, 0\}$

$y_1(n) = x_1(n)$ Ⓝ $h(n) = \{\underbrace{-1, 0}, 3, 2, 2\}$

$y_2(n) = x_2(n)$ Ⓝ $h(n) = \{\underbrace{4, 1}, 0, 4, 6\}$

$y_3(n) = x_3(n)$ Ⓝ $h(n) = \{\underbrace{6, 7}, 5, 3, 3\}$

$y_4(n) = x_4(n)$ Ⓝ $h(n) = \{\underbrace{1, 3}, 4, 3, 1\}$

$\boxed{\cancel{1}\ 0\ \cancel{3}\ \cancel{2}\ \cancel{2}}$

$$
\begin{array}{ccccccccccccc}
\underbrace{-1 \quad 0} & 3 & 2 & 2 & & & & & & & & & \\
& & & \underbrace{4 \quad 1} & 0 & 4 & 6 & & & & & & \\
& & & & & & \underbrace{6 \quad 7} & 5 & 3 & 3 & & & \\
& & & & & & & & & \underbrace{1 \quad 3} & 4 & 3 & 1
\end{array}
$$

$y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\}$

Over lap - Add method:-

In this, zeros are added to each block at end of block.

$x_1(n) = \{\underbrace{3, -1, 0}_{\text{3 data point}}, 0, 0\}$

$x_2(n) = \{1, 3, 2, 0, 0\}$

$x_3(n) = \{0, 1, 2, 0, 0\}$

$x_4(n) = \{1, 0, 0, 0, 0\}$

DSP-2-20

Relation b/w DTFT, DFS, DFT, Z-Transforms:-

① Relation b/w DTFT & DFT:-

$$DTFT[x(n)] = X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\omega n}$$

$$DFT[x(n)] = X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi}{N}kn}, \quad k=0,1,..N-1$$

$$\therefore \boxed{X(k) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}}}, \quad k=0,1,..N-1$$

~~DFT~~ U

→

DFT	DTFT
① Obtained by performing sampling operation in both time & freq. domains	① Sampling is performed only in time domain.
②. It has discrete freq. spectrum	② It has continuous freq. spectrum.
③ DFT gives only positive freq. values	③ It gives both positive & negative freq. values.
④ To get more accurate values of DFT, no of samples N must be very high, but computation time is more.	④

② Relationship b/w DTFT & Z-transform:-

$$DTFT[x(n)] = X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\omega n},$$

$$Z.T[x(n)] = X(Z) = \sum_{n=0}^{N-1} x(n) \cdot Z^{-n}$$

$$\therefore \boxed{X(Z) = X(e^{j\omega})\Big|_{Z = e^{j\omega}}}$$

③ Relationship b/w DFT & Z-transform:-

$$DFT[x(n)] = X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi}{N}nk}$$

$$Z.T[x(n)] = X(Z) = \sum_{n=0}^{N-1} x(n) \cdot Z^{-n} \quad -(1)$$

$$\boxed{X(Z) = X(k)\Big|_{Z = e^{-j\frac{2\pi k}{N}}}}$$

$$IDFT[X(k)] = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) \cdot e^{j\frac{2\pi}{N}nk}$$

Sub in $X(Z)$

$$X(Z) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X(k) \cdot e^{j\frac{2\pi}{N}nk}\right] \cdot Z^{-n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot \sum_{n=0}^{N-1} \left(e^{\frac{j2\pi k}{N}} z^{-1}\right)^n$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot \left(\frac{1-\left(e^{\frac{j2\pi k}{N}} z^{-1}\right)^N}{1- e^{\frac{j2\pi k}{N}} \cdot z^{-1}} \right)$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot \left(\frac{1-z^{-N}}{1-e^{\frac{j2\pi k}{N}} z^{-1}} \right)$$

$$X(z) = \frac{1-z^{-N}}{N} \cdot \sum_{k=0}^{N-1} \frac{X(k)}{1-e^{j2\pi k/N} z^{-1}}$$

(f) Relation b/w DFS & DFT:-

$$DFS[x_p(n)] = \sum_{n=0}^{N-1} x_p(n) \cdot e^{\frac{-j2\pi nk}{N}}$$

$$DFT[x(n)] = \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-j2\pi nk}{N}}$$

$$DFS[x_p(n)] = DFT[x(n)] \Big|_{\text{if } x(n) \text{ is Periodic with } 'N'.}$$

→ Find the 4 DFT of $x(n) = \{1, 2, 3, 4\}$ using DIT & DIF.

Ans. $X(K) = \{10, -2+j2, -2, -2-j2\}$

→ Compute IDFT of $X(K) = \{0, 0, 4, 0\}$.

Ans $x(n) = \{1, -1, 1, -1\}$

$h(n) = \{1, 1, 1, 0, 0\}$ DSP-2-20

$y_1(n) = x_1(n) \textcircled{N} h(n) = \{3, 2, 2, -1, 0\}$

$y_2(n) = x_2(n) \textcircled{N} h(n) = \{1, 4, 6, 5, 2\}$

$y_3(n) = x_3(n) \textcircled{N} h(n) = \{0, 1, 3, 3, 2\}$

$y_4(n) = x_4(n) \textcircled{N} h(n) = \{1, 1, 1, 0, 0\}$

3 2 2 -1 0

add

1 4 6 5 2

add add

0 3 3 3 2

add

1 1 1 0 0

$\therefore y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\}$

Fast Fourier Transform :-

→ The direct evaluation of DFT using formula requires N^2 complex multiplications & $N(N-1)$ complex additions.

So, it requires more no of computations.

→ To reduce the no of computations, many methods were developed.

→ The most popular of these is the FFT (Fast Fourier transform) developed by Cooley & Turkey.

→ The FFT can be defined as an algorithm (or a method) for computing DFT efficiently with divide & conquer approach.

→ Using FFT, for evaluating N-DFT, it requires $\frac{N}{2}\log_2 N$ complex multiplications & $N\log_2 N$ complex additions.

→ Basically there are two algorithms:

(i) DIT (Decimation in time)-FFT

(ii) DIF-FFT (Decimation in frequency)

→ It uses the symmetry & periodicity properties of the phase or twiddle factor W_N.

Symmetry property, $W_N^{k+\frac{N}{2}} = -W_N^k$

Periodicity property, $W_N^{k+N} = W_N^k$

$$W_N = e^{-\frac{j2\pi}{N}}, \quad W_N^k = e^{-\frac{j2\pi}{N}.k}$$

$$W_N^{k+\frac{N}{2}} = W_N^k \cdot W_N^{N/2}$$

$$= W_N^k \cdot e^{-\frac{j2\pi}{N}\cdot N/2} = W_N^k e^{-j\pi} = -W_N^k.$$

$$\therefore W_N^{k+\frac{N}{2}} = -W_N^k$$

$$W_N^{k+N} = W_N^k \cdot W_N^N = W_N^k \cdot e^{\left(-\frac{j2\pi}{N}\right)N} = W_N^k.$$

DIT Radix-2 FFT Algorithm:-

→ Radix-2 means, the no of o/p points N can be expressed as power of 2. i.e $N = 2^m$, where m is an integer which gives no of stages.

If $N = 2$, $N = 2^1$ → 1 stage. $N = 8$, $N = 2^3$ → 3 stages

If $N = 4$, $N = 2^2$ → 2 stages $N = 16 = 2^4$ → 4 stages.

→ Let $x(n)$ is an N-Point Sequence (i.e length of $x(n) = N$), where N is assumed to be a Power of 2.

→ Now decimate or break this Sequence into two sequences of length N/2, where one Sequence consisting of the even-indexed values of $x(n)$ & the other of odd-indexed values of $x(n)$.

$x(n)$

$x_e(n)$ $\quad$ $x_o(n)$

$$x_e(n) = x(2n), \quad n = 0, 1, \ldots \frac{N}{2} - 1$$

$$x_o(n) = x(2n+1), \quad n = 0, 1, \ldots \frac{N}{2} - 1$$

Eg: $N = 8$.

$$x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$$

$$x_e(n) = \{x(0), x(2), x(4), x(6)\}$$

$$x_o(n) = \{x(1), x(3), x(5), x(7)\}.$$

→ We k.T, N-Point DFT of $x(n)$ is

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{\frac{-j2\pi nk}{N}} = \sum_{n=0}^{N-1} x(n) \cdot W_N^{nk}. \quad , k = 0, 1, \ldots N-1.$$

→ Now Seperate $x(n)$ into $x_e(n)$ & $x_o(n)$.

DSP-2-22

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n) \cdot W_N^{2nk} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) \cdot W_N^{(2n+1)k}$$

$$W_N^{2nk} = \left(e^{-\frac{j2\pi}{N}nk}\right)^2 = e^{-\frac{j2\pi}{N/2} \cdot nk} = W_{N/2}^{nk} \qquad k = 0, 1, \ldots \frac{N}{2}-1$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n) \cdot W_{N/2}^{nk} + W_N^k \sum_{n=0}^{N-1} x(2n+1) \cdot W_{N/2}^{nk}.$$

$$X(k) = \underbrace{\sum_{n=0}^{N-1} x_e(n) \cdot W_{N/2}^{nk}}_{\frac{N}{2}\text{-pt DFT of even indexed seq.}} + W_N^k \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_o(n) \cdot W_{N/2}^{nk}}_{\frac{N}{2}\text{-pt DFT of odd indexed seq.}}$$

$$X(k) = X_e(k) + W_N^k \, X_o(k) \quad , \; k = 0, 1, \ldots \frac{N}{2}-1$$

for $k = \frac{N}{2}, \ldots N-1$ $\Rightarrow k \geq \frac{N}{2}$.

$$X(k) = X_e\left(k-\frac{N}{2}\right) + W_N^{k-\frac{N}{2}} \cdot X_o\left(k-\frac{N}{2}\right)$$

$$X(k) = X_e\left(k-\frac{N}{2}\right) - W_N^k \cdot X_o\left(k-\frac{N}{2}\right)$$

from symmetry property

$$\therefore \boxed{\begin{aligned} X(k) &= X_e(k) + W_{N}^{k} X_o(k) \quad \text{for } k = 0, 1, \ldots \tfrac{N}{2}-1 \\ X(k) &= X_e(k-\tfrac{N}{2}) - W_N^{k} X_o(k-\tfrac{N}{2}) \quad \text{for } k = \tfrac{N}{2}, \tfrac{N}{2}+1, \ldots N-1 \end{aligned}}$$

→ Let $N = 8$.

$x_e(0) = x(0)$ $\qquad x_o(0) = x(1)$

$x_e(1) = x(2)$ $\qquad x_o(1) = x(3)$

$x_e(2) = x(4)$ $\qquad x_o(2) = x(5)$

$x_e(3) = x(6)$, $\qquad x_o(3) = x(7)$

by using above eqn.

$$X(k) = X_e(k) + W_8^{k} X_o(k) \quad \text{for } k = 0, 1, \ldots \tfrac{8}{2}-1 \text{ i.e } 0 \text{ to } 3$$

$$X(k) = X_e(k) - W_8^{k}. X_o(k) \quad \text{for } k = 4, 5, \ldots 8-1 \text{ i.e } k \text{ is } 4 \text{ to } 7.$$

$X(0) = X_e(0) + W_8^{0} X_o(0)$

$X(1) = X_e(1) + W_8^{1} X_o(1)$

$X(2) = X_e(2) + W_8^{2} X_o(2)$

$X(3) = X_e(3) + W_8^{3} X_o(3)$

$X(4) = X_e(4) + W_8^{4} X_o(4)$

$= X_e(4-\tfrac{8}{2}) + W_8^{4-\frac{8}{2}} . X_o(4-\tfrac{8}{2})$

$X(4) = X_e(0) + W_8^{0} X_o(0)$

$\text{(1)} \quad X(5) = X_e(5) + W_8^{5} X_o(5) = X_e(1) - W_8^{1} X_o(1)$

$X(6) = X_e(6) + W_8^{6} X_o(6) = X_e(2) - W_8^{2} X_o(2)$

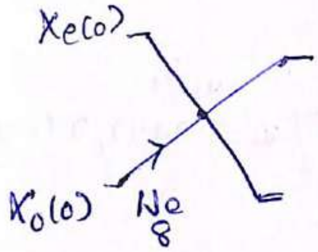
$X(7) = X_e(7) + W_8^{7} X_o(7) = X_e(3) - W_8^{3} X_o(3)$

$\because X_e(k \pm \tfrac{N}{2}) = X_e(k)$ & $X_o(k \pm \tfrac{N}{2}) = X_o(k)$

$\because X_e(k)$ & $X_o(k)$ are periodic with period $\tfrac{N}{2}$.

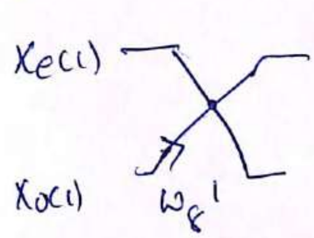
$$W_N^{k \pm \frac{N}{2}} = W_N^{k} . W_N^{\pm N/2} = -W_N^{k}$$

→ These operations can be represented by butterfly



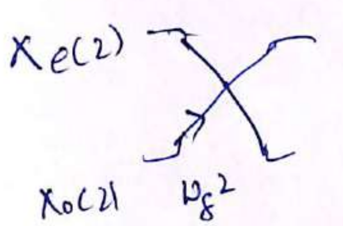
$X_e(0) + W_8^0 X_o(0) = X(0)$

$X_e(0) - W_8^0 X_o(0) = X(4)$.



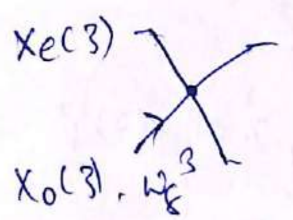
$X_e(1) + W_8^1 X_o(1) = X(1)$

$X_e(1) - W_8^1 X_o(1) = X(5)$



$X_e(2) + W_8^2 X_o(2) = X(2)$

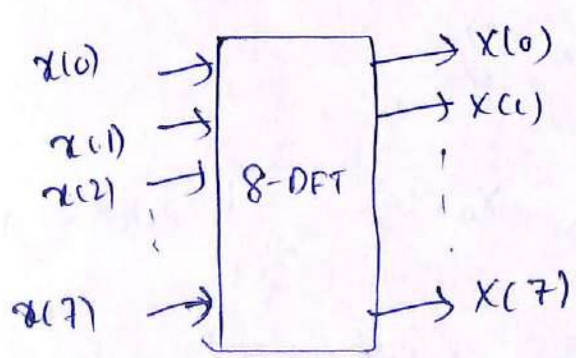
$X_e(2) - W_8^2 X_o(2) = X(6)$



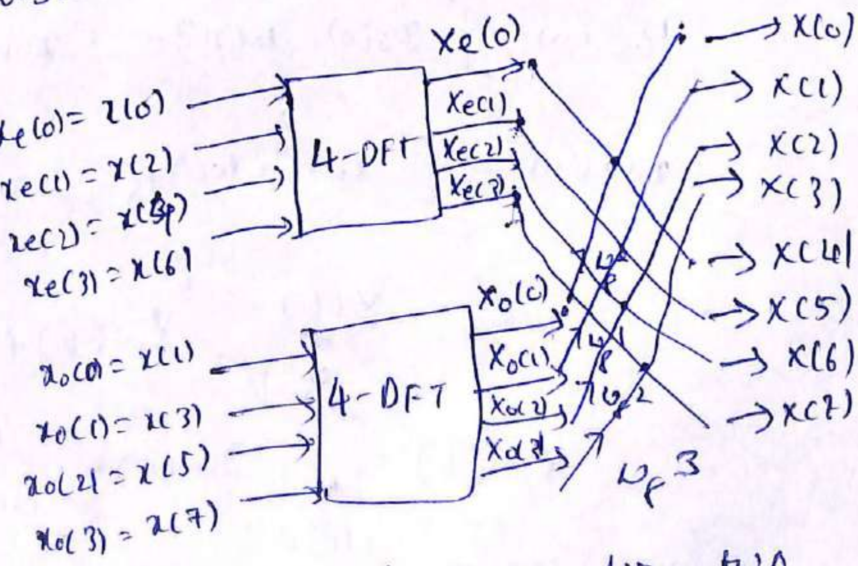
$X_e(3) + W_8^3 X_o(3) = X(3)$

$X_e(3) - W_8^3 X_o(3) = X(7)$.

→ ∴ 8-DFT can be constructed from two 4-point DFTs



fig! 8-DFT



fig! Construction of 8-DFT from two 4 point DFTs.

$\Rightarrow$ Now, apply same approach to decompose each of $\frac{N}{2}$ sample DFT & construct $\frac{N}{2}$ DFT using the $\frac{N}{4}$ DFT's.

$$x_e(n) = \{x_e(0), x_e(1), x_e(2), x_e(3)\}, = \{x(0), x(2), x(4), x(6)\}.$$

$x(n)$

$x(2n) = x_e(n)$ $\qquad$ $x_o(n) = x(2n+1)$ $\rightarrow$ 2 $\frac{N}{2}$-DFTs

$x_e(2n) = x_{ee}(n)$ $\qquad$ $x_{eo}(n) = x_e(2n+1)$ $\qquad$ $x_{oe}(n)$ $\qquad$ $x_{oo}(n)$ $\rightarrow$ four $\frac{N}{4}$ DFTs

$$x_{ee}(n) = \{x_e(0), x_e(2)\} = \{x(0), x(4)\}$$

$$x_{eo}(n) = \{x_e(1), x_e(3)\} = \{x(2), x(6)\}.$$

$||^{ly}$

$$x_o(n) = \{x_o(0), x_o(1), x_o(2), x_o(3)\} = \{x(1), x(3), x(5), x(7)\}$$

$$x_{oe}(n) = \{x_o(0), x_o(2)\} = \{x(1), x(5)\}$$

$$x_{oo}(n) = \{x_o(1), x_o(3)\} = \{x(3), x(7)\}$$

Now $\quad X(k) = X_e(k) + W_N^k X_o(k). \qquad k = 0, 1, \ldots N-1$

$$X_e(k) = \sum_{n=0}^{\frac{N}{2}-1} x_e(n) \cdot W_{N/2}^{nk} \quad \& \quad X_o(k) = \sum_{n=0}^{\frac{N}{2}-1} x_o(n) \cdot W_{N/2}^{nk}$$

$$X_e(k) = \underset{(\text{even})}{\sum_{n=0}^{\frac{N}{2}-1}} x_e(n)\, W_{N/2}^{nk} + \underset{(\text{odd})}{\sum_{n=0}^{\frac{N}{2}-1}} x_e(n)\, W_{N/2}^{nk}$$

DSP-2-24

$$= \sum_{n=0}^{\frac{N}{4}-1} x_{ee}(n) \cdot W_{N/2}^{nk} + \sum_{n=0}^{\frac{N}{4}-1} x_{eo}(n) \cdot W_{N/2}^{nk}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} x_e(2n)\, W_{N/2}^{2nk} + \sum_{n=0}^{\frac{N}{4}-1} x_e(2n+1) \cdot W_{N/2}^{(2n+1)k}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} x_e(2n) \cdot W_{N/4}^{nk} + W_{N/2}^{k} \sum_{n=0}^{\frac{N}{4}-1} x_e(2n+1) \cdot W_{N/4}^{nk}$$

$$= \underbrace{\sum_{n=0}^{\frac{N}{4}-1} x_{ee}(n) \cdot W_{N/4}^{nk}}_{\frac{N}{4}\text{-DFT of } x_{ee}(n)} + W_{N/2}^{k} \underbrace{\sum_{n=0}^{\frac{N}{4}-1} x_{eo}(n) \cdot W_{N/4}^{nk}}_{\frac{N}{4}\text{ DFT of } x_{eo}(n)}$$

$$X_e(k) = X_{ee}(k) + W_{N/2}^{k} X_{eo}(k)$$

$$W_{N/4}^{k} = \left(e^{\frac{-j2\pi}{N/2}}\right)^{k} = e^{\left(\frac{-j2\pi}{N}\right)2k} = W_N^{2k}$$

$$X_e(k) = X_{ee}(k) + W_N^{2k} X_{eo}(k), \quad k = 0, 1, \ldots \frac{N}{2}-1$$

$||^{ly}$

$$X_o(k) = X_{oe}(k) + W_N^{2k} X_{oo}(k), \quad k = 0, 1, \ldots \frac{N}{2}-1$$

for $k \geq \frac{N}{4}$,

$$X_e(k) = X_{ee}\left(k-\frac{N}{4}\right) - W_N^{2k} \cdot X_{eo}\left(k-\frac{N}{4}\right)$$

$$X_o(k) = X_{oe}\left(k-\frac{N}{4}\right) - W_N^{2k} X_{oo}\left(k-\frac{N}{4}\right)$$

DSP-2.25

$N=8$

$k=0,1,\ldots,\frac{N}{2}-1$

$$X_e(0) = X_{ee}(0) + W_8^0 X_{eo}(0)$$

$$X_e(1) = X_{ee}(1) + W_8^2 X_{eo}(1)$$

$$X_e(2) = X_{ee}(2) + W_8^4 X_{eo}(2) = X_{ee}(0) - W_8^0 X_{eo}(0)$$

$$X_e(3) = X_{ee}(3) + W_8^6 X_{eo}(3) = X_{ee}(1) - W_8^2 X_{eo}(1)$$

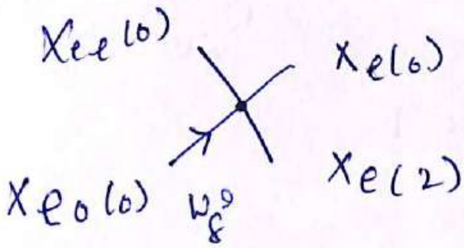
lly

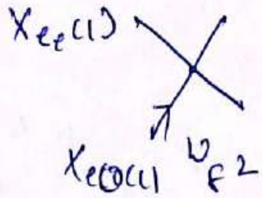
$$X_o(0) = X_{oe}(0) + W_8^0 X_{oe}(0)$$

$$X_o(1) = X_{oe}(1) + W_8^2 X_{oo}(1)$$

$$X_o(2) = X_{oe}(2) + W_8^4 X_{oo}(2) = X_{oe}(0) - W_8^0 X_{oo}(0)$$

$$X_o(3) = X_{oe}(3) + W_8^5 X_{oo}(3) = X_{oe}(1) - W_8^2 X_{oo}(1)$$



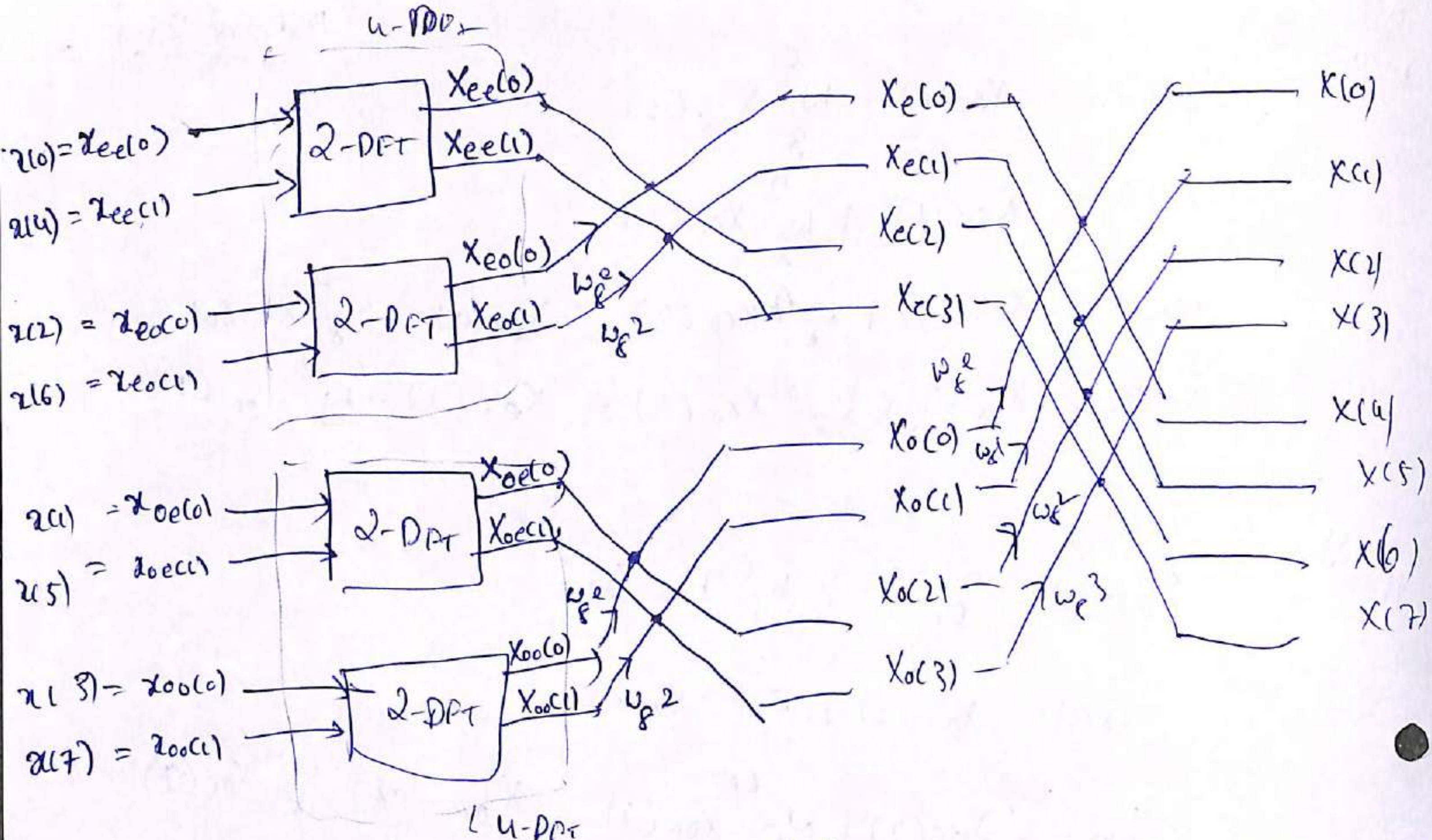


$$X_e(1) = X_{ee}(1) + W_8^2 X_{eo}(1)$$

$$X_e(3) = X_{ee}(1) - W_8^2 X_{eo}(1)$$

lly for $X_o(k)$.

Each 4-pt DFT replaced by 2- 2pt DFT.



$$X_{ee}(k) = \frac{N}{4}\text{ DFT}[x_{ee}(n)] = \sum_{n=0}^{\frac{N}{4}-1} x_{ee}(n) \cdot W_{N/4}^{nk} \qquad 0 \le k \le \frac{N}{4}-1, \ \cancel{k=0,1,\ldots\frac{N}{4}-1}$$

$$W_{N/4}^{nk} = W_N^{4nk}$$

$N=8$

$$0 \le k \le \frac{8}{4}-1, \quad 0 \le k \le 1$$

$$X_{ee}(0) = \sum_{n=0}^{1} x_{ee}(n) \cdot W_8^{4\times n\times 0} = x_{ee}(0) + W_8^0\, x_{ee}(1)$$
$$= x(0) + W_8^0\, x(4)$$

$$X_{ee}(1) = \sum_{n=0}^{1} x_{ee}(n) \cdot W_8^{4n} = x_{ee}(0) + W_8^4\, x_{ee}(1)$$

$$W_N^{k+\frac{N}{2}} = -W_N^k, \quad W_8^{0+\frac{4}{2}} = -W_8^0$$

$$X_{ee}(1) = x_{ee}(0) - W_8^0\, x_{ee}(1) = x(0) - W_8^0\, x(4)$$

DSP-2-26

lly we can write remaining

$$X_{eo}(0) = x_{eo}(0) + W_8^0 x_{eo}(1), \quad X_{eo}(1) = x_{eo}(0) - W_8^0 x_{eo}(1)$$

$$X_{eo}(0) = x(2) + W_8^0 x(6). \quad X_{eo}(1) = x(0) - W_8^0 x(6)$$

lly

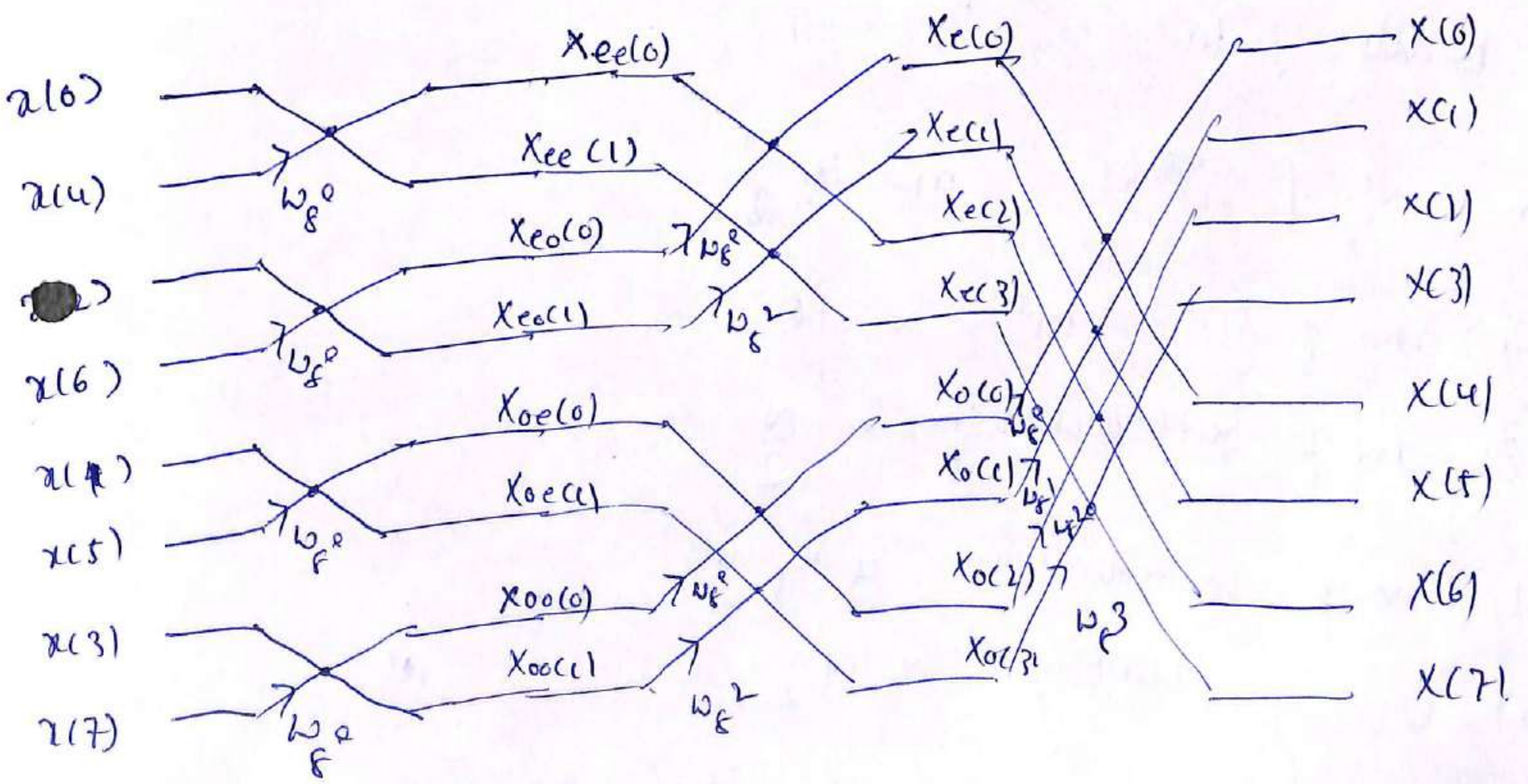
$$X_{oe}(0) = x_{oe}(0) + W_8^0 x_{oe}(1), \quad X_{oe}(1) = X_{oe}(0) - W_8^0 x_{oe}(1)$$

$$X_{oe}(0) = x(1) + W_8^0 x(5), \quad X_{oe}(1) = x(1) - W_8^0 x(5)$$

lly

$$X_{oo}(0) = x_{oo}(0) + W_8^0 x_{oo}(1), \quad X_{oo}(1) = x_{oo}(0) - W_8^0 x_{oo}(1)$$

$$X_{oo}(0) = x(3) + W_8^0 x(7), \quad X_{oo}(1) = x(3) - W_8^0 x(7)$$

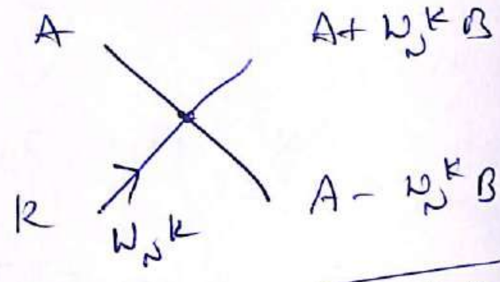


8 - DIT - FFT Algorithm.

→ The i/p is bit reversal order.

Index	Binary	bit rev	rev. index
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

∴ For DIT-FFT, basic butterfly is



12, 17, 18, 32, 35, 36, 46, 46, 49...

for N-DFT:—

twiddle factor : W_N^k (for stages).

1) No of Stages : $m = \log_2 N$.

2) No of i/p samples = N

3) no of butterflies / stage = $\frac{N}{2}$

4) no of multiplications = $\frac{N}{2} \log_2 N$

5) " additions = $N \log_2 N$

6) twiddle factor exponents / stage 'x' is

$$k = \frac{N \cdot t}{2^m}, \quad t = 0, 1, \ldots 2^{m-1} - 1.$$

UNIT- III

IIR DIGITAL FILTERS

→ Comparision of FIR & IIR filters:-

FIR filter

① The impulse response of this filter is restricted to finite no of samples.

② FIR filters are of non-recursive type. ie present O/P sample depends on the present & past I/P samples only.

③ FIR digital filters have the transfer fn. of the form,

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-1}$$

$$H(z) = b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}.$$

④ The poles are fixed at the origin.

⑤ High selectivity can be acheived using a high order T-F

IIR filter

① The impulse response is of infinite duration of samples.

② IIR filters are of recursive type. ie present O/P sample (ie y(n)) depends on the present & past I/P's & also past O/P's.

③ IIR digital filters have the transfer fn of the form

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-M}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

ie

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

④ The poles are placed anywhere.

⑤ High selectivity can be achieved using low order T.F.

(6) Always stable	(6) Not always stable
(7) It has linear phase i.e $\theta(\omega) = -\alpha \cdot \omega$.	(7). do not have linear phase.
(8). Greater flexibility to control the shape of their magnitudes	(8). Less flexibility
(9) The FIR digital filter can be designed directly	(9) Can't be directly designed. First analog filter is to be designed & then it has to be transformed to a digital filter.
(10) The impulse response can be directly ~~designed~~ converted to T-F	(11) Can't be converted directly to digital T-F..

→ Based on type of i/p — Analog filters / digital filters

→ Based on impulse response — IIR digital filter / FIR "

→ Based on freq. response — LPF / HPF / BPF / BRF

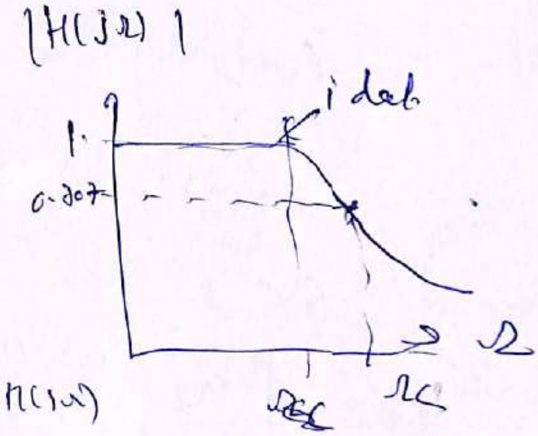
→ Comparision of LPF, HPF, BPF, BRF:-

→ A filter is one, which rejects unwanted frequencies from I/P signal & allow the desired frequencies.

→ The range of freq. that are blocked is called stop band & that are passed through filter is called "pass band".

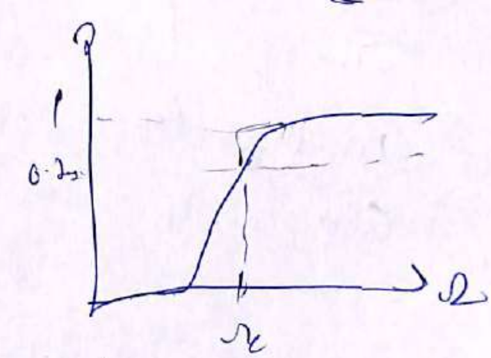
LPF:-

$$|H(j\Omega)| = \begin{cases} 1 & \text{for } \Omega < \Omega_c \\ 0 & \text{for } \Omega > \Omega_c. \end{cases}$$



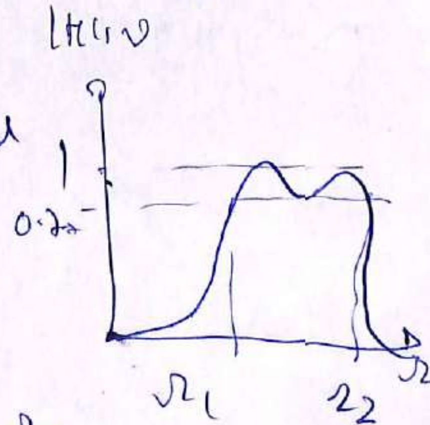
HPF:-

$$|H(j\Omega)| = \begin{cases} 1 & \text{for } \Omega > \Omega_c \\ 0 & \text{for } \Omega < \Omega_c \end{cases}$$



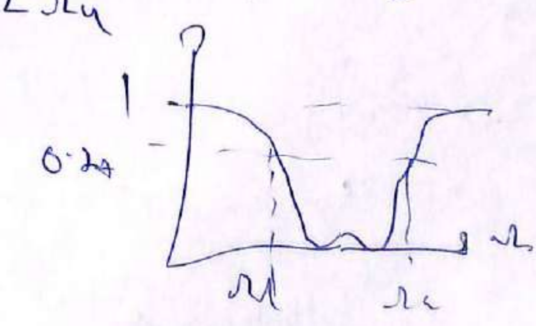
BPF:-

$$H(j\Omega) = \begin{cases} 1, & \text{for } \Omega_l < \Omega < \Omega_u \\ 0, & \text{o.w.} \end{cases}$$



BRF:-

$$H(j\Omega) = \begin{cases} 0, & \text{for } \Omega_l < \Omega < \Omega_u \\ 1, & \text{o.w.} \end{cases}$$



Analog filter	Digital filter
1. Processes analog i/p's & generates analog o/p's	1. Processes digital i/p's & generates digital o/p's.
2. Constructed from active & passive electronic components.	2. from adders, multipliers & delay unit.
3. described by differential eqn.	3. by difference eqn.
4. freq. response of analog filter can be modified by changing the components.	4. by changing filter coefficients.

→ Designing of IIR digital filters:-

→ The IIR digital filters can't be designed directly. First design an analog prototype filter & then transforming into a digital filter.

→ For the given specifications of a digital filter, to design it, it require the following three steps!

(1) Map the desired digital filter specifications into those for an equivalent analog filter.

(2) Derive the analog transfer fn. for the analog prototype.

(3) Transform the transfer fn. of the analog into equivalent digital filter T.F.

- Analog Lowpass filter Design:-

$$H(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=0}^{M} a_i \cdot s^i}{1 + \sum_{i=1}^{N} b_i s^i}$$

where $H(s) = \int_{-\infty}^{\infty} h(t) \cdot e^{-st} \, dt$ & $N \geq M$.

- → For stable analog filter, the poles of $H(s)$ lies in left half of the S-plane.

→ There are two types of analog filter design:

1. Butterworth filter
2. Chebyshev filter $\left\{ \begin{array}{l} \text{type-I} \\ \text{type-II} \end{array} \right.$

→ Analog LP Butterworth filter :-

T.F. of Butterworth LPF is

$$H(j\Omega) = \frac{1}{\left[1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right]^{1/2}} \quad N = 1, 2, 3, \ldots$$

Ω_c → cut-off freq.

N → order of filter.

DSP-3-4

Analog Lowpass filter Design:-

→ The most general form of analog filter transfer fn is

$$H(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=0}^{M} a_i s^i}{1 + \sum_{i=1}^{N} b_i s^i}$$

Where $H(s) = \int_{-\infty}^{\infty} h(t) \cdot e^{-st}\, dt$. & $N \geq M$.

→ For a stable analog filter, the poles of H(s) lie in the left half of the S-plane.

→ we study two types of filter design:

1. Butterworth filter
2. Chebyshev filter

Analog Lowpass Butterworth filter:-

→ The magnitude function of the Butterworth LPF is

$$|H(j\Omega)| = \frac{1}{\left(1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right)^{1/2}}, \quad N = 1, 2, 3, \ldots$$

$N \rightarrow$ order of filter

$\Omega_c \rightarrow$ cut-off freq.

4, 5, 12, 13, 15, 17, 18, 26, 7, 9, 32, 4, 5, 6, 42, 44, 5, 6, 1e1, 3, 5

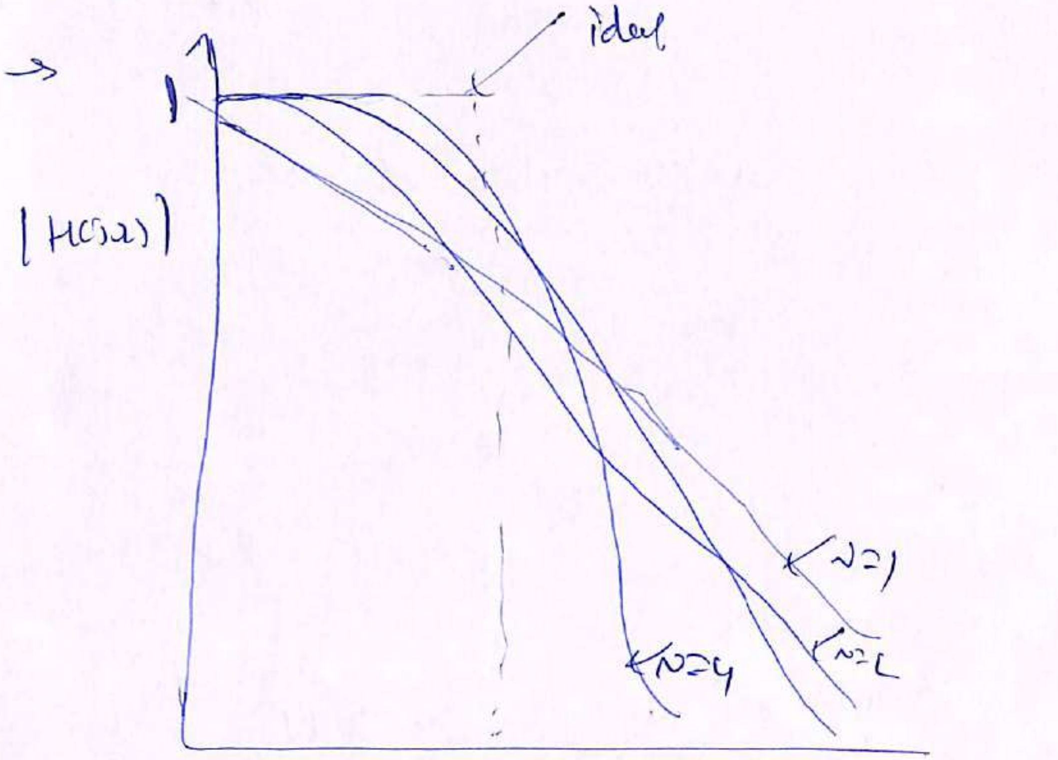


fig: LP Butterworth mag. response $\rightarrow$ freq.

$\Rightarrow$ As shown in fig, the butterworth magnitude function is monotonically decreasing, where the max. response is unity at $\Omega=0$.

$\rightarrow$ $$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

let $\Omega = \frac{s}{j}$. & consider normalized filter i.e $\Omega_c = 1$ rad/s

$$|H(s)|^2 = \frac{1}{1+\left(\frac{s}{j}\right)^{2N}}$$

$$H(s)\cdot H(-s) = \frac{1}{1+(-s^2)^N}$$

→ To find Poles, Dr = 0.

$$1+(-s^2)^N = 0$$

If N is odd, $s^{2N} = 1$

$$s^{2N} = e^{j2\pi k}$$

roots are. $S_k = e^{\frac{j\pi k}{N}}$, $k = 1, 2, \cdots, 2N$.

N is even. $s^{2N} = -1$

$$s^{2N} = e^{j(2k-1)\pi}$$

$$S_k = e^{\frac{j(2k-1)\pi}{2N}}, \quad k = 1, 2, \cdots, 2N$$

If $\underline{\underline{N=3}}$

$$s^6 = e^{j2\pi k}, \quad S_k = e^{\frac{j\pi k}{3}}, \quad k = 1, 2, \cdots 2N$$

$\underline{\underline{k=1.}}$

$$S_1 = e^{\frac{j\pi}{3}} = 0.5 + j0.866$$

$$S_2 = e^{j\frac{2\pi}{3}} = -0.5 + j0.866$$

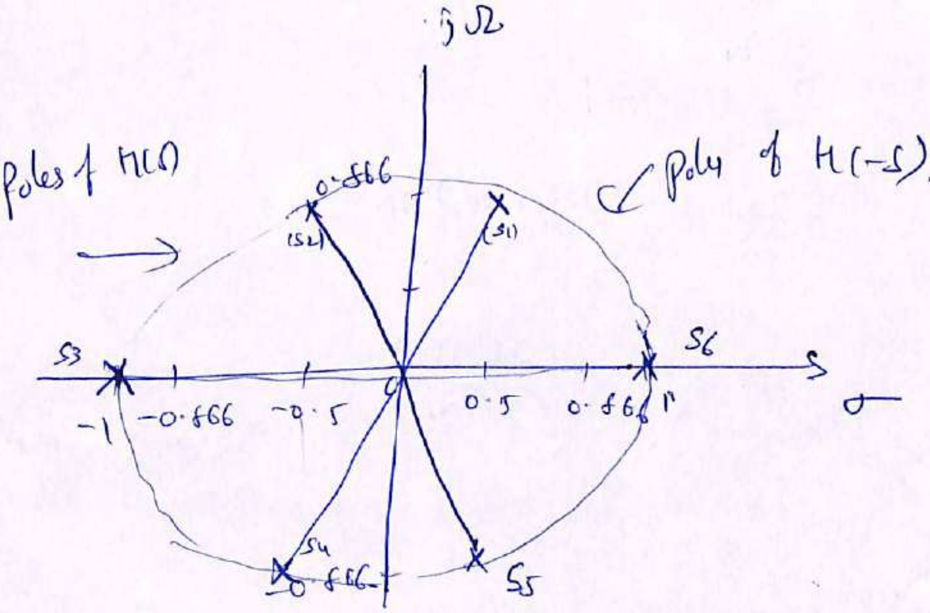
$$S_3 = e^{j\pi} = -1$$

$$S_4 = e^{j\frac{4\pi}{3}} = -0.5 - j0.866$$

$$S_5 = e^{j\frac{5\pi}{3}} = 0.5 - j0.866$$

$$S_6 = e^{j2\pi} = 1.$$

→ Locate all poles in S-plane. The angular seperation b/w poles $\frac{360}{2N}$.



→ To ensure stability, all poles that lie on LHP.

So, consider the poles only on LHP.

$$H(s) = \frac{1}{(S-S_1)(S-S_2)(S-S_3)}$$

$$(S+1)(S+0.5+j0.866)(S+0.5-j0.866) = (S+1)(S^2+S+1).$$

$$H(s) = \frac{1}{(S+1)(S^2+S+1)}$$

DSP-3-6

If N is even, $N=4$.

$$S_k = e^{j(2k-1)\pi/2N}, \quad k=1,2,\ldots,2N$$

$k=1$

$$S_1 = e^{j\frac{\pi}{8}} = \cos\frac{\pi}{8} + j\sin\frac{\pi}{8} =$$

$$S_2 = e^{j\frac{3\pi}{8}} =$$

✓ $$S_3 = e^{j\frac{5\pi}{8}} = -0.3827 + j0.9239$$

✓ $$S_4 = e^{j\frac{7\pi}{8}} = -0.9239 + j0.3827$$

✓ $$S_5 = e^{j\frac{9\pi}{8}} = -0.9239 + -j0.3827$$

✓ $$S_6 = e^{j\frac{11\pi}{8}} = -0.3827 - j0.9239$$

$$S_7 = e^{j\frac{13\pi}{8}} =$$

$$S_8 = e^{j\frac{15\pi}{8}} =$$

Consider only poles on L.H.S.

Dr. of $H(s)$.

$$Dr = (S-S_3)(S-S_4)(S-S_5)(S-S_6)$$

$$Dr = (S^2+1.84776S+1)(S^2+0.765365S+1)$$

4, 5, 7, 13, 25, 8, 3
8, 40, 4, 6, 7, 8
1 → 1, 3, 4, 5

∴ for $N = 4$.

$$H(s) = \frac{1}{(s^2 + 0.765365s + 1)(s^2 + 1.847765s + 1)}$$

∴ List of Butter Polynomials for $\Omega_c = 1$ rad/sec

N	Denominator of H(s)
1	$s+1$
2	$s^2 + \sqrt{2}s + 1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.765375s+1)(s^2+1.84775s+1)$
5	$(s+1)(s^2+0.618035s+1)(s^2+1.618035s+1)$
6	$(s^2+1.931855s+1)(s^2+\sqrt{2}s+1)(s^2+0.517645s+1)$
7	$(s+1)(s^2+1.80194s+1)(s^2+1.247s+1)(s^2+0.4455s+$

→ for unnormalized fn. (i.e $\Omega_c \neq 1$ rad/sec),

$S_k' = \Omega_c \cdot S_k \Rightarrow S_k = \frac{S_k'}{\Omega_c}$.

Replace in $H(s)$, $s \to s/\Omega_c$

Finding order (N) of Butterworth filter:-

→ eq. $|H(j\Omega)| = \frac{1}{\left(1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}\right)^{1/2}}$ is restricted to -3db attenuation at Ω_c.

→ Now, to find the 'N' (order of filter) for the given specifications, α_p, α_s at Ω_p & Ω_s, we have to Consider the magnitude function as.

$$|H(j\Omega)| = \frac{1}{\left(1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}\right)^{1/2}}$$

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}$$

where

α_p → max. pass band attenuation in positive db at pass band freq. Ω_p.

α_s → min. stop band attenuation in positive db at stop band freq. Ω_s.

ε → parameter related to pass band

λ → " " " stop band

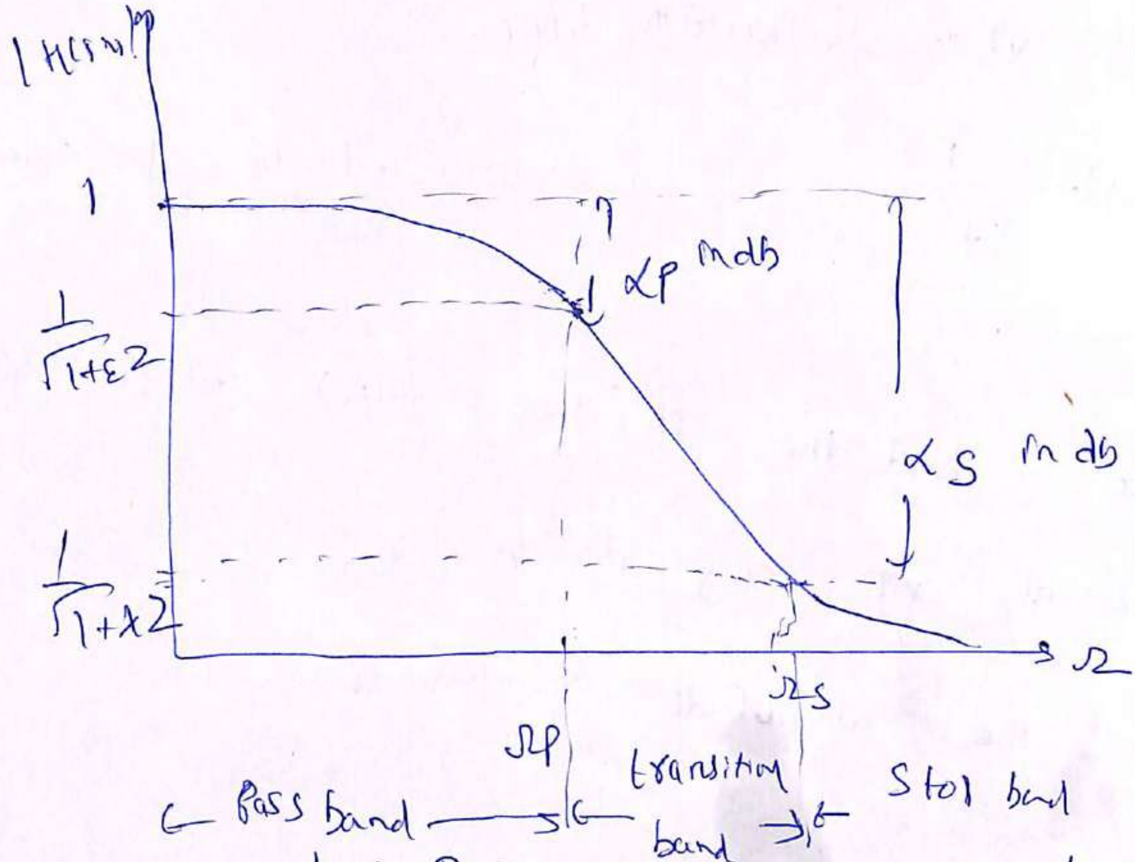


fig: Butterworth approximation of mag response

Squaring

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}$$

$$20\log|H(j\Omega)| = 10\log 1 - 10\log\left[1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}\right]$$

$$\underbrace{20\log|H(j\Omega)|}_{} = -10\log\left[1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}\right]$$

At $\Omega = \Omega_p$, attenuation $= \alpha_p$.

$$-\alpha_p = -10\log\left(1+\varepsilon^2\left(\frac{\Omega_p}{\Omega_p}\right)^{2N}\right)$$

$$\alpha_p = 10\log(1+\varepsilon^2)$$

$$0.1\alpha p = \log(1+\varepsilon^2)$$

$$\Rightarrow 1+\varepsilon^2 = 10^{0.1\alpha p}$$

$$\varepsilon^2 = 10^{0.1\alpha p} - 1$$

$$\boxed{\varepsilon = \sqrt{10^{0.1\alpha p} - 1}}$$

(ii) At $\Omega = \Omega s$, atte $= -\alpha s$.

$$-\alpha s = -10 \log \left[1 + \varepsilon^2 \left(\frac{\Omega s}{\Omega p}\right)^{2N}\right]$$

$$\alpha s = 10 \log \left(1 + \varepsilon^2 \left(\frac{\Omega s}{\Omega p}\right)^{2N}\right).$$

$$\Rightarrow \varepsilon^2 \left(\frac{\Omega s}{\Omega p}\right)^{2N} = 10^{0.1\alpha s} - 1.$$

$$\therefore \left(\frac{\Omega s}{\Omega p}\right)^{2N} = \frac{10^{0.1\alpha s} - 1}{10^{0.1\alpha p} - 1} \quad (\because \varepsilon^2 = 10^{0.1\alpha p} - 1)$$

$$\Rightarrow \boxed{N = \frac{\log \sqrt{\dfrac{10^{0.1\alpha s} - 1}{10^{0.1\alpha p} - 1}}}{\log \dfrac{\Omega s}{\Omega p}}}$$

→ To get integer value, round off N to next higher integer

$$N \geq \frac{\log \sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log \dfrac{\Omega_s}{\Omega_p}}$$

$$\boxed{N \geq \frac{\log \dfrac{\lambda}{\varepsilon}}{\log \dfrac{\Omega_s}{\Omega_p}}}$$

$$\varepsilon = \left(10^{0.1\alpha_p}-1\right)^{1/2}$$

$$\lambda = \left(10^{0.1\alpha_s}-1\right)^{1/2}$$

$$\boxed{N \geq \frac{\log A}{\log (1/k)}}$$

$$A = \frac{\lambda}{\varepsilon}$$

$$k = \frac{\Omega_p}{\Omega_S}$$ → transition ratio

Finding Ω_c from given filter specifications :—

$$\boxed{\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p}-1\right)^{1/2N}} = \frac{\Omega_s}{\left(10^{0.1\alpha_s}-1\right)^{1/2N}}}$$

DSP-3-9

We K.T

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \quad \text{if } \Omega_c \text{ given}$$

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}$$

$$\frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} = \frac{1}{1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}}$$

$$\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N} = \left(\frac{\Omega}{\Omega_c}\right)^{2N}$$

$$\Rightarrow \left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = \varepsilon^2$$

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = 10^{0.1\alpha_P} - 1$$

$$\boxed{\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_P}-1\right)^{1/2N}}}$$

we get

$$\left(\frac{\Omega s}{\Omega p}\right)^{2N} = \frac{10^{0.1\alpha s} - 1}{10^{0.1\alpha p} - 1}$$

$$\Omega s = \Omega p \left[\frac{10^{0.1\alpha s} - 1}{10^{0.1\alpha p} - 1}\right]^{1/2N}$$

But $\Omega p = \Omega_c (10^{0.1\alpha p} - 1)^{1/2N}$

$$\Omega s = \Omega c \,(10^{0.1\alpha s} - 1)^{1/2N}$$

$$\Rightarrow \Omega c = \frac{\Omega s}{(10^{0.1\alpha s} - 1)^{1/2N}}$$

$$\therefore \boxed{\Omega_c = \frac{\Omega p}{(10^{0.1\alpha p} - 1)^{1/2N}} = \frac{\Omega s}{(10^{0.1\alpha s} - 1)^{1/2N}}}$$

→ Steps to design an analog Butterworth LPF :-

1) From the given specifications, find the order of filter 'N'.
2) Round off it to the next higher integer.
3) Find the transfer function H(s) for $\Omega_c = 1$ rad/sec. for the values of 'N'.
4) Calculate the value of cut-off freq. Ω_c
5) Find $H_a(s)$ by sub. $S \rightarrow \frac{S}{\Omega_c}$ in H(s)

Q:- Design an analog Butterworth filter has a −2 db Passband attenuation at a frequency of 20 rad/sec. & at least −10 db stop band attenuation at 30 rad/sec

Sol Given $\alpha_p = 2$ db, $\Omega_p = 20$ rad/s

$\alpha_s = 10$ db, $\Omega_s = 30$ rad/sec

1) find N.

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log \frac{\Omega_s}{\Omega_p}}$$

$$N \geq \frac{\log \sqrt{\frac{10 - 1}{10^{0.2} - 1}}}{\log \frac{30}{20}} \geq 3.37$$

$\underline{\underline{N = 4}}$

(2) $N = 4$

(3) Normalized $H(s)$ for $N=4$ is

$$H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.84775s+1)}$$

(4) $$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} \text{ or } \frac{\Omega_s}{(10^{0.1\alpha_s}-1)^{1/2N}}$$

$$\Omega_c = 21.3868$$

(5) $$s \to \frac{s}{\Omega_c} = \frac{s}{21.3868} \quad \text{in } H(s)$$

$$H_a(s) = \frac{1}{\left(\left(\frac{s}{21.3868}\right)^2 + 0.76537\times\frac{s}{21.3868}+1\right)\left(\left(\frac{s}{21.3868}\right)^2+1.84775\frac{s}{21.3868}+1\right)}$$

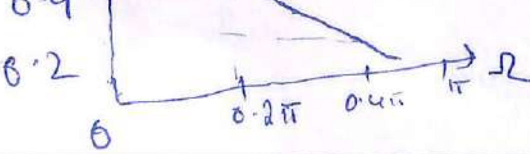
$$H_a(s) = \frac{0.20921\times 10^6}{(s^2+16.3686s+457.394)(s^2+39.517s+457.394)}$$

(2) For the given specifications design an analog Butterworth filter.

$0.9 \le |H(j\Omega)| \le 1$ for $0 \le \Omega \le 0.2\pi$.

$|H(j\Omega)| \le 0.2$ for $0.4\pi \le \Omega \le \pi$

Sol



$\therefore \frac{1}{\sqrt{1+\varepsilon^2}} = 0.9, \Rightarrow \varepsilon = 0.484$

$\frac{1}{\sqrt{1+\lambda^2}} = 0.2 \Rightarrow \lambda = 4.898$

$N = 2$; $\Omega_c = 0.24\pi$

Analog Low Pass Chebyshev Filters :-

→ There are two types — type-I, type-II.

→ Type-I Chebyshev filters are all-pole filters that exhibits equiripple behaviour in the pass band & a monotonic characteristics in the denominator.

→ Type-II filter contains both poles & zeros & exhibits a monotonic behavior in the pass band & an equiripple behaviour in the stop band.

→ Type-I :-

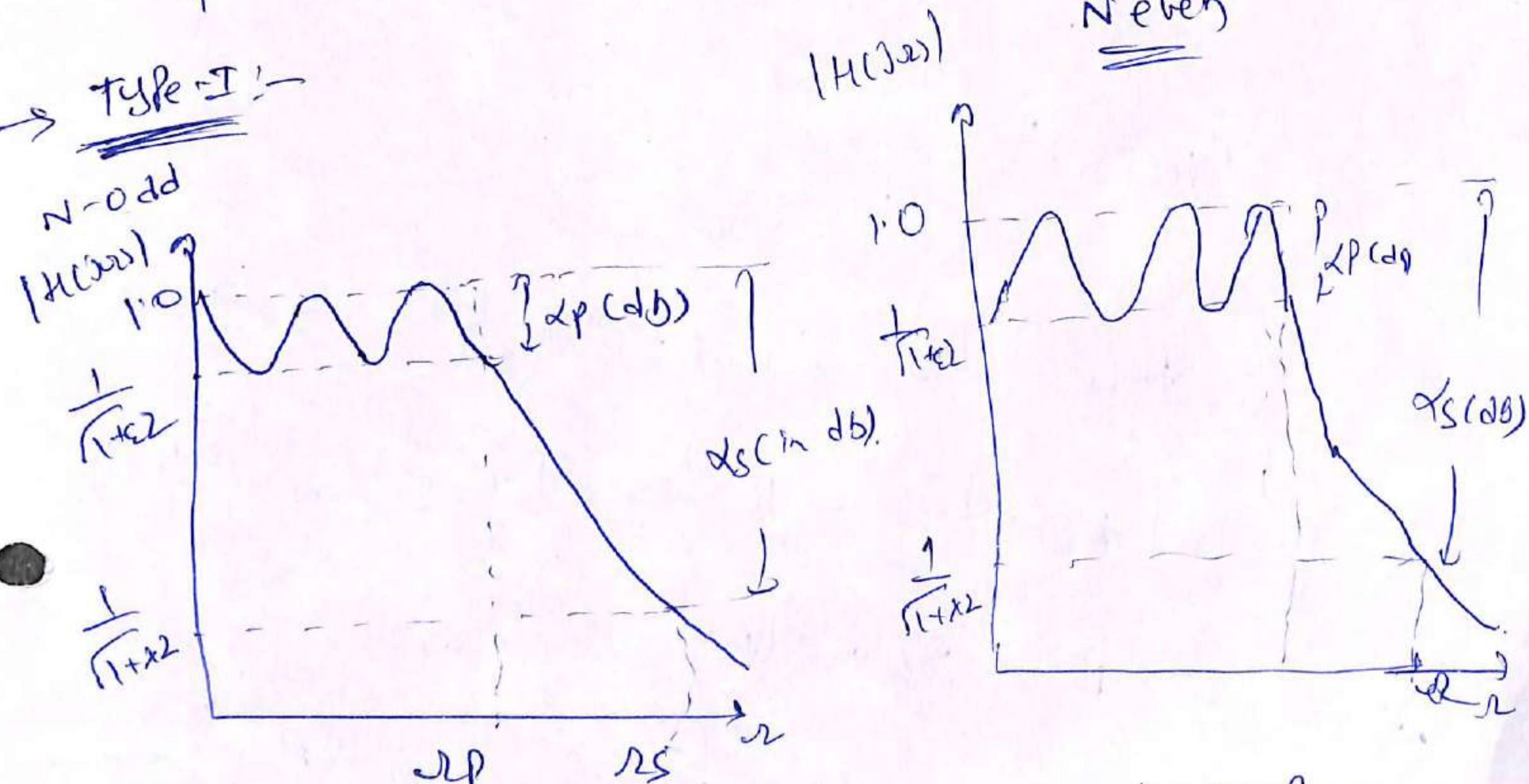


fig: Low pass Chebyshev filter magnitude response

→ The magnitude response of Nth order type-I filter is

$$\boxed{|H(j\Omega)| = \frac{1}{\left(1+\epsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)\right)^{1/2}}} \quad -(1)$$

$N = 1, 2, 3, \ldots$

Where $C_N(x) \rightarrow N^{th}$ order Chebyshev polynomial

$$C_N(x) = \cos(N \cos^{-1} x), \quad |x| \leq 1 \text{ (Pass band)}$$

$$C_N(x) = \cosh(N \cosh^{-1} x), \quad |x| > 1, \text{ (Stop band)}$$

$\varepsilon \rightarrow$ Parameter related to ripple in the pass band.

1) Finding order of filter:-

take square of eq-(1)

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)}$$

take log & multiply with 10 on B-S

$$20 \log |H(j\Omega)| = -10 \log\left(1+\varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)\right).$$

At $\Omega = \Omega_p$, $20 \log |H(j\Omega)| = -\alpha_p$

$$-\alpha_p = -10 \log\left(1+\varepsilon^2 C_N^2\left(\frac{\Omega_p}{\Omega_p}\right)\right)$$

$$\alpha_p = 10 \log(1+\varepsilon^2 C_N^2(1))$$

$$C_N(1) = \cos(N \cdot \cos^{-1} 1) = \cos(0) = 1.$$

$$\alpha_p = 10 \log(1+\varepsilon^2)$$

$$0.1\alpha_p = \log(1+\varepsilon^2)$$

take Antilog

$$10^{0.1\alpha p} = 1+\varepsilon^2$$

$$\Rightarrow \boxed{\varepsilon^2 = 10^{0.1\alpha p} - 1}, \quad \varepsilon = \sqrt{10^{0.1\alpha p} - 1}$$

At $\Omega = \Omega s$

$$\alpha s = 10 \log \left(1 + \varepsilon^2 C_N^2 \left(\frac{\Omega s}{\Omega p}\right)\right]$$

$$\because \frac{\Omega s}{\Omega p} > 1, \quad C_N(x) = \cosh (N \cosh^{-1} x) \text{ if } |x| > 1$$

$$\alpha s = 10 \log \left(1 + \varepsilon^2 . (\cosh (N \cosh^{-1} (\frac{\Omega s}{\Omega p}))^2\right)$$

take Antilog

$$10^{0.1\alpha s} = 1 + \varepsilon^2 (\cosh (N \cosh^{-1} \frac{\Omega s}{\Omega p})))^2$$

$$\varepsilon^2 . (\cosh (N \cosh^{-1} \frac{\Omega s}{\Omega p}))^2 = 10^{0.1\alpha s} - 1$$

$$(\cosh (N \cosh^{-1} \frac{\Omega s}{\Omega p}))^2 = \frac{10^{0.1\alpha s} - 1}{10^{0.1\alpha p} - 1} \quad (\because \text{Sub } \varepsilon^2$$

take $\cosh^{-1}$

$$N \cosh^{-1} \frac{\Omega s}{\Omega p} = \cosh^{-1} \left(\sqrt{\frac{10^{0.1\alpha s} - 1}{10^{0.1\alpha p} - 1}}\right)$$

$$\boxed{N = \frac{\cosh^{-1}\sqrt{\dfrac{10^{0.1\alpha s}-1}{10^{0.1\alpha p}-1}}}{\cosh^{-1}\dfrac{\Omega s}{\Omega p}}}$$

$$\boxed{N \geq \frac{\cosh^{-1} A}{\cosh^{-1} 1/k}}$$

$$A = \frac{\lambda}{\varepsilon}$$

$$\lambda = \sqrt{10^{0.1\alpha s}-1}$$

$$N \geq \frac{\cosh^{-1} \frac{\lambda}{\varepsilon}}{\cosh^{-1} 1/k}$$

$$\varepsilon = \sqrt{10^{0.1\alpha p}-1}$$

$$k = \frac{\Omega p}{\Omega s}.$$

we can cal.

$$\cosh^{-1}(x) = \ln\left[x + \sqrt{x^2-1}\right]$$

(2) Pole locations of chebyshev filter:-

-> we kn Poles of Butterworth filter lies on circle.

-> Poles of chebyshev filter lies on Ellipse.

To cal. $1+\varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right) = 0$

Sub $j\Omega = s \Rightarrow \Omega = \frac{s}{j} = -sj$

DSP-3-13

$$y_k = \frac{\Omega_s \Omega_k}{\sigma_k^2 + \Omega_k^2}$$

Where

$$\sigma_k = a \cos \phi_k$$

$$\Omega_k = b \cos \phi_k$$

$$a = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2} \right],$$

$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2} \right]$$

$$\phi_k = \frac{\pi}{2} + \left(\frac{2k-1}{2N} \right) \pi$$

$$\cancel{\mu = \varepsilon^{-1} + \sqrt{\varepsilon^{-2} + 1}}$$

$$\mu = \lambda + \sqrt{1 + \lambda^2}$$

$\rightarrow$ $$N = \frac{\cosh^{-1} \frac{1}{\varepsilon}}{\cosh^{-1} \frac{\Omega_s}{\Omega_p}}.$$

Steps to design an analog Chebyshev LPF:-

1) From the given Specifications, find the order of the filter 'N'.

2) Round off it to the next higher integer

3) Find a, b, μ, ε from given Specifications

$$\varepsilon = \sqrt{10^{0.1\alpha_P} - 1}$$

$$\mu = \varepsilon^{-1} + \sqrt{\varepsilon^{-2} + 1}$$

$$a = \Omega_P \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2} \right]$$

$$b = \Omega_P \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2} \right]$$

4) Calculate Poles of Chebyshev filter lie on ellipse by using

$$S_k = a\cos\phi_k + j\, b\sin\phi_k = \sigma_k + j\,\Omega_k.$$

where

$$\phi_k = \frac{\pi}{2} + \left(\frac{2k-1}{2N}\right)\pi \qquad k = 1, 2, \ldots, N.$$

5, find the denominator polynomial of T-F of above poles.

The Numerator of T-F depends on the value of 'N'.

6, The Numerator of T-F depends on the value of 'N'.

→ N odd

for N odd, substitute $S=0$ in the denominator polynomial & find the value. This is equal to the numerator of T-F.

→ N even

for N even, Sub. $S=0$ in Dr & divide the result by $\sqrt{1+\varepsilon^2}$. This value is equal to Nr.

Eg:- Given the specifications, $\alpha_p = 3\,db$, $\alpha_s = 16\,db$, $f_p = 1\,kHz$ & $f_s = 2\,kHz$. Determine the order of the filter using Chebyshev approximation. Find $H(S)$.

So) Given

$\Omega_p = 2\pi \times 1000\,Hz = 2000\pi$ rad/Sec

$\Omega_s = 2\pi \times 2000\,Hz = 4000\pi$ rad/Sec

$\alpha_p = 3\,db$, $\alpha_s = 16\,db$

Step 1:-

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$N \geq 1.91$$

Step 2:-

$$N = 2. \quad (\text{Round off})$$

Step 3:-

$$\varepsilon = \sqrt{(10^{0.1\alpha_p}-1)} = 1$$

$$\mu = \varepsilon^{-1} + \sqrt{\varepsilon^{-2}+1} = 2.414$$

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 910\pi$$

$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 2197\pi$$

Step 4:-

The pole are

$$s_k = a\cos\phi_k + b\sin\phi_k, \quad k = 1, 2.$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)}{2N}\pi$$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{4} = 135^\circ, \quad \phi_2 = \frac{\pi}{2} + \frac{3\pi}{4} = 225^\circ.$$

$$s_1 = a\cos\phi_1 + b\sin\phi_1 = -643.46\pi + j\,1554\pi$$

DSP-3-15

$S_2 = a\cos\phi_2 + j\,b\sin\phi_2 = -643.46\pi - j\,1554\pi$

Step 5:- Dr. of $H(S) = \dfrac{1}{(S-S_1)(S-S_2)} = (S+643.46\pi)^2 + (1554\pi)^2$

Step 6:- Nr. N = even, put $S=0$ & divide $\sqrt{1+\varepsilon^2}$

$$\text{Nr of } H(S) = \frac{(643.46\pi)^2 + (1554\pi)^2}{\sqrt{1+(1)^2}}$$

$$= (1414.38)^2\,\pi^2$$

Step 7:-

$$H(S) = \frac{(1414.38)^2\,\pi^2}{(S+643.46\pi)^2 + (1554\pi)^2}$$

eg 2:- Obtain an analog Chebyshev filter T.F that satisfies the constraints:

$$\frac{1}{\sqrt{2}} \le |H(j\Omega)| \le 1;\quad 0 \le \Omega \le 2$$

$$|H(j\Omega)| < 0.1,\quad \Omega \ge 4$$

Sol $\dfrac{1}{\sqrt{1+\varepsilon^2}} = \dfrac{1}{\sqrt{2}},\ \dfrac{1}{\sqrt{1+\lambda^2}} = 0.1,\quad \Omega_p = 2,\ \Omega_s = 4$

$N = 2.269$

Design of IIR filters from analog filters :-

→ There are several methods that can be used to design digital filter having an infinite duration unit sample response. All are based on converting an analog filter into a digital filter.

→ The techniques to convert an analog filter into a digital filter should possess the following desirable properties.

(1) The $j\Omega$-axis in the S-plane map into the unit circle in the Z-plane. Thus there will be a direct relationship between the two frequency variables in the two domains.

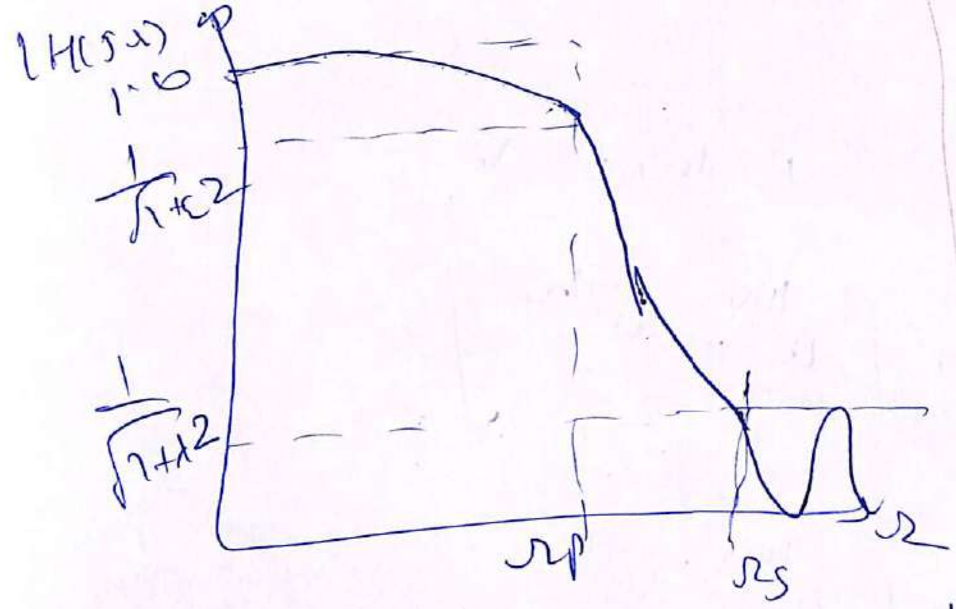
(2) The LHP of S-plane should map into the inside of unit circle in the Z-plane. Thus a stable analog filter will be converted to a digital filter.

→ Mostly used methods for digitizing the analog into digital filter are!

1) Impulse Invariance technique
2) Bilinear transformation technique
3) Step invariance technique
   Approximation of derivatives
4) (5) Matched z-transform technique.

~~Steps~~

Chebyshev type-2 filter:-



$\sigma_k = a\cos\phi_k$

$\Omega_k = b\sin\phi_k$

$\mu = \lambda + \sqrt{1+\lambda^2}$

$N = \frac{\cosh^{-1}\left(\frac{A}{\epsilon}\right)}{\cosh^{-1}\frac{\Omega_s}{\Omega_p}}$

fig! Type-II Chebyshev filter mag. response.

→ The magnitude response is,

$$|H(j\Omega)| = \frac{1}{\left(1+\epsilon^2\left[\frac{C_N^2\left(\frac{\Omega_s}{\Omega_p}\right)}{C_N^2\left(\frac{\Omega_s}{\Omega}\right)}\right]\right)^{1/2}}$$

→ The zeros are located on the imaginary axis.

at

$$S_k = j\frac{\Omega_s}{\sin\phi_k}, \quad k = 1,2,\ldots,N$$

→ The poles are located at (x_k, y_k) are

$$x_k = \frac{\Omega_s \cdot \sigma_k}{\sigma_k^2 + \Omega_k^2}, \quad k = 1,2,\ldots,N$$

$y_k = \frac{\Omega_s \cdot \Omega_k}{\sigma_k^2 + \Omega_k^2}$

Simplify.

$$\boxed{S_K = a\cos\phi_k + j b \sin\phi_k = \sigma_k + j\Omega_k}$$

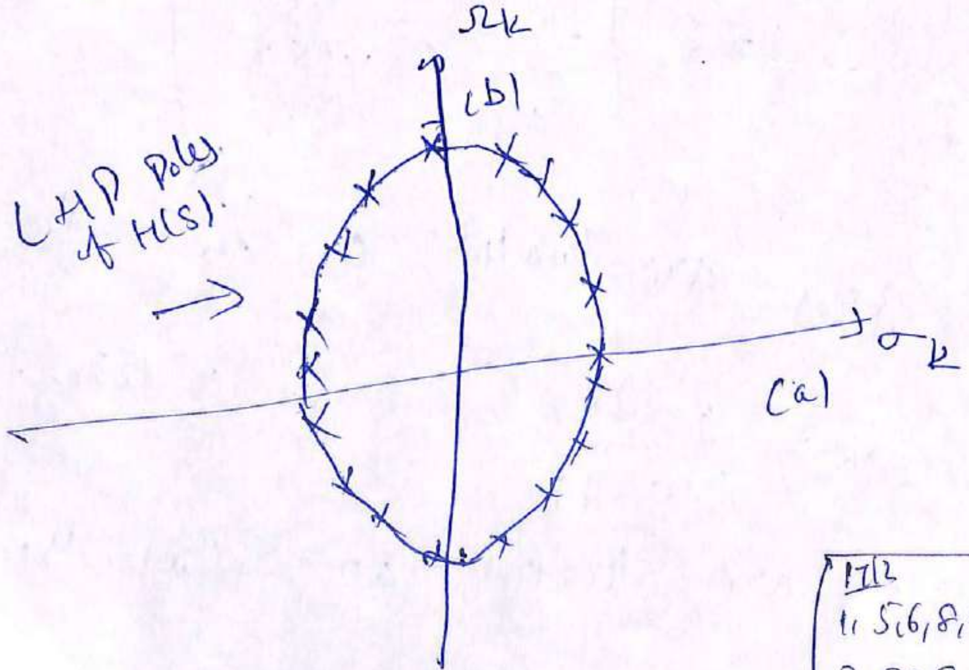
$k = 1, 2, \ldots N.$

minor axis of ellipse → $a = \Omega_p \left[\dfrac{\mu^{1/N} - \mu^{-1/N}}{2} \right],$

major axis of lobe → $b = \Omega_p \left[\dfrac{\mu^{1/N} + \mu^{-1/N}}{2} \right]$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}$$



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1, 5, 6, 8, 9, 12, 13, 18, 19, 20, 2, 3, 30, 3, 40, 3, 4, 6, 7, 8, 9, 51, 61, 2, 3, 4, 5.

Design of IIR filter using Impulse Invariance technique:-

→ In impulse invariance method the IIR filter is designed such that the unit impulse response h(n) of digital filter is the sampled version of the impulse response of analog filter.

$$ZT[h(n)] = H(z) = \sum_{n=-\infty}^{\infty} h(n) \cdot z^{-n}$$

$$H(z)\Big|_{z=e^{sT}} = \sum_{n=-\infty}^{\infty} h(n) \cdot e^{-sTn}$$

→ Consider the mapping of points from S-plane into the z-plane by

$$z = e^{sT}$$

$$s = \sigma + j\Omega \quad \& \quad z \text{ in polar form as, } z = r \cdot e^{j\omega}$$

$$r e^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$r e^{j\omega} = e^{\sigma T} \cdot e^{j\Omega T}$$

$$\therefore r = e^{\sigma T}, \quad \omega = \Omega t$$

∴ Analog pole is mapped to a place in the z-plane of radius $e^{\sigma T}$ & angle ΩT.

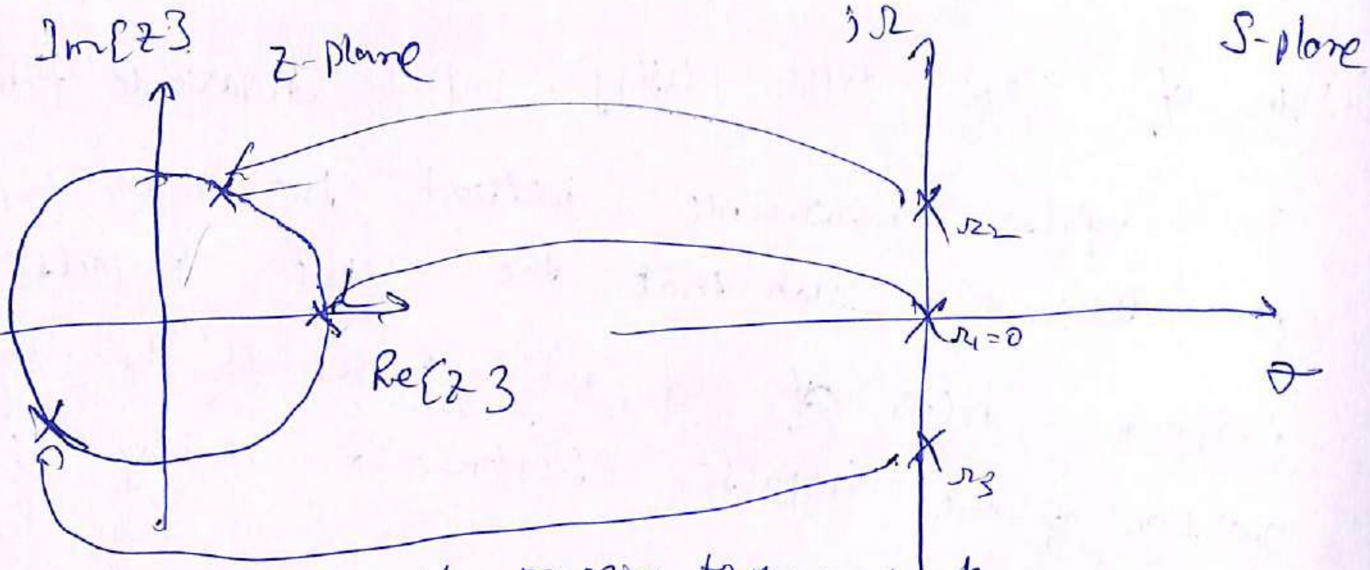


Fig: jΩ-axis mapping to the unit circle

i)

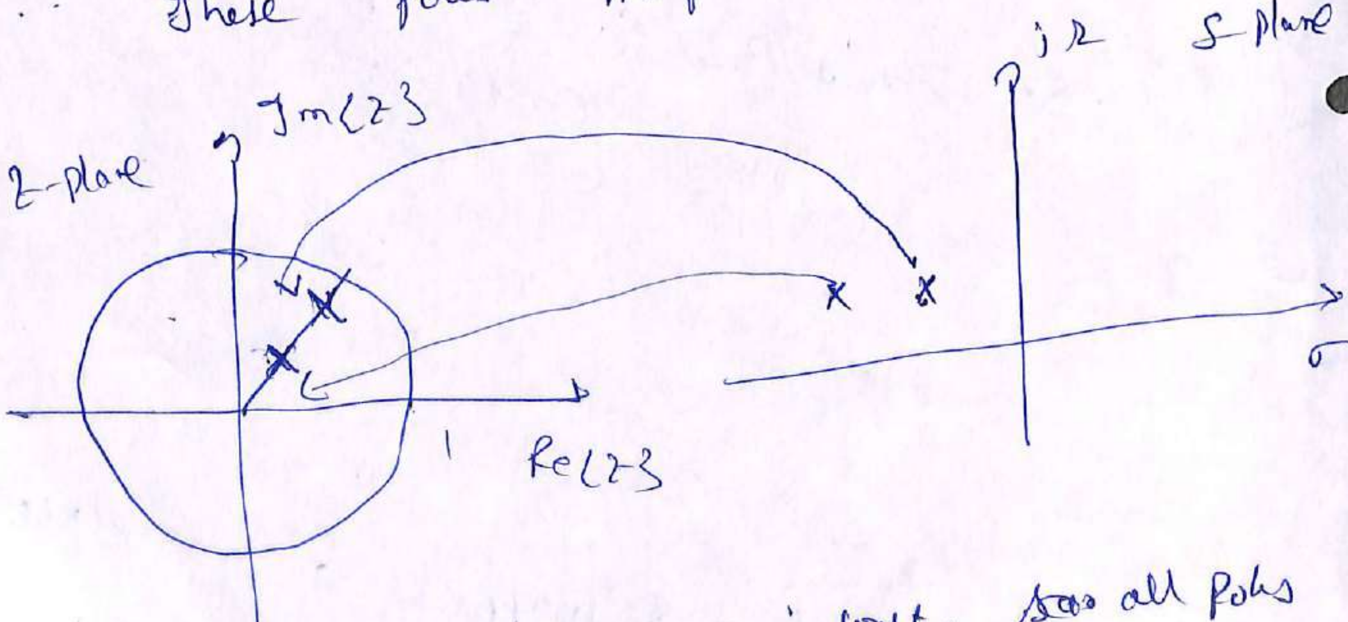
$r = e^{\sigma T}$,

if $\sigma = 0$ (real part = 0), $r = 1$

ii) Consider poles in L.H.P where $\sigma < 0$.

$r = e^{\sigma T} < 1$, if $\sigma < 0$

∴ These poles map inside the unit circle.



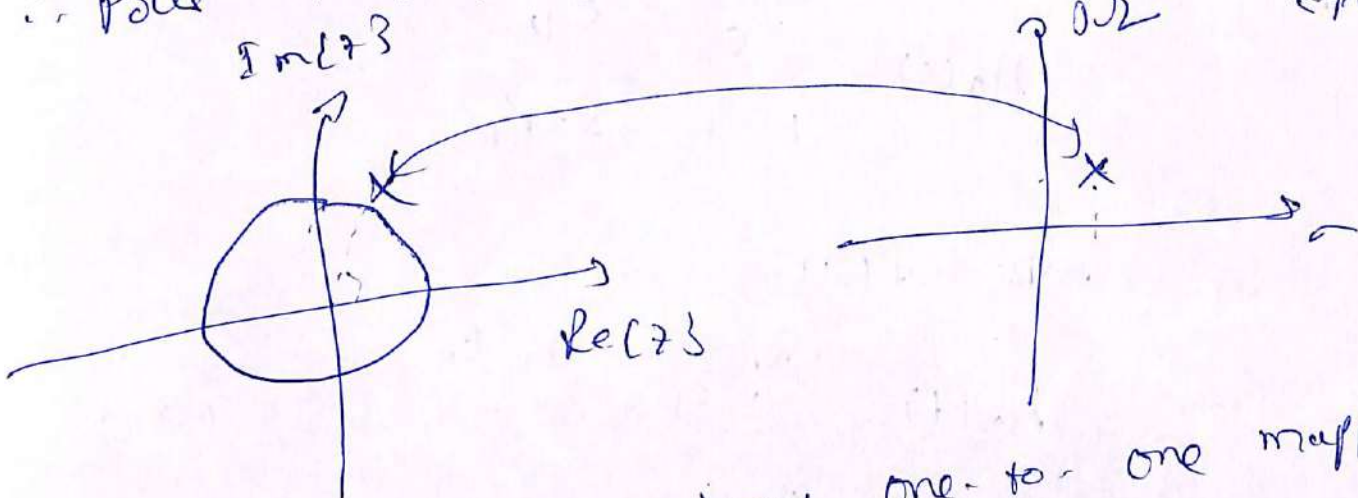
iii) $\omega = \Omega T$, Ω is const, so all poles at same angle

DSP-3-18

(ii) $\sigma > 0$ ie poles in R.H.P.

$r = e^{\sigma T} > 1$ if $\sigma > 1$.

$\therefore$ Poles in RHP are mapped out-side unit circle in z-plane.



→ But it is not always one-to-one mapping. rather it has many to one mapping. i.e many poles in S plane are mapped into a single poles in z-plane if all poles have same real part & imaginary part is differed by some integer multiple of $\frac{2\pi}{T}$.

eg: $s_1 = \sigma + j\Omega$

$s_2 = \sigma + j\left(\Omega + \frac{2\pi}{T}\right)$

$z_1 = e^{s_1 T} = e^{\sigma T} \cdot e^{j\Omega T}$

$z_2 = e^{s_2 T} = e^{\left(\sigma + j\left(\Omega + \frac{2\pi}{T}\right)\right)} = e^{\sigma T} \cdot e^{j\Omega T}$.

→ Let $H_a(s)$ is the system function of an analog filter. Express that in partial fraction as.

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{s - P_k}.$$

Take ILT

$$h_a(t) = \sum_{k=1}^{N} C_k \cdot e^{P_k t}, \quad t \geq 0.$$

→ Sample $h_a(t)$ periodically at $t = nT$,

$$h_a(nT) = \sum_{k=1}^{N} C_k \cdot e^{P_k \cdot nT}$$

if $T = 1$ sec.

$$h(n) = h_a(nT)$$

we KT

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) \cdot z^{-n}$$

$$H(z) = \sum_{n=0}^{\infty} \sum_{k=1}^{N} C_k \cdot e^{P_k nT} \cdot z^{-n}$$

$$= \sum_{k=1}^{N} C_k \cdot \sum_{n=0}^{\infty} \left(e^{P_k T} \cdot z^{-1}\right)^n$$

$$\boxed{H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}}$$

DSP-3-19

∴ To convert $H_a(s)$ to $H(z)$.

If $H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s-p_k}$ then $H(z) = \sum_{k=1}^{N} \frac{c_k}{1-e^{p_k T} \cdot z^{-1}}$.

→ Drawback :- due to aliasing, it is unsuccessful for HP, BRF filters.

Steps to design a digital filter using Impulse Invariance.

1) For the given specifications, find $H_a(s)$

2) Select the sampling rate of digital filter (T).

3) Express $H_a(s)$ as partial fractions.

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s-p_k}.$$

4) Compute $H(z)$ by

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1-e^{p_k T} z^{-1}}.$$

→ for high sampling rate (i.e. for T less value)

$$H(z) = \sum_{k=1}^{N} \frac{T \cdot c_k}{1-e^{p_k T} z^{-1}}.$$

Eg1 For the analog T.F given below, determine digital T.F using impulse invariance method. Assume T=1 sec

$$H(s) = \frac{2}{(s+1)(s+2)}$$

Sol Using Partial fraction, we can write

$$H(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

$$H(s) = \frac{2}{s+1} - \frac{2}{s+2}$$

$$= \frac{2}{s-(-1)} - \frac{2}{s-(-2)}$$

Using impulse invariance techniques, we have

$$H(s) = \sum_{k=1}^{N} \frac{C_k}{s-P_k} \quad \text{then} \quad H(z) = \sum_{k=1}^{N} \frac{C_k}{1-e^{P_k T} z^{-1}}$$

$P_1 = -1$, $P_2 = -2$, $T = 1$ sec, $C_1 = 2$, $C_2 = -2$

$$H(z) = \frac{2}{1-e^{-1} z^{-1}} - \frac{2}{1-e^{-2} z^{-1}}$$

$$= \frac{2}{1-0.3678 z^{-1}} - \frac{2}{1-0.1353 z^{-1}}$$

$$H(z) = \frac{0.465 z^{-1}}{1-0.503 z^{-1} + 0.04976 z^{-2}}$$

DSP-3-20.

(2) Using impulse invariance with $T = 1$ sec. determine $H(z)$ if $H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$.

Ans. $H(z) = \frac{0.4532^{-1}}{1 - 0.7497z^{-1} + 0.2432z^{-2}}$

(3) Design a third order Butterworth digital filter using impulse invariant technique. Assume sampling period $T = 1$ sec

Sol for $N = 3$

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

$$= \frac{A}{s+1} + \frac{B}{s+0.5+j0.866} + \frac{C}{s+0.5-j0.866}$$

$$H(s) = \frac{1}{s-(-1)} + \frac{-0.5+0.288j}{s-(-0.5-j0.866)} + \frac{-0.5-0.288j}{s-(-0.5)+j0.866)}$$

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

$$H(z) = \frac{1}{1-0.368z^{-1}} + \frac{-1+0.66z^{-1}}{1-0.786z^{-1}+0.368z^{-2}}$$

(4) An analog filter has a transfer function $H(s) = \frac{10}{s^2+7s+10}$.
Design a digital filter equivalent to this using Impulse invariant method for $T = 0.2$ sec.

Sol) $H(s) = \frac{10}{s^2+7s+10} = \frac{-3.33}{s-(-5)} + \frac{3.33}{s-(-2)}$

$$H(z) = \sum_{k=1}^{N} \frac{T \cdot c_k}{1-e^{p_k T} z^{-1}}$$

$$= 0.2 \left[\frac{-3.33}{1-e^{-1} z^{-1}} + \frac{3.33}{e^{-0.4} z^{-1}} \right]$$

$$= \frac{0.2012 z^{-1}}{1 - 1.0378 z^{-1} + 0.247 z^{-2}}$$

1, 3, 5, 7, 8, 12, 13, 14, 15, 17,
18, 19, 20, 4, 5, 6, 7, 8, 9,
12-1, 3, 5 - 32, 4, 5, 6, 7, 8, 10, 2, 4, 5,

(DSP-3-2)

(ii) Bilinear Transformation Method:-

→ The bilinear transformation is a conformal mapping that transforms the $j\Omega$ axis into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components.

→ Consider analog T.F is

$$H(s) = \frac{b}{s+a}$$

$$\frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

$$s\,Y(s) + a\,Y(s) = b\,X(s)$$

[3, 15, 18, 34, 7, 40, 4, 6-3, 5]

take I.L.T.

$$\frac{dy(t)}{dt} + a\,y(t) = b\,x(t) = y'(t) + a\,y(t) + b\,x(t) \quad —(1)$$

we can write

$$y(t) = \int_{t_0}^{t} y'(\tau)\,d\tau + y(t_0) \qquad \text{here } y'(t) = \frac{dy(t)}{dt}$$

→ Approximate above using trapezoidal formula

at $t = nT$ { $t_0 = nT - T$.

(at trap.)

$$y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT-T)\right] + y(nT-T). \quad —(2)$$

Sub $t = nT$ in eq-(1)

$$y'(nT) = -a\,y(nT) + b\,x(nT) \quad —(3)$$

Sub eq-(3) in eq-(2)

$$y(nT) = \frac{T}{2}\left[-a\,y(nT) + b\,x(nT) + a\,y(nT-T) + b\,x(nT-T)\right] + y(nT-T)$$

$$y(nT)\left[1 + \frac{aT}{2}\right] - y(nT-T)\left[1 - \frac{aT}{2}\right] = \frac{bT}{2}\left[x(nT) + x(nT-T)\right]$$

Let $y(nT) = y(n)$, $y(nT-1) = y(n-1)$, $x(nT-T) = x(n-1)$, $x(nT) = x(n)$

for $T = 1$.

$$y(n)\left(1 + \frac{aT}{2}\right) - y(n-1)\left(1 - \frac{aT}{2}\right) = \frac{bT}{2}\left[x(n) + x(n-1)\right]$$

take Z.T on B.T

$$\left(1 + \frac{aT}{2}\right)\cdot Y(z) - \left(1 - \frac{aT}{2}\right) z^{-1}\,Y(z) = \frac{bT}{2}\left[X(z) + z^{-1}X(z)\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{bT}{2}\left(1 + z^{-1}\right)}{1 + \frac{aT}{2} - \left(1 - \frac{aT}{2}\right) z^{-1}}$$

$$= \frac{\frac{bT}{2}\left(1 + z^{-1}\right)}{1 - z^{-1} + \frac{aT}{2}\left(1 + z^{-1}\right)}$$

DSP-3.22

$\Rightarrow$

$$H(z) = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a}$$

Compare with $H(s) = \frac{b}{s+a}$

$$H(s) = H(z) \Big|_{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$\therefore \boxed{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$\rightarrow$ This relationship is known as bilinear transformation

$z = re^{j\omega}$. $\quad s = \sigma + j\Omega$

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

$$= \frac{2}{T}\left[\frac{re^{j\omega}-1}{re^{j\omega}+1}\right] = \frac{2}{T}\left[\frac{r(\cos\omega + j\sin\omega) - 1}{r(\cos\omega + j\sin\omega) + 1}\right]$$

$$s = \frac{2}{T}\left[\frac{r^2 - 1 + j\,2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right]$$

$s = \sigma + j\Omega$

Compare real & imaginary parts,

$$\sigma = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega}\right]; \quad \Omega = \frac{2}{T}\left[\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

i) $r=1$,

$\sigma=0$. Poles lies on unit cle in z-plane

ii) $r<1$,

$\sigma<0$, poles lies inside unit cle.

iii) $r>1$

$\sigma>1$, poles lies outside unit cle.

If $r=1$,

$$\sigma=0, \quad \Omega = \frac{2}{T}\cdot\frac{\sin\omega}{1+\cos\omega} = \frac{2}{T}\cdot\tan\frac{\omega}{2}.$$

$$\boxed{\Omega = \frac{2}{T}\tan\frac{\omega}{2} \quad \text{or} \quad \omega = 2\tan^{-1}\frac{\Omega T}{2}}$$

→ This is called wrapping effect.

Wrapping effect:-

$\Omega \rightarrow$ analog freq, $\omega \rightarrow$ digital freq.

9, 14, 15, 16, 25, 7, 30, 3, 4, 8, 40, 4, 5, 1e-4, 5

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

$\Omega = \frac{2}{T}\cdot\frac{\omega}{2}$ for ω small. $\omega = \Omega T$ - linear relation

⇒ But for high frequencies, $\tan\frac{\omega}{2} \neq \frac{\omega}{2}$, the relationship b/w Ω & ω is not linear. So distortion is introduced in freq. scale of digital filter to that of the analog filter. This is known as warping effect.

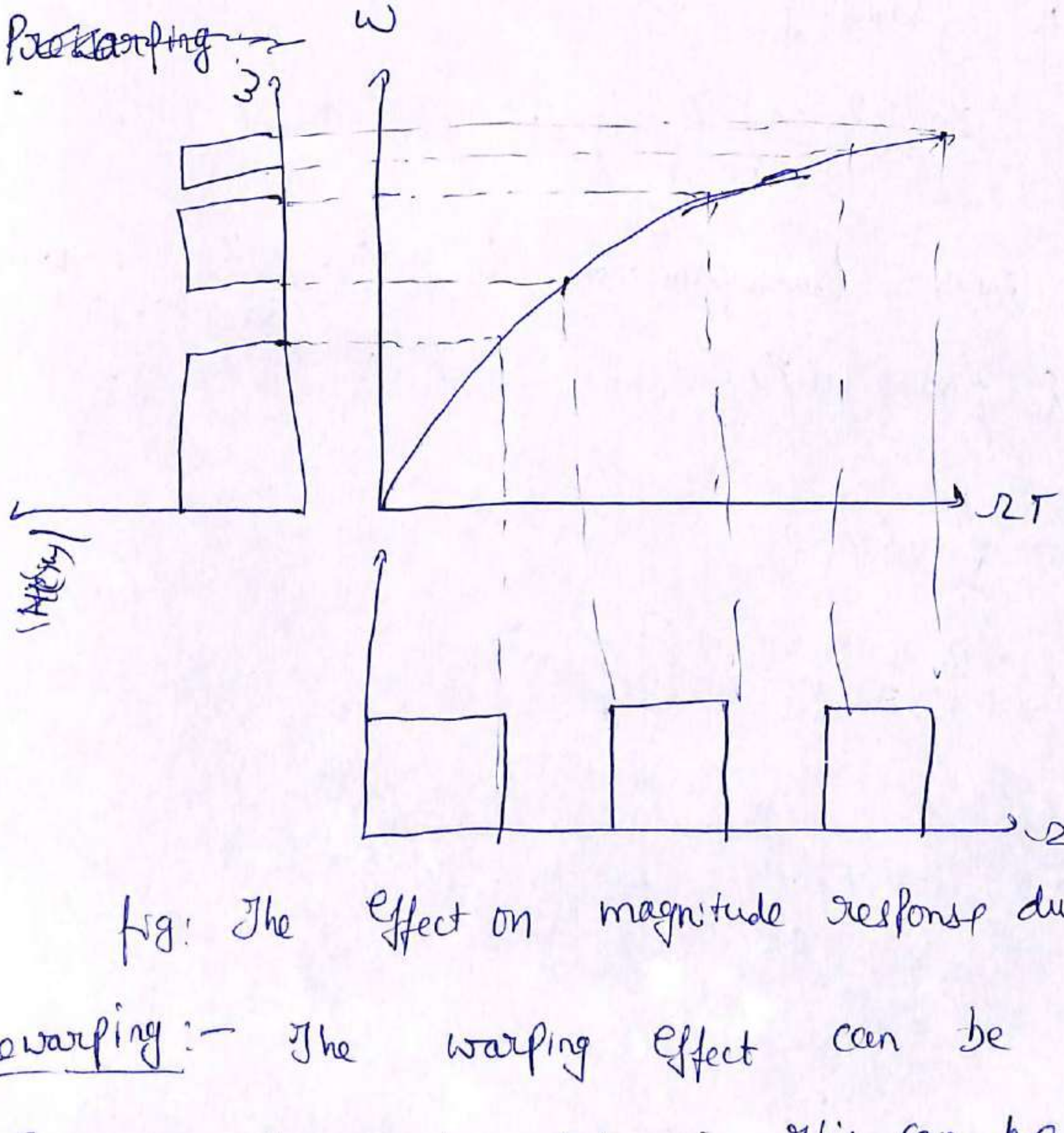


fig: The effect on magnitude response due to warping effect

Prewarping :- The warping effect can be eliminated by prewarping the analog filter. This can be done by finding prewarping analog frequencies.

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}.$$

$$\therefore \Omega_P = \frac{2}{T} \tan\frac{\omega_P}{2}, \quad \Omega_S = \frac{2}{T} \tan \frac{\omega_S}{2}.$$

Steps to design digital filter using bilinear transform technique

(1) From the given specifications, find prewarping analog freq. using $\Omega = \frac{2}{T} \cdot \tan\frac{\omega}{2}$.

(2) Using the analog frequencies, find H(s) of the analog filter.

(3) Select the sampling rate of the digital filter. T sampl/sec

(4) Substitute $s = \frac{2}{T} \cdot \left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ in H(s) to get H(z).

① Apply bilinear transformation to $H(s) = \frac{2}{(s+1)(s+2)}$ with T = 1 sec & find H(z).

Sol

$$H(s) = \frac{2}{(s+1)(s+2)}$$

$$s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] \text{ in } H(s).$$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \frac{2}{\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1\right)\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\right)}$$

$$H(z) = \frac{0.166\,(1+z^{-1})^2}{1-0.33z^{-1}}$$

(iii) Step Invariant method :-

→ In this approach, the sampled filter is designed so its step response is a sampled version of the step response of analog filter.

→ let $H(s)$ is analog T.F.

→ Multiply $H(s)$ by $1/s$ to get the step response of the analog filter.

$$S(s) = G(s) = \frac{H(s)}{s}$$

→ take Inverse L.T of $\frac{H(s)}{s}$.

$$s(t) = g(t) = I.L.T\left[\frac{H(s)}{s}\right] = \mathcal{L}^{-1}\left[\frac{H(s)}{s}\right]$$

→ Next find sampled version of $s(t)$.

Sub $t = nT$.

$$s(nT) = \mathcal{L}^{-1}\left[\frac{H(s)}{s}\right]$$

→ take Z.T of sampled version of $s(t)$

$$S(z) = G(z) = Z.T[s(nT)].$$

→ to get ~~stable~~ T.F $H(z)$, ~~need to~~ divide $S(z)$ by $\frac{z}{z-1}$

~~Hl~~ ∵ $s(n) = h(n) * u(n)$

$$S(z) = H(z).U(z)$$

$$\Rightarrow H(z) = \frac{S(z)}{U(z)} = \frac{S(z)}{z/z-1}$$

$$\therefore H(z) = \frac{z-1}{z} \cdot Z \cdot T\left[\mathcal{L}^{-1}\left[\frac{H(s)}{s}\right]\right]$$

$$H(z) = \frac{z-1}{z} \cdot S(z)$$

Steps to design digital filter using step invariance method:-

(1) find analog T.F $H(s)$ from the given specifications

(2) divide $H(s)$ by S. ie $\rightarrow \frac{H(s)}{S}$

3) Represent $\frac{H(s)}{S}$ in partial fractions.

$$\frac{H(s)}{S} = \sum_{k=1}^{N} \frac{C_k}{S-P_k}.$$

(4) Find $S(z)$ by using

$$S(z) = \sum_{k=1}^{N} \frac{C_k}{1-e^{P_k T} z^{-1}}$$

(5) find $H(z)$ by multiplying $S(z)$ with $\frac{z-1}{z}$.

$$H(z) = \frac{z-1}{z} S(z)$$

Problem:- for the analog T.F $H(s) = \frac{2}{(S+1)(S+2)}$ determine $H(z)$ using step invariant method. Assume $T = 1$ sec.

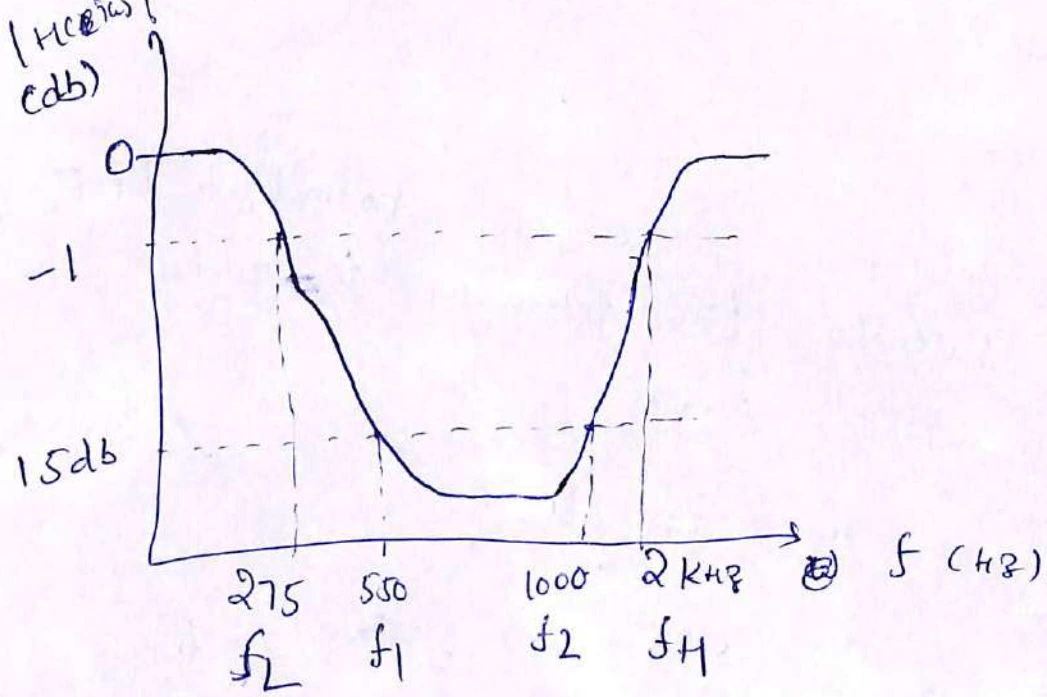
Q, Design a ~~digital~~ ~~IIR~~ ~~bandreject~~ ~~filter~~ following specifications.

Passband d.c to 275 Hz & 2kHz to ∞

Stopband 550 Hz to 1000 Hz.

$\alpha_p = 1\,db$, $\alpha_s = 15\,dB$, $F = 8\,kHz$

Sol)



$$\omega_l = \frac{2\pi f_L}{f_s} = \frac{2\pi \times 275}{8000} = 0.034375\pi \times 2$$

$$\omega_H = \frac{2\pi f_H}{f_s} = \frac{2\pi \times 2 \times 10^3}{8000} = 0.5\pi$$

$$\omega_1 = \frac{2\pi f_1}{f_s} = \frac{2\pi \times 550}{8000} = 0.06875\pi \times 2$$

$$\omega_2 = \frac{2\pi f_2}{f_s} = \frac{2\pi \times 1000}{8000} = \frac{\pi}{4}.$$

4, 5, 7, 9, 11,
13, 16, 17, 18, 19, 25,
27, 9, 32, 3, 4, 6,
9, 40, 1, 3, 4, 5,
50,

Using Bilinear transformation:—

1) Prewarp the analog frequencies

$$\Omega_1 = \frac{2}{T} \cdot \tan\frac{\omega_1}{2} = \frac{2}{1} \cdot \tan 0.034375\pi = 0.1084$$

$\Omega_1 =$

$\Omega_2 =$

$\Omega_4 =$

2. First design a prototype normalized LPF & use suitable transformation to obtain the T.F of band reject filter.

$$\Omega_r = \min\{|A|, |B|\}$$

$$A = \frac{\Omega_1(\Omega_4 - \Omega_1)}{-\Omega_1^2 + \Omega_1\Omega_4} = 3.246$$

$$B = \frac{\Omega_2(\Omega_4 - \Omega_1)}{-\Omega_2^2 + \Omega_1\Omega_4} = -5.847$$

$$\frac{\Omega_s}{\Omega_p} = \Omega_r = 3.246 \quad (\because \Omega_p = 1)$$

3)

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_p} - 1}{10^{0.1\alpha_s} - 1}}}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)} = 1.666$$

$\therefore N = 2$

DSP-3-26

(4) Find $H(S)$.

$$S_K = a\cos\phi_k + bj\sin\phi_k, \quad k = 1,2,\dots,N.$$

$k = 1, 2.$

$$S_1 = a\cos\phi_1 + j\,b\sin\phi_1$$

$$S_2 = a\cos\phi_2 + j\,b\sin\phi_2$$

$\Omega p = 1$ rad/sec for normalized Chebyshev filter

$$a = \Omega p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 0.776$$

$$b = \Omega p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 1.261$$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 4.17$$

$$\varepsilon = \sqrt{10^{0.1\alpha p} - 1} = 0.508$$

$$\phi_K = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$$

$$S_1 = -0.5487 + j\,0.895$$

$$S_2 = -0.5487 - j\,0.895$$

$$H_L(S) = \frac{0.9825}{S^2 + 1.0974S + 1.102}$$

(5)

→ To get T.F of band reject filter.

$$S \rightarrow \frac{S(\Omega_u - \Omega_l)}{S^2 + \Omega_l\Omega_u} = \frac{0.8916S}{S^2 + 6.1084}$$

$$H(s) = \frac{0.88156\,(s^2 + 0.2168 s^2 + 0.01175)}{s^4 + 0.88785 s^3 + 0.9382 s^2 + 0.09618 s + 0.01174}$$

(5) $$H(z) = H(s)\Big|_{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \frac{0.3732\,(1 - 3.2176 z^{-1} + 4.588 z^{-2} - 3.2176 z^{-3} + z^{-4})}{1 - 1.8869 z^{-1} + 1.429 z^{-2} - 0.8077 z^{-3} + 0.3292 z^{-4}}$$

DSP-3-27

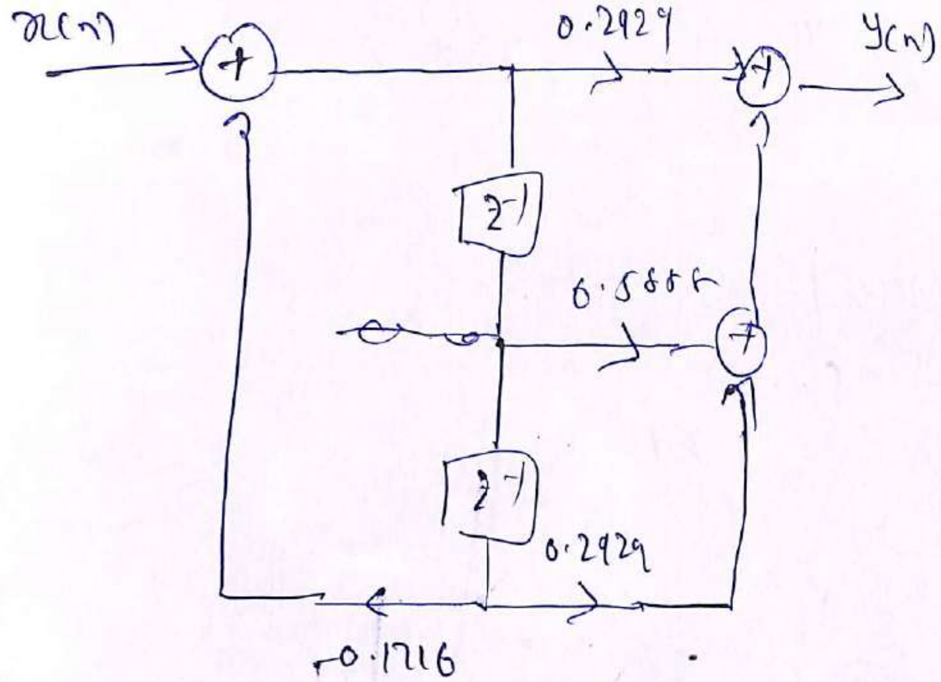


fig: Direct form-II realization

(b) <u>Impulse</u> <u>Invariant Method</u>:-

Here $\omega = \Omega T$.

$T = 1$ sec.

(1)

$\omega = \Omega$.

$\therefore \Omega p = \omega p, \quad \Omega s = \omega s.$

$\alpha = 4.898, \ \varepsilon = 1.$

(2) $\therefore N \geq \dfrac{\log \frac{A}{\varepsilon}}{\log \frac{\Omega s}{\Omega p}} = 3.924$

$\therefore N = 4.$

(3) $H(S) = \dfrac{1}{(S^2 + 0.76537S + 1)(S^2 + 1.8477S + 1)}$

3) $\Omega_c = \frac{\Omega_P}{(\varepsilon)^{1/N}} = \frac{\pi}{2} = 0.5\pi = 1.57.$

4) $H_a(s) = H(s)\Big|_{s \to \frac{s}{1.57}}$

$$\therefore H_a(s) = \frac{1}{\left(\left(\frac{s}{1.57}\right)^2 + 0.76537 \times \frac{s}{1.57} + 1\right)\left(\left(\frac{s}{1.57}\right)^2 + 1.84776 \times \frac{s}{1.57} + 1\right)}$$

$$= \frac{(1.57)^4}{(s^2 + 1.2025 s + 2.465)(s^2 + 2.9025 + 2.465)}$$

5) $H_a(s)$ in partial fractions.

$$H_a(s) = \frac{0.7253 + j1.754}{s - (-1.45 - j0.6)} + \frac{0.7253 - j1.754}{s - (-1.45 + j0.6)}$$

$$+ \frac{-0.7253 - 0.3j}{s - (-0.6 - j1.45)} + \frac{-0.7253 + 0.3j}{s - (-0.6 + j1.45)}$$

6) $H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$

1, 4, 6, 7, 14, 17, 18, 19, 22, 23, 30, 43, 6, 424, 9, 59, (c-1), 5

DSP-3-28

→ Convert the single pole LPF with system function $H(z) = \dfrac{0.5(1+z^{-1})}{1-0.302z^{-2}}$ into BPF with upper & lower cut-off frequencies ω_u & ω_l respectively. The LPF has 3 db B.W. $\omega_p = \frac{\pi}{6}$, $\omega_u = \frac{3\pi}{4}$ & $\omega_l = \frac{\pi}{4}$

Sol)

$$z^{-1} \Rightarrow -\left(\frac{z^{-2}+a_1 z^{-1}+a_2}{a_2 z^{-2}+a_1 z^{-1}+1}\right)$$

$$a_1 = -\frac{2\alpha k}{k+1}, \quad a_2 = \frac{k-1}{k+1}$$

$$\alpha = \frac{\cos((\omega_u+\omega_l)/2)}{\cos((\omega_u-\omega_l)/2)} = 0.$$

$$K = \cot\left(\frac{\omega_u-\omega_l}{2}\right). \tan\frac{\omega_p}{2} = 0.268$$

$$z^{-1} \Rightarrow -\left(\frac{z^{-2}-0.577}{-0.577z^{-2}+1}\right)$$

$$H(z) = \frac{0.5\left(1+\left(\frac{-z^{-2}+0.577}{-0.577z^{-2}+1}\right)\right)}{1-0.302\left(\frac{-z^{-2}+0.577}{1-0.577z^{-2}}\right)^2}$$

$$H(z) = \frac{0.955(1-z^{-2})}{1-0.333z^{-2}}$$

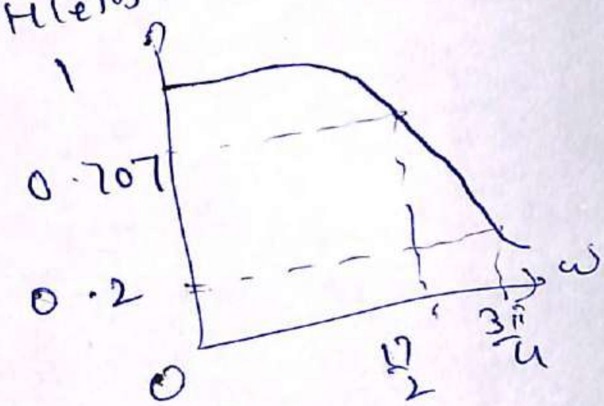
Design a digital Butterworth filter [illegible]

$H(z)$ satisfying the constraints.

$$0.707 \leq |H(e^{j\omega})| \leq 1 \quad \text{for} \quad 0 \leq \omega \leq \frac{\pi}{2}$$

$$|H(e^{j\omega})| \leq 0.2 \quad \text{for} \quad \frac{3\pi}{4} \leq \omega \leq \pi$$

with $T = 1$ sec using (a) the bilinear transformation (b) Impulse invariance. Realize the filter in each case using the most convenient realization form.



Sol (a) Bilinear transformation :-

(1) Given $\dfrac{1}{\sqrt{1+\varepsilon^2}} = 0.707$

$\Rightarrow \varepsilon = 1$

$\dfrac{1}{\sqrt{1+\lambda^2}} = 0.2 \Rightarrow \lambda = 4.898$

$\omega_p = \dfrac{\pi}{2}, \quad \omega_s = \dfrac{3\pi}{4}$

(2) Since using bilinear transformation, we have to prewarping analog freq

$$\frac{\Omega_s}{\Omega_p} = \frac{\frac{2}{T}\tan\frac{\omega_s}{2}}{\frac{2}{T}\tan\frac{\omega_p}{2}} = 2.414$$

(3)

$$N \geq \frac{\log \lambda/\varepsilon}{\log \frac{\Omega_s}{\Omega_p}} = 1.803$$

$\boxed{N = 2}$

(3) $\Omega_c^* = \frac{\Omega_p}{(\varepsilon)^{1/N}} = \Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = 2$ rad/sec

(4) $\because N = 2$,

Normalized T.F $H(s) = \frac{1}{s^2+\sqrt{2}s+1}$

$\Omega_c = 1$ rad/sec.

(5) To find $H(s)$ for $\Omega_c^* = 2$ rad/sec,

$$s \rightarrow \frac{s}{\Omega_c^*} = \frac{s}{2}$$

$$H_a(s) = \frac{1}{\left(\frac{s}{2}\right)^2 + \sqrt{2}\cdot\frac{s}{2} + 1} = \frac{4}{s^2 + 2.828s + 4}$$

(6)

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

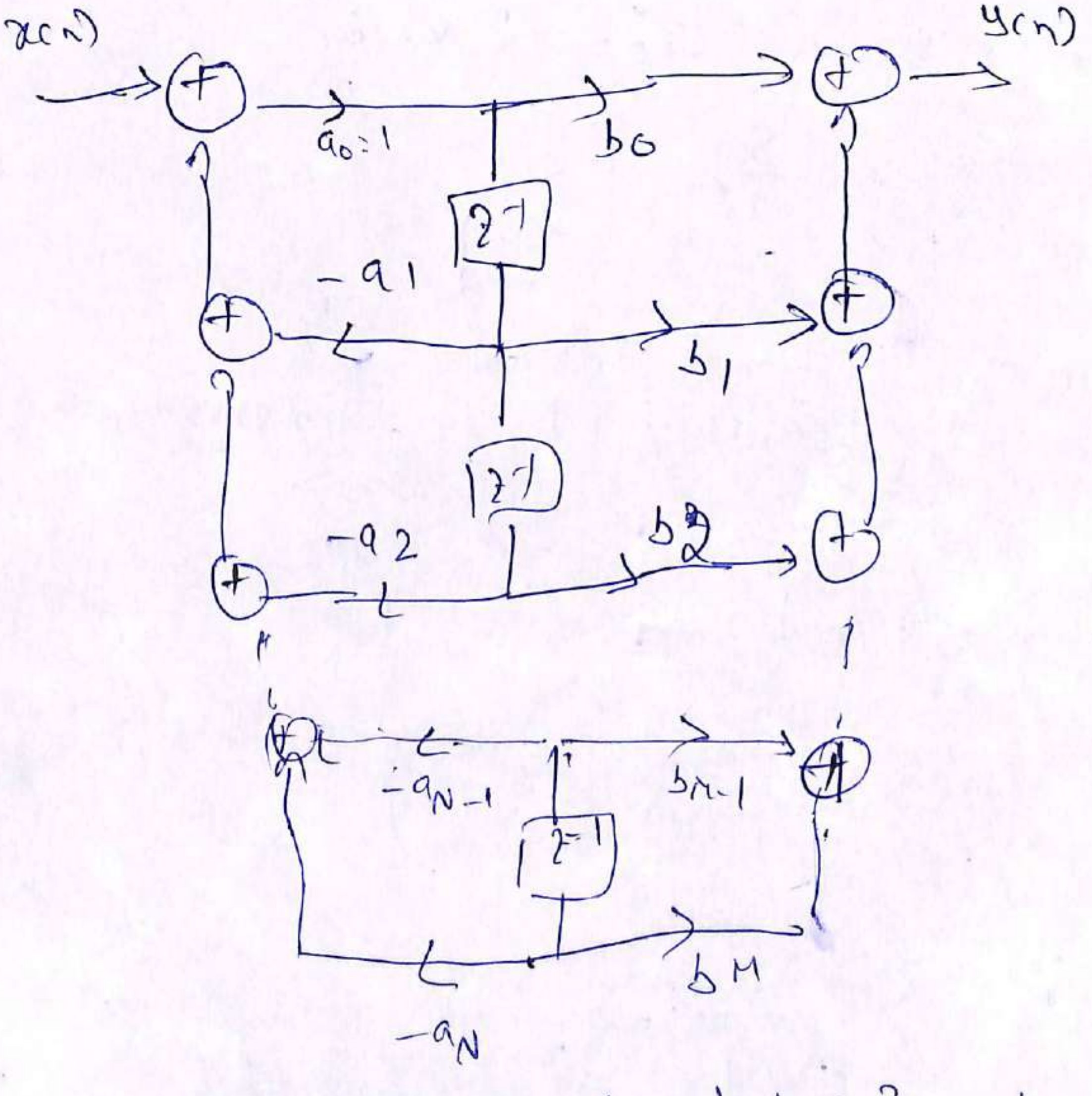
$$H(z) = \frac{4}{\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)^2 + 2.828\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) + 4}$$

$$H(z) = \frac{0.2929\,(1+z^{-1})^2}{1+0.1716\,z^{-2}}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.2929 + 0.5858 z^{-1} + 0.2929 z^{-2}}{1 + 0.1716 z^{-2}}$$

Realize using direct - form - II

$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}.$$



$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

Here, ∴ $b_0 = 0.2929$, $b_1 = 0.5858$, $b_2 = 0.2929$

$a_0 = 1$. $a_1 = 0$, $a_2 = 0.1716$

Spectral transformations :-

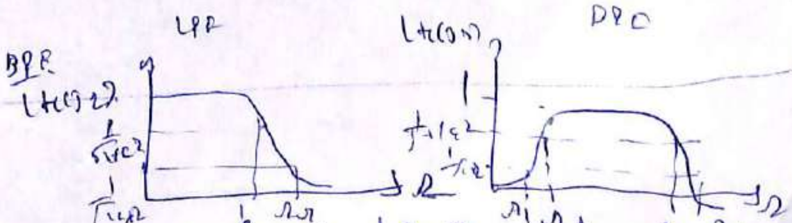
(i) Frequency or Spectral transformations in Analog domain:-

-> From a normalized LP analog filter, we can design filters with diff Ω_c.

(i) ~~Low pass to Low pass.~~

Analog freq. transformation from normalized analog LPF.

type	transformation
Low Pass	$s \to \frac{s}{\Omega_c^*}$ or $\frac{\Omega_c \cdot \Omega_c^*}{s}$ here $\Omega_c = 1$ rad/s $\Omega_c^* = \Omega_c$.
High Pass	$s \to \frac{\Omega_c^*}{s}$ or $\frac{\Omega_c \cdot \Omega_c^*}{s}$
BPF	$s \to \frac{(s^2 + \Omega_l \Omega_u)}{s(\Omega_u - \Omega_l)}$ or $\frac{\Omega_c \cdot (s^2 + \Omega_l \Omega_u)}{s(\Omega_u - \Omega_l)}$
BRF	$s \to \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$ or $\frac{\Omega_c s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$



For BPF, $\Omega_a = \frac{\Omega_s}{\Omega_p} = \min\{|A|, |B|\}$

$\Omega_0 \to$ centre freq.

$\Omega_0 = \sqrt{\Omega_l \Omega_u}$

$Q = \frac{\Omega_0}{\Omega_u - \Omega_l}$

$A = \frac{\Omega_1^2 + \Omega_l \Omega_u}{\Omega_1 (\Omega_u - \Omega_l)}$, $B = \frac{\Omega_2^2 - \Omega_l \Omega_u}{\Omega_2 (\Omega_u - \Omega_l)}$

Spectral transformations :-

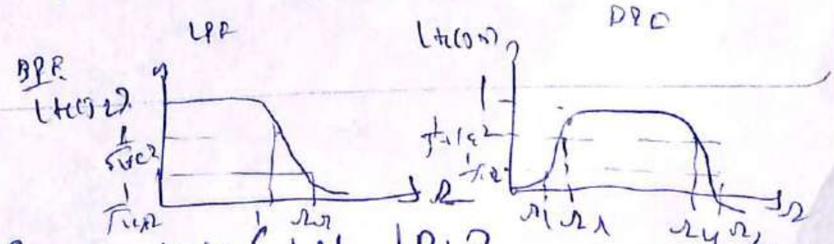
(i) Frequency or Spectral transformations in Analog domain:-

-> From a normalized LP analog filter, we can design filters with diff Ω_c.

~~(i) Low pass to Low Pass.~~

Analog freq. transformation from normalized analog LPF.

type	transformation
Low Pass	$s \to \frac{s}{\Omega_c^*}$ or $\frac{\Omega_c \cdot \Omega_c^*}{s}$ we $\Omega_c = 1$ rad/s ($\Omega_c^* = \Omega_c$.)
High Pass	$s \to \frac{\Omega_c^*}{s}$ or $\frac{\Omega_c \cdot \Omega_c^*}{s}$
BPF	$s \to \frac{(s^2 + \Omega_l \Omega_u)}{s(\Omega_u - \Omega_l)}$ or $\frac{\Omega_c \cdot (s^2 + \Omega_l \Omega_u)}{s(\Omega_u - \Omega_l)}$
BRF	$s \to \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$ or $\frac{\Omega_c s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$



For BPF, $\Omega_a = \frac{\Omega_s}{\Omega_p} = \min\{|A|, |B|\}$

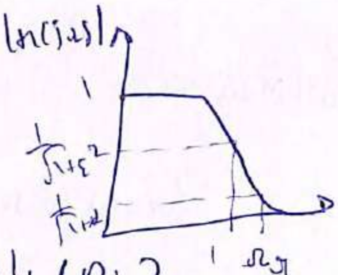
$\Omega_0 \to$ centre freq.

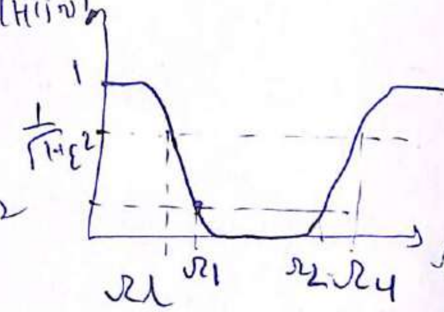
$\Omega_0 = \sqrt{\Omega_l \Omega_u}$

$Q = \frac{\Omega_0}{\Omega_u - \Omega_l}$

$$A = \frac{\Omega_1^2 + \Omega_l \Omega_u}{\Omega_1(\Omega_u - \Omega_l)}, \quad B = \frac{\Omega_2^2 - \Omega_l \Omega_u}{\Omega_2(\Omega_u - \Omega_l)}$$

For BSF,





$$\Omega_r = \frac{\Omega_s}{\Omega_p} = \min\{|A|, |B|\}$$

$$, A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_l\Omega_u}, \quad B = \frac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_l\Omega_u}.$$

(ii) Freq. transformation in digital domain:-

Type	Transformation	Design Parameter
Low-Pass	$z^{-1} \to \frac{z^{-1} - a}{1 - az^{-1}}$	$a = \frac{\sin[(\omega_p - \omega_p^*)/2]}{\sin[(\omega_p + \omega_p^*)/2]}$
High Pass	$z^{-1} \to -\left(\frac{z^{-1} + a}{1 + az^{-1}}\right)$	$a = -\frac{\cos[(\omega_p' + \omega_p^*)/2]}{\cos[(\omega_p' - \omega_p^*)/2]}$
Band pass	$z^{-1} \to -\left(\frac{z^{-2} + a_1 z^{-1} + a_2}{a_2 z^{-2} + a_1 z^{-1} + 1}\right)$ $\boxed{z^{-1} \to -\frac{z^{-2} - \frac{2\alpha k}{1+k} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}}$	$a_1 = -\frac{2\alpha k}{k+1}$ $a_2 = \frac{(k-1)}{k+1}$ $\alpha = \frac{\cos[(\omega_1 + \omega_2)/2]}{\cos[(\omega_2 - \omega_1)/2]}$ $k = \cot\left(\frac{\omega_2 - \omega_1}{2}\right) \cdot \tan\frac{\omega_c}{2}$
Band Stop	$z^{-1} \to \frac{z^{-2} + a_1 z^{-1} + a_2}{a_2 z^{-2} + a_1 z^{-1} + 1}$	$a_1 = -2\alpha / k+1$ $a_2 = 1-k / 1+k$ $\alpha = \frac{\cos[(\omega_2 + \omega_1)/2]}{\cos(\omega_2 - \omega_1)/2}$ $k = \tan\left(\frac{\omega_2 - \omega_1}{2}\right) \tan\left(\frac{\omega_p}{2}\right)$

UNIT - IV

FIR DIGITAL FILTERS

characteristics of FIR digital filter, freq. response, Design of FIR digital filters, fourier method, digital filters using window technique, freq. Sampling technique, Comparision of IIR & FIR filters, Illustrative problems.

-x-

→ Differences b/w FIR & IIR filters :-

IIR filter	FIR filter
1) An IIR filter is a digital filter designed by considering all infinite samples of impulse response.	1) An FIR filter is a digital filter designed by considering only a finite no of samples of impulse response.
2) The impulse res, digital filter can't be directly designed. First analog filter is to be designed & then it has to be transformed to a digital filter.	2) The digital filter can be directly designed.
3) IIR filters are easily realized recursively.	3) FIR can be realized recursively & non-recursively.
4) IIR filters do not have linear phase	4) FIR filters can be easily designed to have perfectly linear phase.

5. The Specifications include the desired characteristics for magnitude response only.	5. The Specifications include the desired characteristics for both magnitude & phase response.
6. Not always stable	6. Always stable.
7. The errors due to round off noise in FIR filters are more	7. Errors due to round off noise are less severe.
8. Less flexibility to control the shape of their magnitude response	8. More flexibility.
9) In these filters, the poles are placed anywhere	9. In these filters, the poles are fixed at the origin for non-recursive structures
10) High selectivity can be acheived with low-order transfer fns.	10). High selectivity can be acheived using a relatively high order for the T.F.
11) The Implementation of IIR filter involves fewer Parameters, less memory requirements & lower Computational Complexity	12) Memory requirement and execution time are very high.

Characteristics of FIR digital filters :-

The T.F of a FIR causal filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n) \, z^{-n}$$

$h(n) \rightarrow$ Impulse response of the filter.

→ The freq. response (F.T of $h(n)$) is,

$$H(\omega) = H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) \cdot e^{-j\omega n}.$$

→ $H(\omega)$ is Periodic with Period 2π.

i.e $H(\omega) = H(\omega + 2k\pi)$, $k = 0, 1, 2, \ldots$

→ Since $H(\omega)$ is a complex, it can be expressed as,

$$H(\omega) = \pm |H(\omega)| \cdot e^{j\theta(\omega)}$$

$|H(\omega)| \rightarrow$ mag. response

$\theta(\omega) \rightarrow$ phase response.

→ (ii) Phase delay & group delay is defined as:

$$\tau_p = -\frac{\theta(\omega)}{\omega} \quad \& \quad \tau_g = -\frac{d\theta(\omega)}{d\omega}$$

i) FIR filter with Constant phase & group delay :-

→ For FIR filters with linear phase, we can define,

$\theta(\omega) = -\alpha\omega, \qquad -\pi \le \omega \le \pi.$

$\alpha \rightarrow$ Const. Phase delay in samples.

$$\tau_g = -\frac{\theta(\omega)}{\omega} = \frac{\alpha\omega}{\omega} = \alpha,$$

$$\tau_p = -\frac{d\,\theta(\omega)}{d\omega} = \alpha.$$

$\therefore \tau_g = \tau_g = \alpha$. & α is independent of freq

$$\sum_{n=0}^{N-1} h(n).\, e^{-j\omega n} = \pm H(\omega).\, e^{j\theta(\omega)}$$

$$\sum_{n=0}^{N-1} h(n).\, [\cos\omega n - j\sin\omega n] = \pm H(\omega)[\cos\theta(\omega) + j\sin\theta(\omega)]$$

$$\Rightarrow \sum_{n=0}^{N-1} h(n).\cos\omega n - j\sum_{n=0}^{N-1} h(n).\sin\omega n = \pm H(\omega).\cos\theta(\omega) \pm j\, H(\omega).\sin\theta(\omega)$$

Compare real & imag. parts.

$$\sum_{n=0}^{N-1} h(n).\cos\omega n = \pm H(\omega).\cos\theta(\omega)$$

$$-\sum_{n=0}^{N-1} h(n).\sin\omega n = \pm H(\omega).\sin\theta(\omega)$$

$\theta(\omega) = -\alpha\omega$

$$\frac{\sum_{n=0}^{N-1} h(n).\cos\omega n}{-\sum_{n=0}^{N-1} h(n)\sin\omega n} = \frac{\cos\theta(\omega)}{\sin\theta(\omega)} = \frac{\cos\alpha\omega}{-\sin\alpha\omega}$$

$$\sum_{n=0}^{\infty} h(n).\sin\omega n.\cos\alpha\omega = -\sum_{n=0}^{\infty} h(n).\cos\omega n.\sin\alpha\omega$$

$$\Rightarrow \sum_{n=0}^{\infty} h(n)\left[\sin\omega n.\cos\alpha\omega - \cos\omega n.\sin\alpha\omega\right] = 0$$

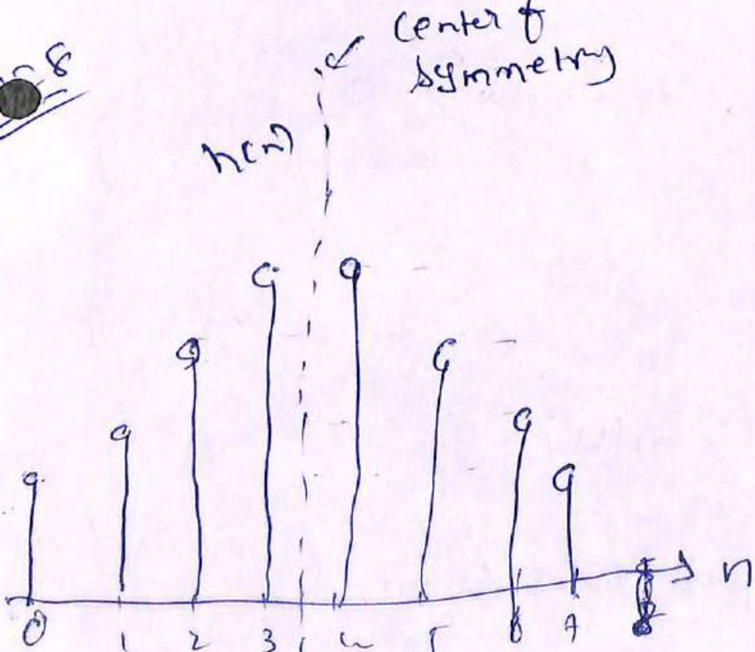
$$\Rightarrow \sum_{n=0}^{N-1} h(n).\sin(\alpha - n)\omega = 0$$

The above eqn. is true, when

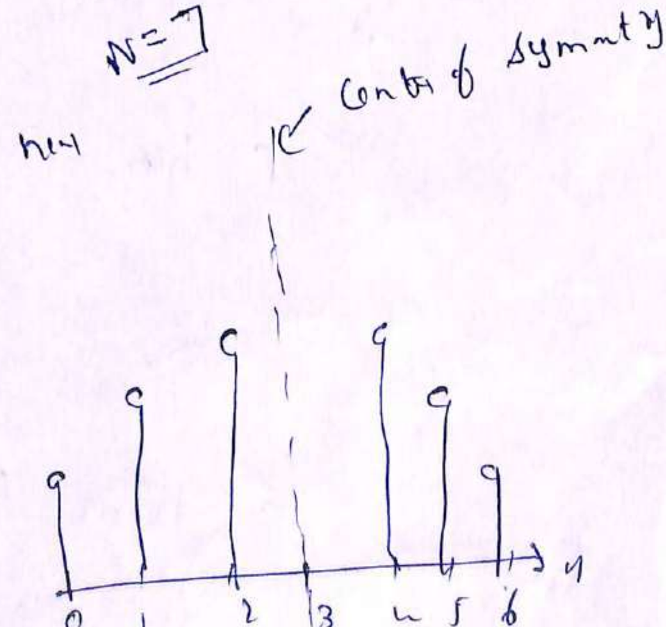
$$h(n) = h(N-1-n) \text{ \& } \alpha = \frac{N-1}{2}, \text{ for } 0 \le n \le N-1.$$

→ * This shows that FIR filters will have constant phase & group delay when the impulse response is symmetrical about $\alpha = \frac{N-1}{2}$.

N=8



N=7



N = even

$\alpha = \frac{8-1}{2} = 3\frac{1}{2}$

$\alpha = \frac{7-1}{2} = \frac{6}{2} = 3$

5, 19, 21, 4, 7, 3&4, 40, 41 le-5

(ii) FIR filters with only const. group delay but not Phase delay:-

$$\theta(\omega) = \beta - \alpha\omega$$

$$\tau_p = \frac{-\theta(\omega)}{\omega} = -\frac{\beta}{\omega} + \alpha,$$

$$\tau_g = \frac{-d\theta(\omega)}{d\omega} = -\frac{d}{d\omega}(\beta - \alpha\omega) = \alpha.$$

then

$$\sum_{n=0}^{N-1} h(n)\cdot e^{-j\omega n} = \pm H(\omega)\cdot e^{j(\beta - \alpha\omega)}$$

$$\because \boxed{\theta(\omega) = \beta - \alpha\omega}$$

$$\Rightarrow \sum_{n=0}^{N-1} h(n)\left[\cos\omega n - j\sin\omega n\right] = \pm H(\omega)\left[\cos(\beta - \alpha\omega) + j\sin(\beta - \alpha\omega)\right].$$

After simplifying,

$$\sum_{n=0}^{N-1} h(n)\cdot \sin(\beta - (\alpha - n)\omega) = 0$$

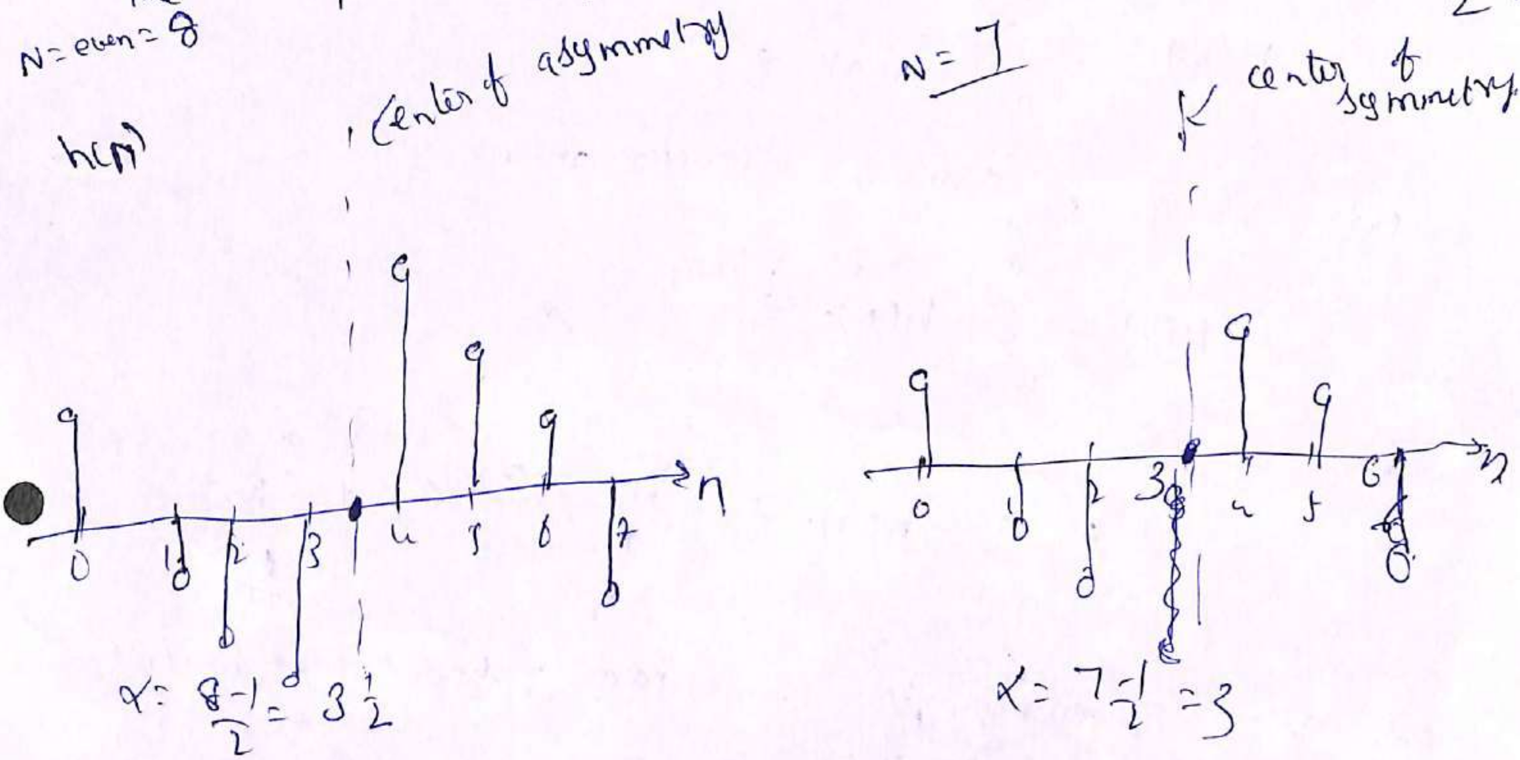
if $\beta = \frac{\pi}{2}$,

$$\sum_{n=0}^{N-1} h(n)\cdot \cos(\alpha - n)\omega = 0$$

The above is satisfied only when

$$\boxed{h(n) = -h(N-1-n), \quad \alpha = \frac{N-1}{2}.}$$

→ This shows that FIR filters have constant group delay but not phase delay when the impulse response is asymmetric about $\alpha = \frac{N-1}{2}$.



III. <u>Frequency Response</u> of <u>linear phase FIR filters</u> :-

→ The freq. response of the filter is the F.T. of its impulse response.

→ Depending on the value of N, & the type of symmetry of the filter impulse sequence, there are four types of impulse response for linear phase FIR filters.

1) Symmetrical impulse response when N is odd
2) " " " N is even
3) Asymmetric " " N is odd
4) " " " N is even

i) Freq. Response of linear phase FIR filter when impulse response is symmetrical & N is odd :-

$$H(\omega) = \sum_{n=0}^{N-1} h(n).\, e^{-j\omega n}$$

let $N = 7$, center of symmetry at $n = \frac{N-1}{2}$.

$$H(\omega) = \sum_{n=0}^{6} h(n).\, e^{-j\omega n}$$

$$= \sum_{n=0}^{2} h(n).\, e^{-j\omega n} + h(3)\, e^{-j\omega 3} + \sum_{n=4}^{6} h(n).\, e^{-j\omega n}$$

→ In general, if N is odd ~~& h(n) is symmetric~~, we can write it as

$$H(\omega) = \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right). e^{-j\omega\left(\frac{N-1}{2}\right)} +$$

$$\sum_{n=\frac{N+1}{2}}^{N-1} h(n).\, e^{-j\omega n}.$$

let $m = N-1-n$, ⇒ $n = N-1-m$ in 3rd one.

For $n = \frac{N+1}{2}$, $m \rightarrow (N-1) - \left(\frac{N+1}{2}\right) = \frac{N-3}{2}$

$n = N-1$, $m \rightarrow N-1-(N-1) = 0$.

$$H(\omega) = \sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot e^{-j\omega n} + h\left(\frac{N-1}{2}\right) \cdot e^{-j\omega\left(\frac{N-1}{2}\right)} +$$

$$\sum_{m=0}^{\frac{N-3}{2}} h(N-1-m) \cdot e^{-j\omega\left(N-1-m\right)}$$

replace m by n

$$H(\omega) = \sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot e^{-j\omega n} + h\left(\frac{N-1}{2}\right) \cdot e^{-j\omega\left(\frac{N-1}{2}\right)} +$$

$$\sum_{n=0}^{\frac{N-3}{2}} h(N-1-n) \cdot e^{-j\omega(N-1-n)}$$

→ for a symmetrical impulse response,

$$h(n) = h(N-1-n)$$

$$H(\omega) = \sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot e^{-j\omega n} + h\left(\frac{N-1}{2}\right) \cdot e^{-j\omega\left(\frac{N-1}{2}\right)} +$$

$$\sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot e^{-j\omega(N-1-n)}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot \left\{ e^{j\omega\left(\frac{N-1}{2}-n\right)} \right.\right.$$

$$+ \sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot e^{-j\omega\left(\frac{N-1}{2}-n\right)}\Bigg]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right]\right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} 2h(n) \cdot \cos\left(\left(\frac{N-1}{2}-n\right)\omega\right)\right]$$

Let $k = \frac{N-1}{2} - n \Rightarrow n = \frac{N-1}{2} - k$

for $n = 0$, $k = \frac{N-1}{2}$

$n = \frac{N-3}{2}$, $k = 1$

$$\therefore H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + \sum_{k=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-k\right) \cdot \cos k\omega\right]$$

replace k by n

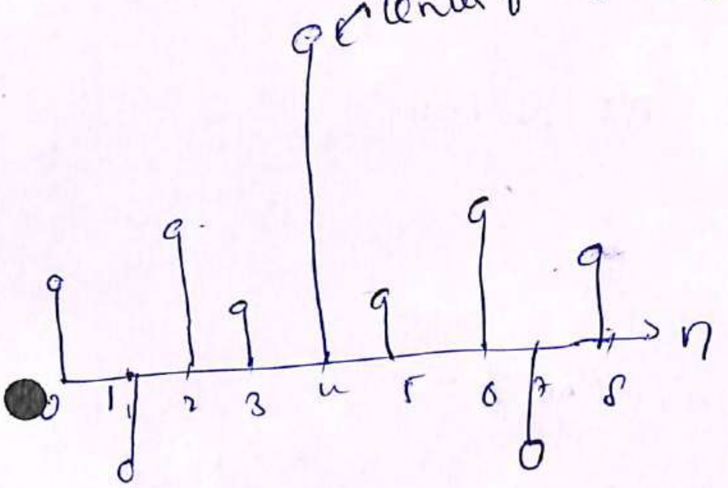
$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right) \cdot \cos n\omega\right]$$

$$|H(\omega)| = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right) \cdot \cos n\omega$$

$$\angle H(\omega) = -\omega\left(\frac{N-1}{2}\right) = -\omega\alpha, \quad \alpha = \frac{N-1}{2}.$$

→ ∴ The magnitude function of $H(\omega)$, i.e. $|H(\omega)|$ is Symmetric with $\omega = \pi$.

Eg:- $N=9$



$\alpha = \frac{N-1}{2} = \frac{9-1}{2} = 4.$

fig: Symmetric impulse response for $N=9$

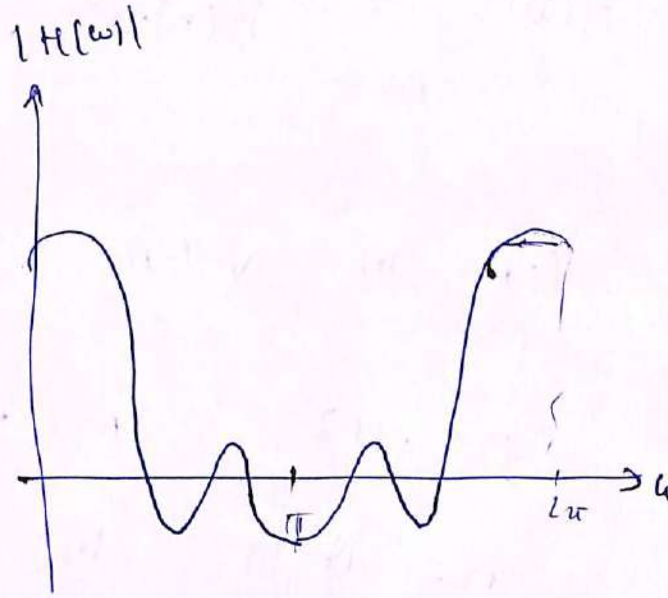


fig: its mag. response.

(ii) Frequency Response of Linear Phase FIR filter when impulse response is Symmetrical & N is even –

$$H(\omega) = \sum_{n=0}^{N-1} h(n) \cdot e^{-j\omega n}.$$

If $N=6$

$$H(\omega) = \sum_{n=0}^{5} h(n) \cdot e^{-j\omega n} = \sum_{n=0}^{\frac{N}{2}-1=2} h(n) \cdot e^{-j\omega n} + \sum_{n=N/2=3}^{N-1=5} h(n) \cdot e^{-j\omega n}$$

$\therefore$ In general,

$$H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n) \cdot e^{-j\omega n} + \sum_{n=N/2}^{N-1} h(n) \cdot e^{-j\omega n}$$

Let $m = N-1-n$, $\Rightarrow n = N-1-m$

$n = N/2$, $m = \frac{N}{2} - 1$

$n = N-1$, $m = (N-1)-(N-1) = 0$.

$$\therefore H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n) \cdot e^{-j\omega n} + \sum_{m=0}^{\frac{N}{2}-1} h(N-1-m) \cdot e^{-j\omega(N-1-m)}$$

replace n by m.

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n) \cdot e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(N-1-n) \cdot e^{-j\omega(N-1-m)}$$

For symmetric impulse resp, $h(n) = h(N-1-n)$.

$$H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} h(n) \cdot e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(n) \cdot e^{-j\omega(N-1-n)}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n) \left[e^{-j\omega n} + e^{-j\omega(N-1-n)} \right]$$

(4-7)

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=0}^{\frac{N}{2}-1} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right] \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot \sum_{n=0}^{\frac{N}{2}-1} 2h(n) \cdot \cos\left(\omega\left(\frac{N-1}{2}-n\right)\right).$$

let $\frac{N}{2} - n = k$. $\Rightarrow$ $n = \frac{N}{2} - k$

$n = 0$, $k = \frac{N}{2}$.

$n = \frac{N}{2} - 1$, $k = \frac{N}{2} - \left(\frac{N}{2} - 1\right) = 1$.

$$\therefore H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot \sum_{k=1}^{\frac{N}{2}} 2h\left(\frac{N}{2}-k\right) \cdot \cos\omega\left(k-\frac{1}{2}\right)$$

replace k by n

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)} \cdot \sum_{n=1}^{N/2} 2h\left(\frac{N}{2}-n\right) \cdot \cos\omega\left(n-\frac{1}{2}\right)$$

$$|H(\omega)| = \sum_{n=1}^{N/2} 2h\left(\frac{N}{2}-n\right) \cdot \cos\omega\left(n-\frac{1}{2}\right)$$

$$\angle H(\omega) = -\omega\left(\frac{N-1}{2}\right) = -\omega \cdot \alpha, \quad \alpha = \frac{N-1}{2}.$$

∴ The magnitude function of $H(\omega)$ is Asymmetric with $\omega = \pi$ when impulse response is symmetric & N is even no.

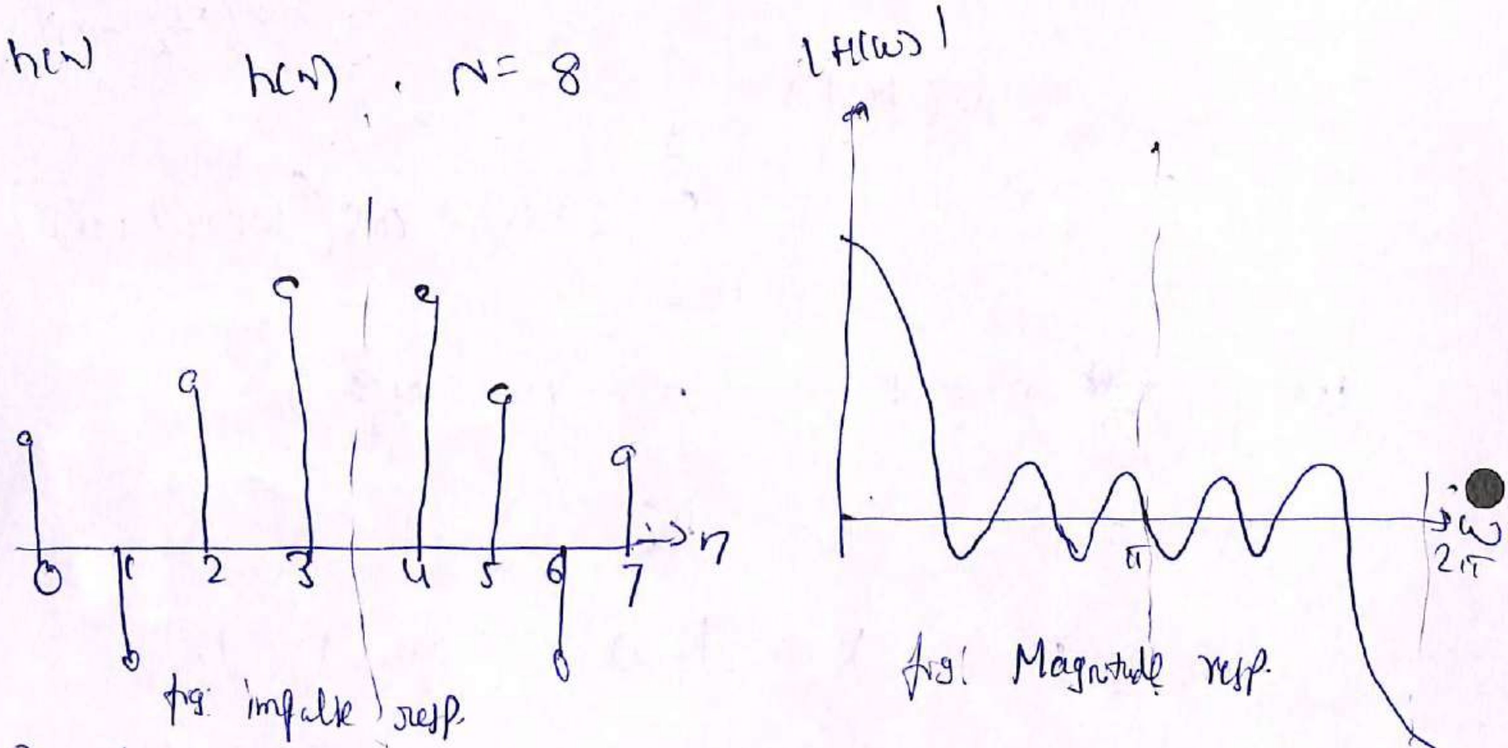


fig: impulse resp.

fig: Magnitude resp.

3) Freq Response of Linear Phase FIR filter when Impulse response is Asymmetric & N is odd:-

$$H(\omega) = \sum_{n=0}^{\frac{N-1}{2}} h(n) \cdot e^{-j\omega n}$$

The Impulse response is antisymmetric with center of antisymmetry at $n = \frac{N-1}{2}$ & also $h(\frac{N-1}{2}) = 0$.

$$H(\omega) = \sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot e^{-j\omega n} + h(\tfrac{N-1}{2}) \cdot e^{-j\omega(\frac{N-1}{2})} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) \cdot e^{-j\omega n}.$$

$\because h(\frac{N-1}{2}) = 0$

let $m = N-1-n$, $\Rightarrow n = N-1-m$ $\boxed{4-8}$

for $n = \frac{N+1}{2}$, $m = \frac{N-3}{2}$

$n = N-1$, $m = 0$

$$\therefore H(\omega) = \sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot e^{-j\omega n} + \sum_{m=0}^{\frac{N-3}{2}} h(N-1-m) \cdot e^{-j\omega(N-1-m)}$$

replace m by n

$$H(\omega) = \sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot e^{-j\omega n} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n) \cdot e^{-j\omega(N-1-n)}.$$

For asymmetric, $h(N-1-n) = -h(n)$.

$$\therefore H(\omega) = \sum_{n=0}^{\frac{N-3}{2}} h(n) \cdot e^{-j\omega n} + \sum_{n=0}^{\frac{N-3}{2}} -h(n) \cdot e^{-j\omega(N-1-n)}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=0}^{\frac{N-3}{2}} h(n) \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} - e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right] \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=0}^{\frac{N-3}{2}} 2h(n) j \cdot \sin\left(\omega\left(\frac{N-1}{2}-n\right)\right) \right]$$

$\because j = e^{j\pi/2}$

$$H(\omega) = e^{j\left(\frac{\pi}{2} - \omega\frac{(N-1)}{2}\right)} \left[\sum_{n=0}^{\frac{N-3}{2}} 2h(n) \sin\left(\omega\left(\frac{N-1}{2} - n\right)\right) \right]$$

5, 13, 19, 27, ~~X~~, 5, ~~X~~, ~~X~~

4, 6, 6-4, ~~X~~, 5.

let $k = \frac{N-1}{2} - n$. $\qquad n = \frac{N-1}{2} - k$ $\qquad$ 4 -

for $n=0$, $\quad k = \frac{N-1}{2}$

$n = \frac{N-3}{2}$, $\quad k = 1$

$$\therefore H(\omega) = e^{j\left(\frac{\pi}{2} - \omega\left(\frac{N-1}{2}\right)\right)} \cdot \left[\sum_{k=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - k\right) \cdot \sin\omega k\right]$$

Replace k by n we get

$$H(\omega) = e^{j\left(\frac{\pi}{2} - \omega\left(\frac{N-1}{2} - 1\right)\right)} \left[\sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right) \sin\omega n\right]$$

Mag. Response.

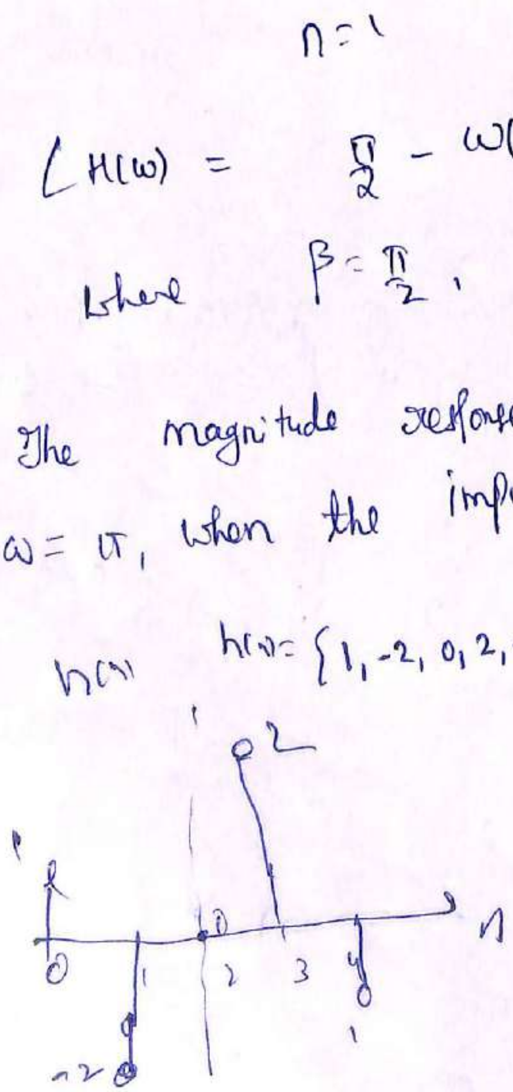
$$|H(\omega)| = \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right) \cdot \sin\omega n,$$

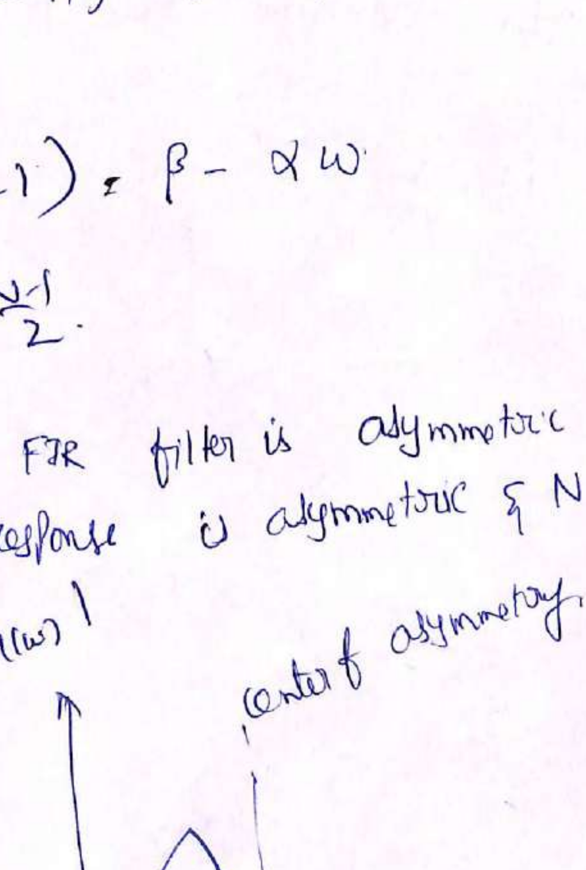
$$\angle H(\omega) = \frac{\pi}{2} - \omega\left(\frac{N-1}{2} - 1\right) = \beta - \alpha\omega$$

where $\beta = \frac{\pi}{2}$, $\alpha = \frac{N-1}{2}$.

* $\therefore$ The magnitude response of FIR filter is asymmetric with $\omega = \pi$, when the impulse response is asymmetric & N odd.

$h(n)$ $\quad h(n) = \{1, -2, 0, 2, -1\}$ $\qquad |H(\omega)|$ $\qquad$ center of asymmetry.





22, 5, 34, 40, 1,
3, 4, 7, 9, let,
4, 5

(4) Freq. Response of Linear phase FIR filter when Impulse is Asymmetric & N is even:-

Design of FIR filters using windows:-

→ In window method:

1) first choose the desired frequency response specification $H_d(\omega)$

2) Determine the corresponding unit sample (a impulse) response $h_d(n)$ by take Inverse Fourier transform of $H_d(\omega)$.

3) The unit sample response $h_d(n)$ will be an infinite sequence. but to get finite sequence, we have to truncate $h_d(n)$ to yield an FIR filter of length 'N'.

4) The truncation is acheived by multiplying $h_d(n)$ by a window sequence $\omega(n)$. then the resultant sequence will be of length 'N'. & can be denoted by $h(n)$.

5) The Z.T of $h(n)$ will give the filter T.F $H(z)$.

→ However, the window chosen for truncating the infinite impulse response should have the some desirable characteristics. They are:

1) The central lobe of the frequency response of the window should contain most of the energy & should be narrow.

2) The highest side lobe level of the freq. response should be small.

3, The side lobes of the freq. response should decrease in energy as ω tends to π.

(1) <u>Rectangular window</u> :-

The rectangular window sequence is given by:

$$w_R(n) = \begin{cases} 1 & \text{for } -\left(\frac{N-1}{2}\right) \le n \le \frac{N-1}{2} \\ 0 & \text{o.w} \end{cases}$$

or

$$w_R(n) = \begin{cases} 1 & \text{for } 0 \le n \le N-1 \\ 0 & \text{o.w.} \end{cases}$$

The spectrum (or freq. response) of rectangular window is given by the Fourier transform of $w_R(n)$.

$$F.T[x(n)] = X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$$

$$W_R(e^{j\omega}) = W_R(\omega) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} w_R(n) \cdot e^{-j\omega n}$$

$$= \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} 1 \cdot e^{-j\omega n}$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)} + \cdots + e^{j\omega} + 1 + e^{-j\omega} + \cdots + e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)} \left[1 + e^{-j\omega} + \cdots + e^{-j\omega\left(\frac{N-1}{2}\right)}\right]$$

$$= e^{j\omega\left(\frac{N-1}{2}\right)} \left[\frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}\right]$$

$$= \frac{e^{j\omega\frac{N}{2}}}{e^{j\omega/2}} \left[\frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}\right]$$

$$= \frac{e^{j\omega\frac{N}{2}} - e^{-j\omega N/2}}{e^{j\omega/2} - e^{-j\omega/2}}$$

$$\boxed{W_R(e^{j\omega}) = \frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}}$$

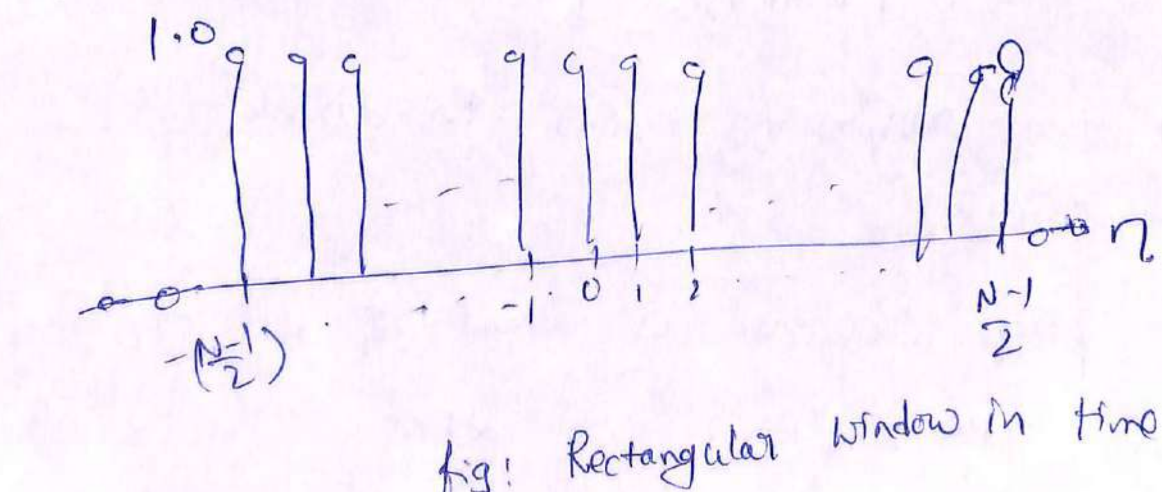


fig: Rectangular window in time domain

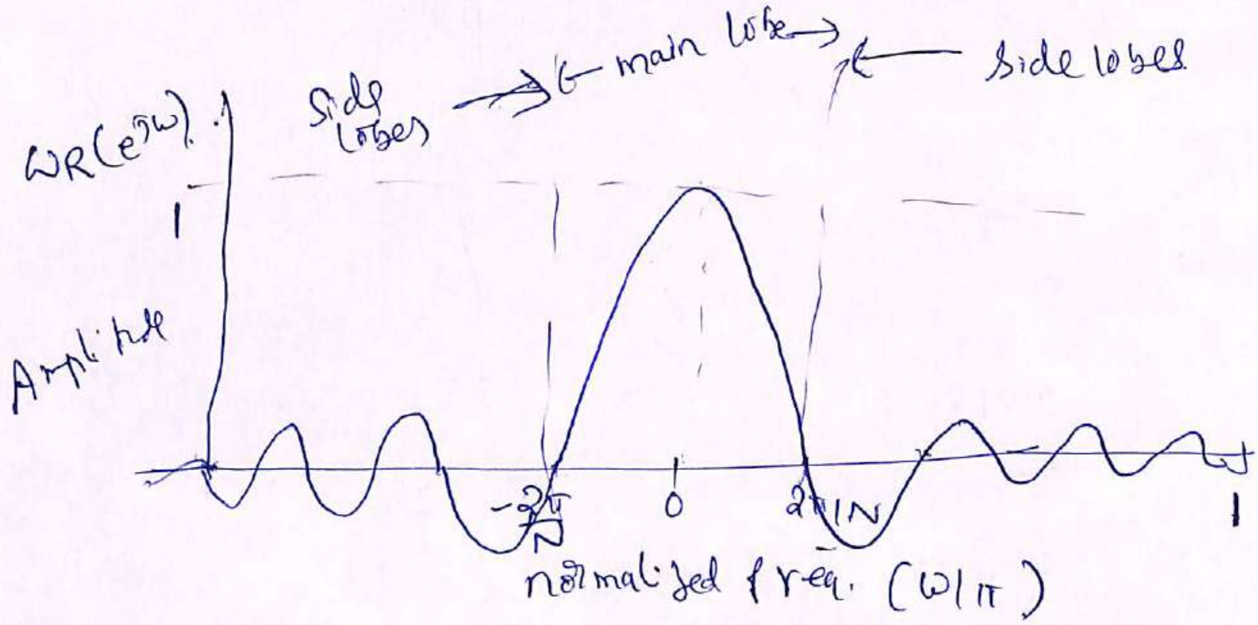


fig: freq response of $W_R(n)$ (or spectrum)

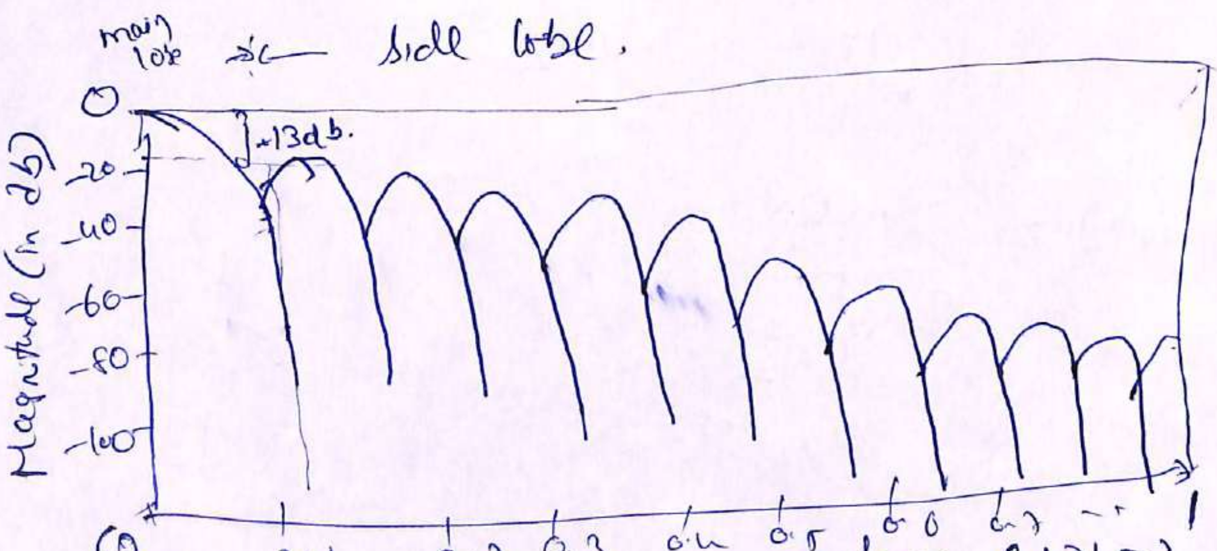


fig: freq response of $W_R(n)$ in db. (a Log mag response).

→ The freq. response is real & its zero occur when $\frac{N\omega}{2} = k\pi$ or $\omega = \frac{2k\pi}{N}$.

where k is an integer

→ The response of ω b/w $-\frac{2\pi}{N}$ & $\frac{2\pi}{N}$ is called the main lobe & the other lobes are known as side lobes.

→ The main lobe of the response is the portion that lies b/w the 1st two zero crossings.

→ The main lobe for rectangular window = $\frac{4\pi}{N}$.

→ The first side lobe will be 13 db down the peak of the main lobe & roll off will be 20 db/decade.

→ As the window is made longer, the main lobe becomes narrower & higher & the side lobe become more concentrated around $\omega = 0$.

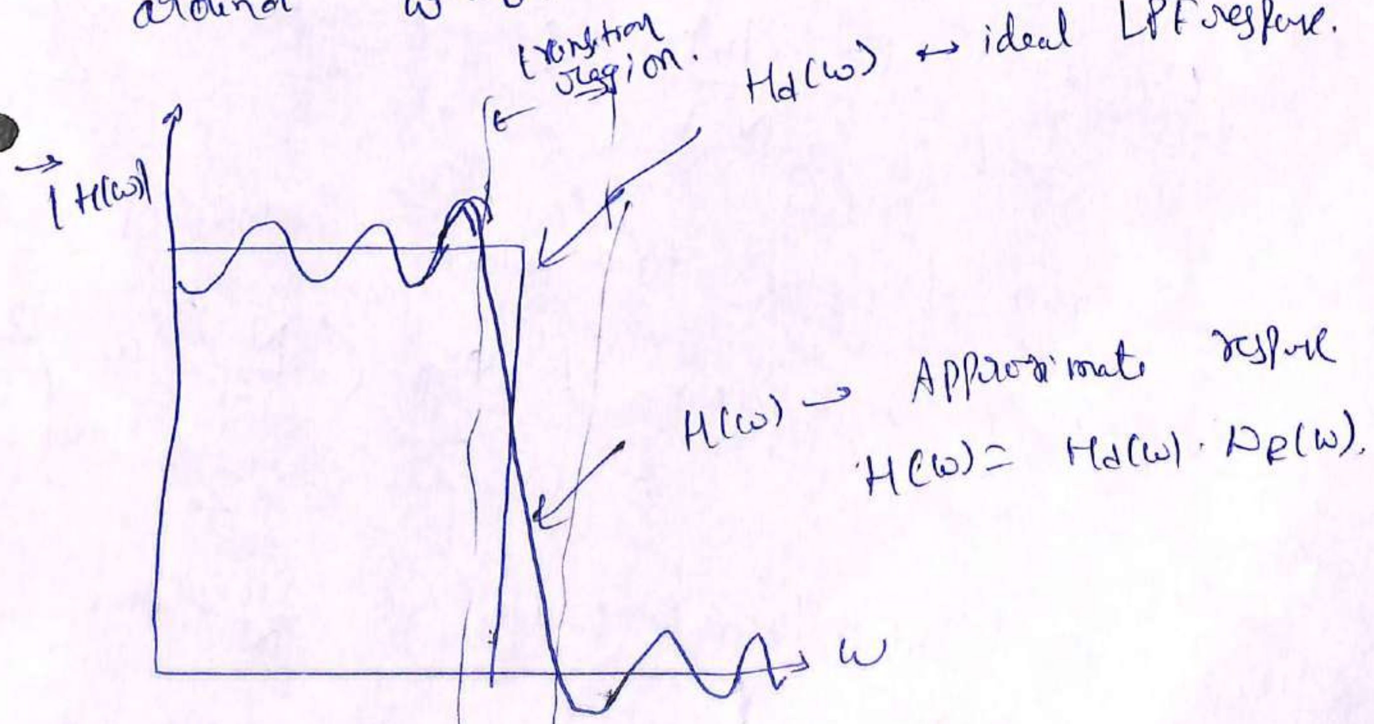


fig: freq. response of LPF.

→ Ideal, freq. response of LPF changes abruptly from pass to stop band. But here the freq. response changes slowly & called "transition region".

→ The width of transition region depends on the width of main lobe.

→ The effect, where max. ripple occurs just before & just after transition band is known as Gibb's phenomenon. & can be reduced by using a less abrupt truncation of filter coefficients.

(2) Triangular window:- (or Bartlett window)

→ The N-point triangular window is given by,

$$W_T(n) = \begin{cases} 1 - \dfrac{2|n|}{N-1} & \text{for } -\left(\dfrac{N-1}{2}\right) \le n \le \dfrac{N-1}{2} \\ 0 & \text{o.w} \end{cases}$$

→ The F.T of the triangular window is

$$W_T(e^{j\omega}) = W_T(\omega) = \left[\frac{\sin\left(\frac{N-1}{4}\right)\omega}{\sin\frac{\omega}{2}}\right]^2.$$

→ Here, the main lobe width is $\frac{8\pi}{N}$ (i.e $-\frac{4\pi}{N}$ to $\frac{4\pi}{N}$) i.e twice of rectangular window

→ The side lobe level is smaller than [4-13] that of rectangular window & reduced from −13 db to −25 db.

→ It produces a smooth mag. response in both pass band & stop band.

→ But drawbacks!

1) The transition region is more

2) The attenuation in stop band is less

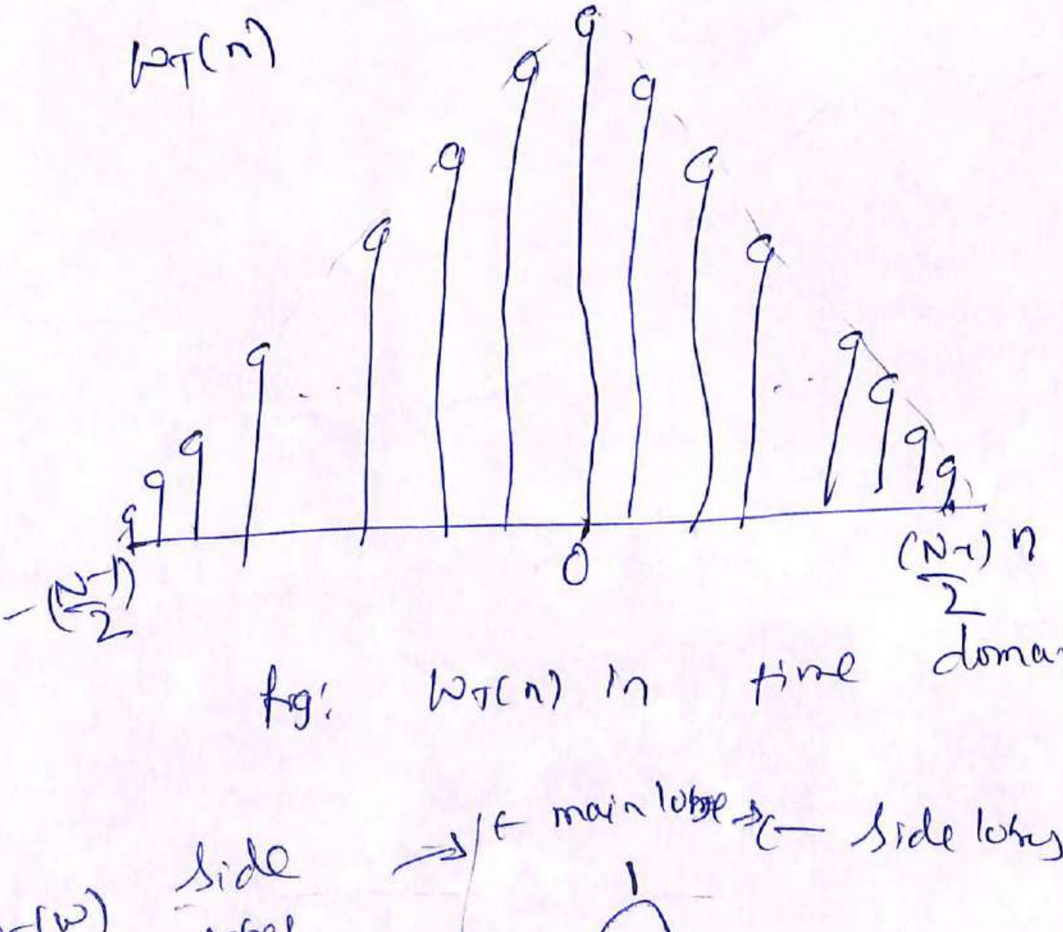


fig: $w_T(n)$ in time domain.

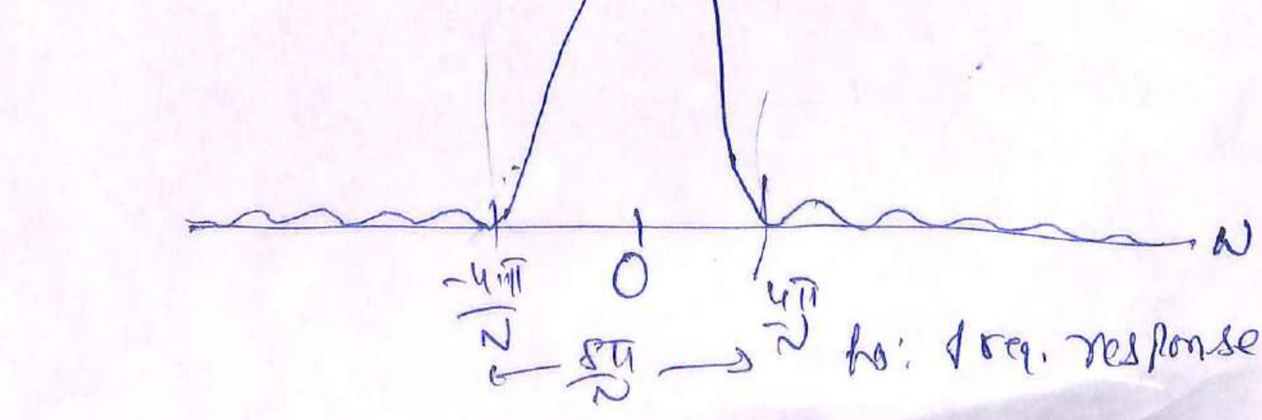
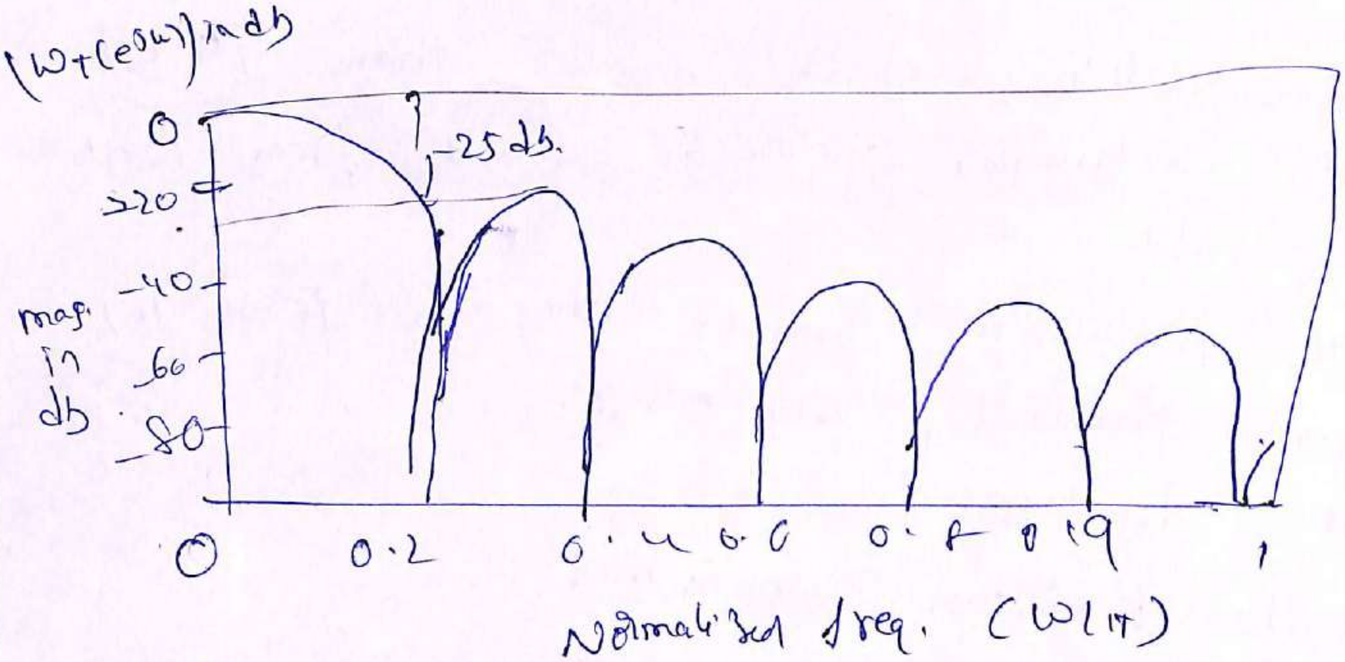
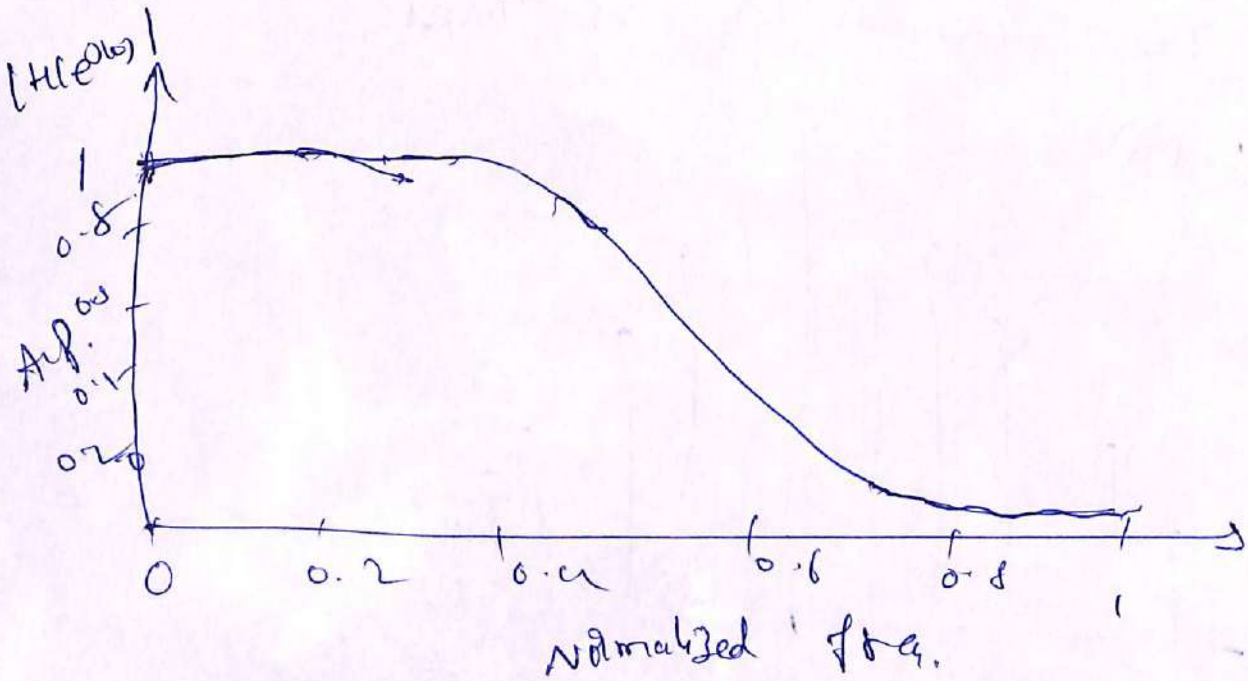


fig: freq. response of $w_T(n)$



|W_T(e^{jω})| in db
0
-25 db.
-20
-40
mag. in db
-60
-80
0
0.2
0.4
0.6
0.8
0.9
1
Normalized freq. (ω/π)



|H(e^{jω})|
1
0.8
0.6
Amp.
0.4
0.2
0
0.2
0.4
0.6
0.8
1
Normalized freq.

3, Raised Cosine Window :-

→ The raised cosine window multiplies the central coefficients by approximately unity & smoothly truncate the Fourier coefficients towards the ends of the filter.

→ The window sequence is of the form.

$$w_\alpha(n) \begin{cases} = \alpha + (1-\alpha) \cdot \cos \frac{2\pi n}{N-1}, & \text{for } -\left(\frac{N-1}{2}\right) \le n \le \frac{N-1}{2} \\ = 0 & \text{ow} \end{cases}$$

→ The freq. response $w_\alpha(n)$ is given by,

$$W_\alpha(e^{j\omega}) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} \left(\alpha + (1-\alpha) \cdot \cos \frac{2\pi n}{N-1}\right) \cdot e^{-j\omega n}$$

$$= \alpha \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} e^{-j\omega n} + (1-\alpha) \left(\sum_{n=-\frac{(N-1)}{2}}^{\frac{N-1}{2}} \left(\frac{e^{j\frac{2\pi n}{N-1}} + e^{-j\frac{2\pi n}{N-1}}}{2} \right) \cdot e^{-j\omega n} \right)$$

$$= \alpha \cdot \frac{\sin \frac{\omega N}{2}}{\sin \omega/2} + \frac{(1-\alpha)}{2} \left[\sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} e^{-j\left(\omega - \frac{2\pi}{N-1}\right)n} + \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} e^{-j\left(\omega + \frac{2\pi}{N-1}\right)n} \right]$$

B # Second term.

$$= \frac{1-\alpha}{2}\left[e^{j(\omega - \frac{2\pi}{N-1})\cdot\frac{N-1}{2}} + \cdots + e^{j(\omega-\frac{2\pi}{N-1})} + 1 + e^{-j(\omega-\frac{2\pi}{N-1})} + \cdots + e^{-j(\omega-\frac{2\pi}{N-1})\cdot(\frac{N-1}{2})}\right]$$

$$= \frac{1-\alpha}{2}\cdot e^{j(\omega-\frac{2\pi}{N-1})\cdot\frac{N-1}{2}}\left[1 + e^{-j(\omega-\frac{2\pi}{N-1})} + \cdots + e^{-j(\omega-\frac{2\pi}{N-1})(N-1)}\right]$$

$$= \frac{1-\alpha}{2}\cdot e^{j(\omega-\frac{2\pi}{N-1})\cdot\frac{N-1}{2}}\left(\frac{1-e^{-j(\omega-\frac{2\pi}{N-1})N}}{1-e^{-j(\omega-\frac{2\pi}{N-1})}}\right)$$

$$= \frac{1-\alpha}{2}\cdot\frac{e^{j(\frac{\omega N}{2}-\frac{\pi N}{N-1})}}{e^{j(\frac{\omega}{2}-\frac{\pi}{N-1})}}\left[\quad '' \quad\right]$$

$$= \frac{1-\alpha}{2}\cdot\frac{\sin\left(\frac{\omega N}{2}-\frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2}-\frac{\pi}{N-1}\right)}$$

11y 3rd term

$$C = \frac{1-\alpha}{2}\left[\frac{\sin\left(\frac{\omega N}{2}+\frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2}+\frac{\pi}{N-1}\right)}\right]$$

$$\therefore W_R(e^{j\omega}) = \alpha \cdot \frac{\sin\frac{\omega N}{2}}{\sin\omega/2} + \frac{1-\alpha}{2} \cdot \frac{\sin\left(\frac{\omega N}{2} - \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{N-1}\right)}$$

$$+ \frac{1-\alpha}{2} \cdot \frac{\sin\left(\frac{\omega N}{2} + \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{N-1}\right)}$$

(4) Hanning window: —

→ The Hanning window seq. can be obtained by $\alpha = 0.5$ in raised cosine window.

$$W_{Hn}(n) = \begin{cases} 0.5 + 0.5\cos\frac{2\pi n}{N-1}, & \text{for } -\frac{N-1}{2} \le n \le \frac{N-1}{2} \\ = 0 & \text{o.w.} \end{cases}$$

→ The freq. response of Hanning window is

$$W_{Hn}(e^{j\omega}) = 0.5\,\frac{\sin\frac{\omega N}{2}}{\sin\omega/2} + 0.25 \cdot \frac{\sin\left(\frac{\omega N}{2} - \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{N-1}\right)}$$

$$+ 0.25 \cdot \frac{\sin\left(\frac{\omega N}{2} + \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{N-1}\right)}$$

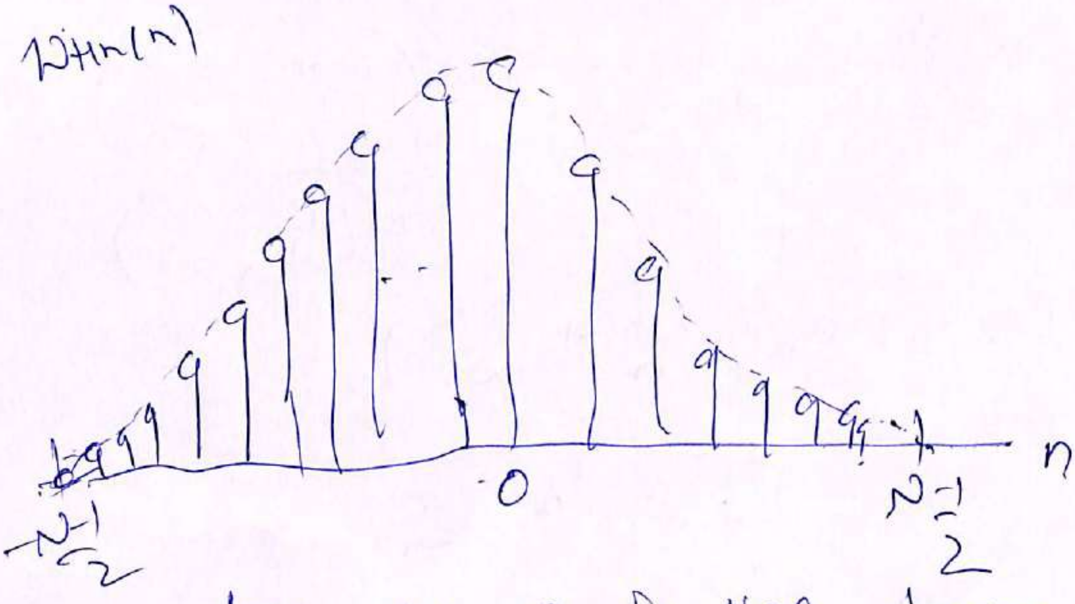


fig: $W_{Hn}(n)$ in time domain

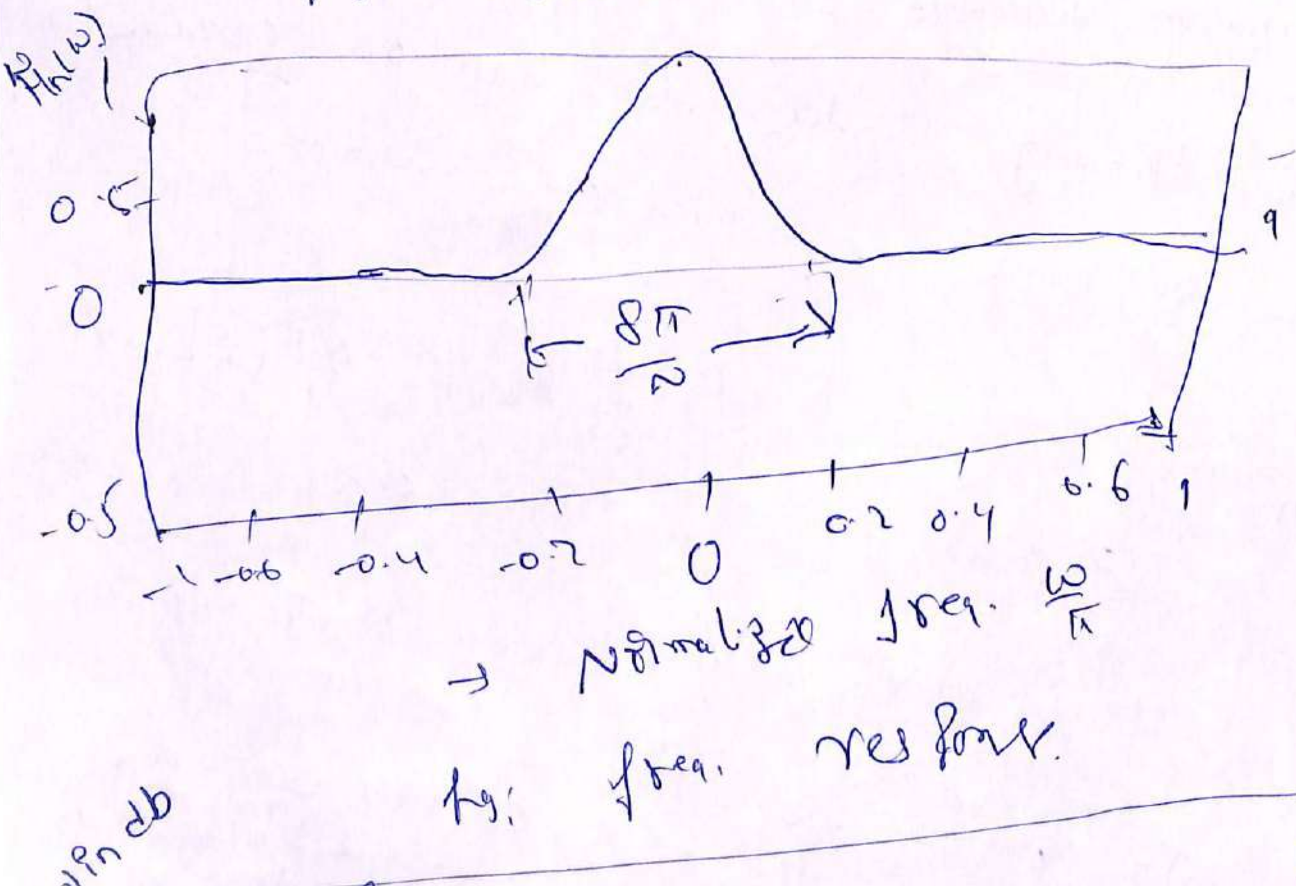
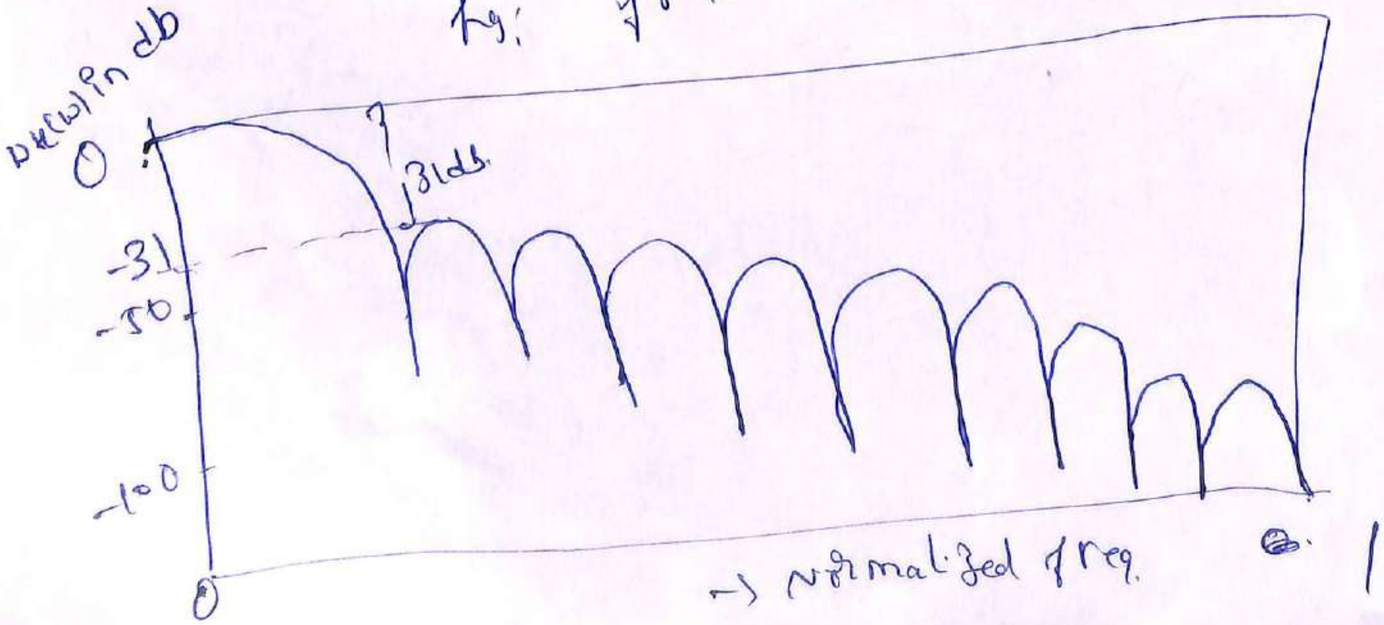


fig: freq. response



→ The min. stop band attenuation of the filter is 44 dB.

(5) Hamming window:–

Sol. $\alpha = 0.54$ in Raised cosine window

$$W_H(n) = \begin{cases} 0.54 + 0.46\cos\left(\frac{2\pi n}{N-1}\right). & \text{for } -\frac{(N-1)}{2} \le n \le \frac{N-1}{2} \\ 0 & \text{o.w} \end{cases}$$

The freq. response of Hamming window is

$$W_H(e^{j\omega}) = 0.54 \cdot \frac{\sin\frac{\omega N}{2}}{\sin\omega/2} + 0.23 \cdot \frac{\sin\left(\frac{\omega N}{2} - \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{N-1}\right)}$$

$$+ 0.23 \cdot \frac{\sin\left(\frac{\omega N}{2} + \frac{\pi N}{N-1}\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{N-1}\right)}$$

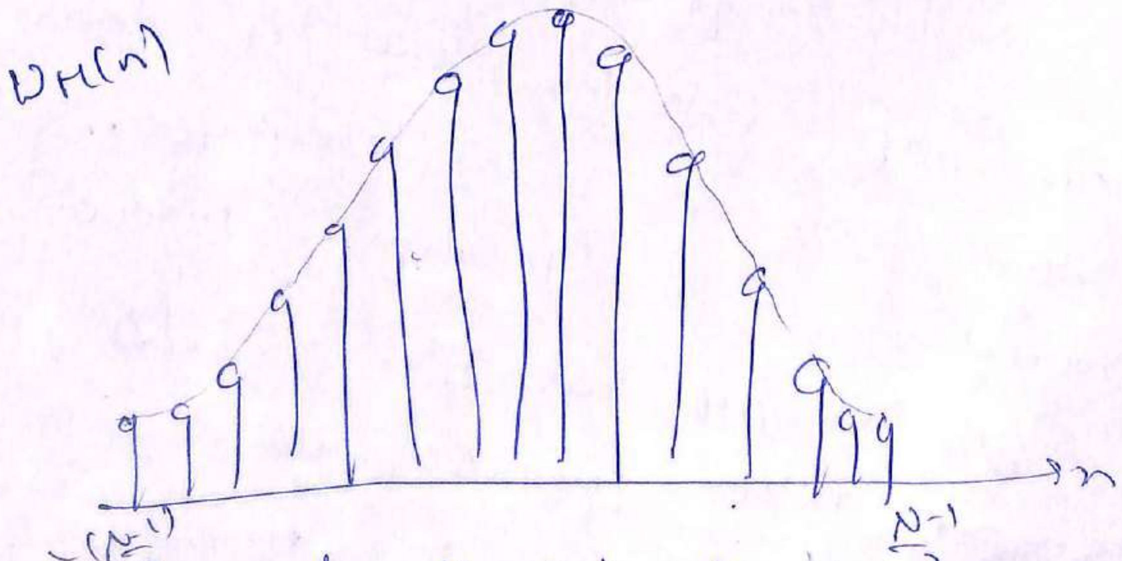
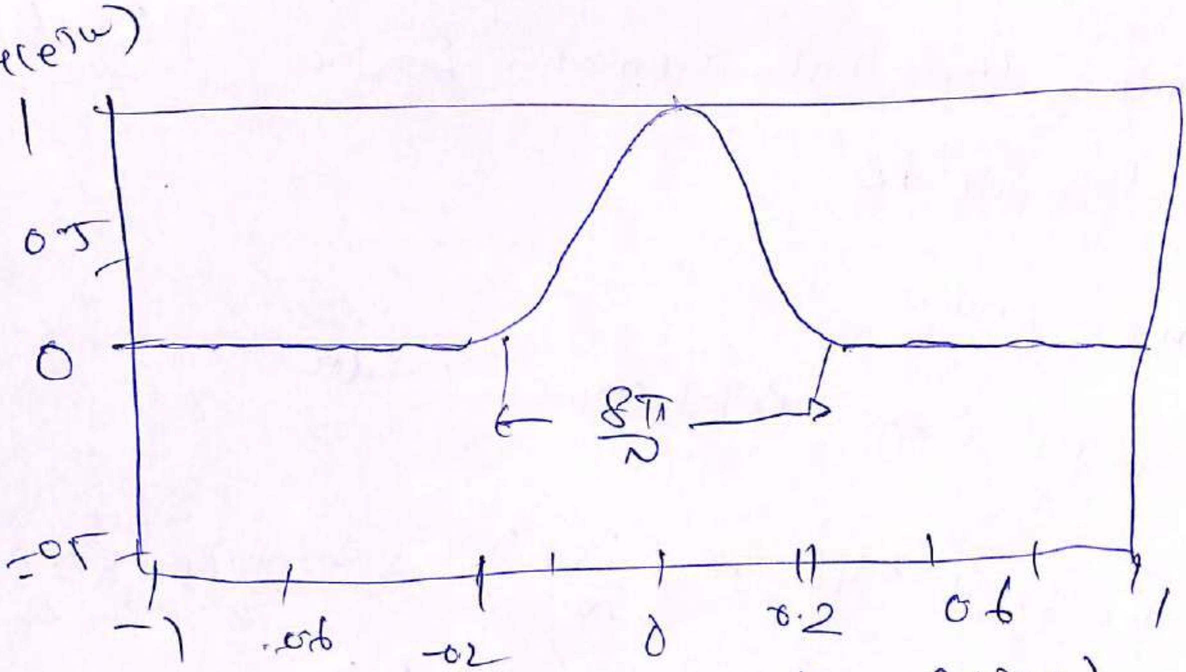


fig: $W_H(n)$ in time domain.



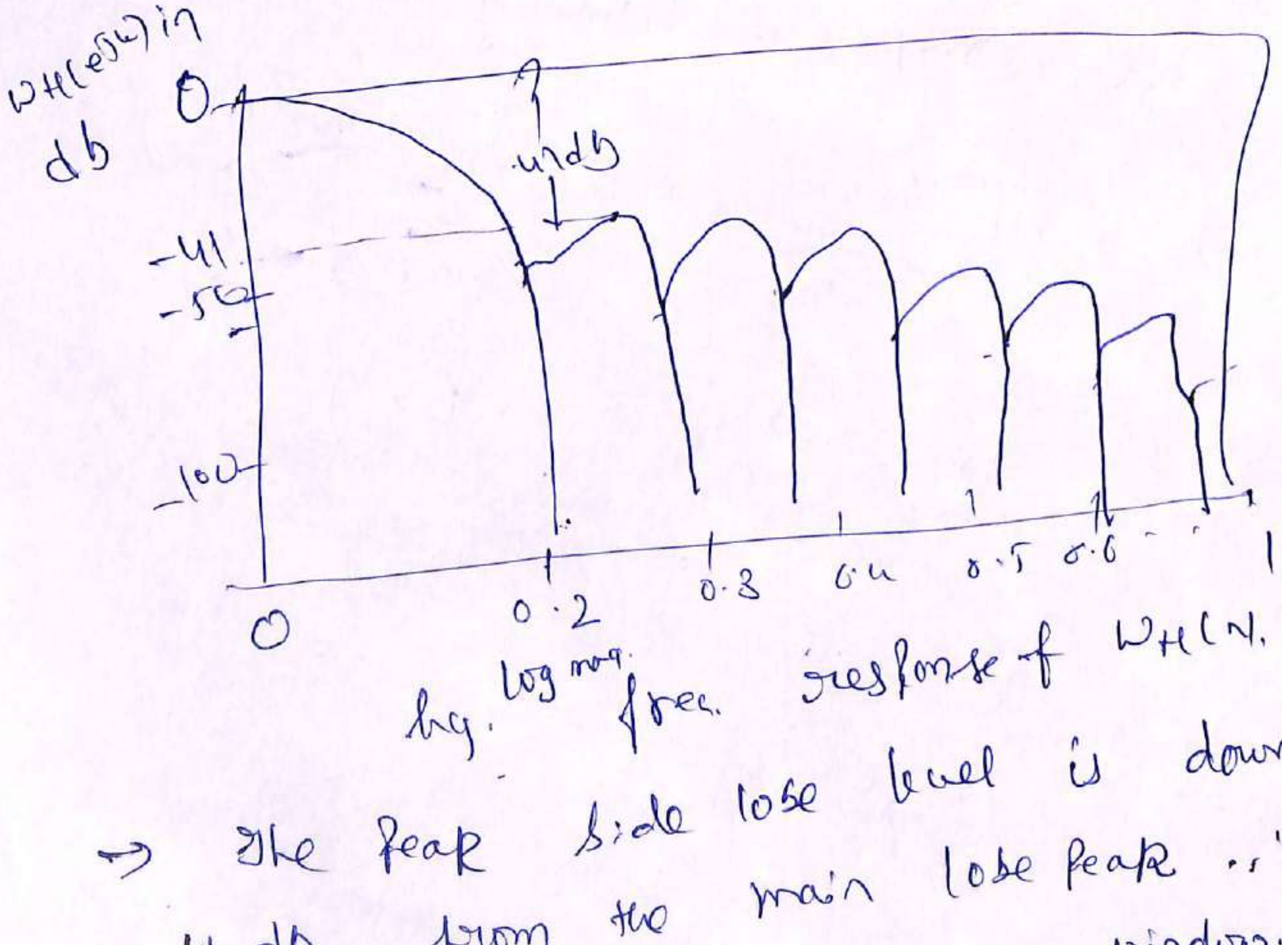


fig. log mag. freq. response of WH(n).

→ The peak side lobe level is down about 41 dB from the main lobe peak ∴ An improvement of 10 dB to Hanning window. The first side lobe peak level is −53 dB.

→ But, at higher frequencies, the stop band attenuation is low compared to Hanning window.

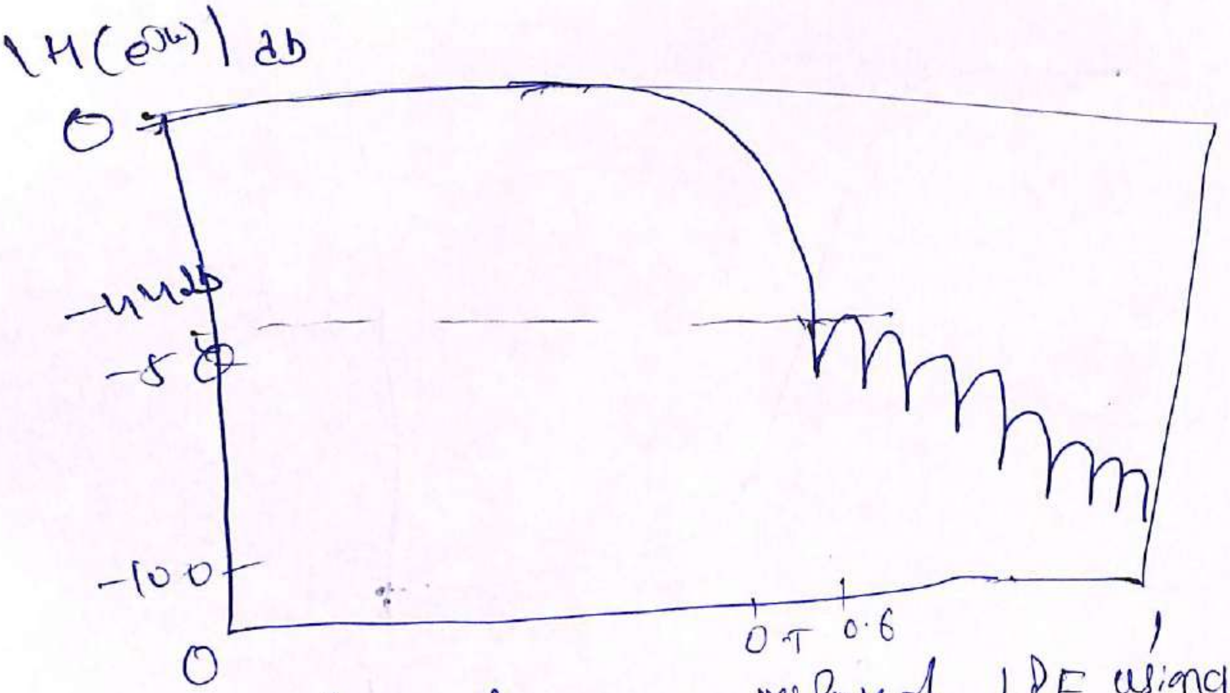


fig: Log mag. response of LPF using Hanning Window.

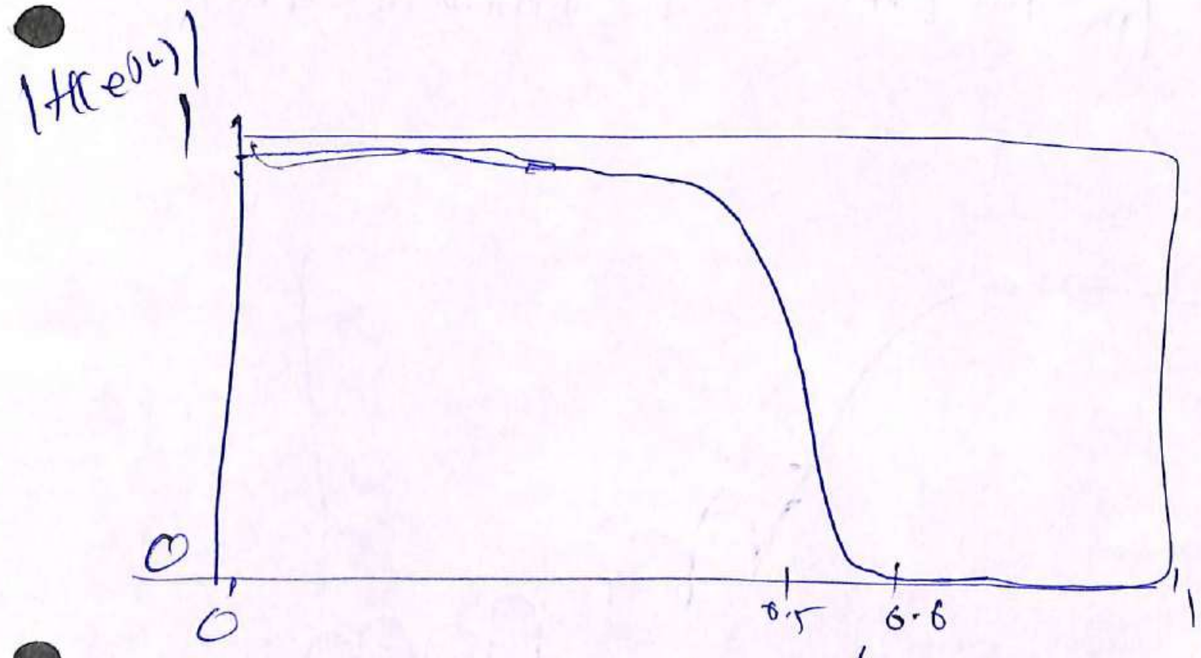


fig: freq. response of LPF using Hanning Window

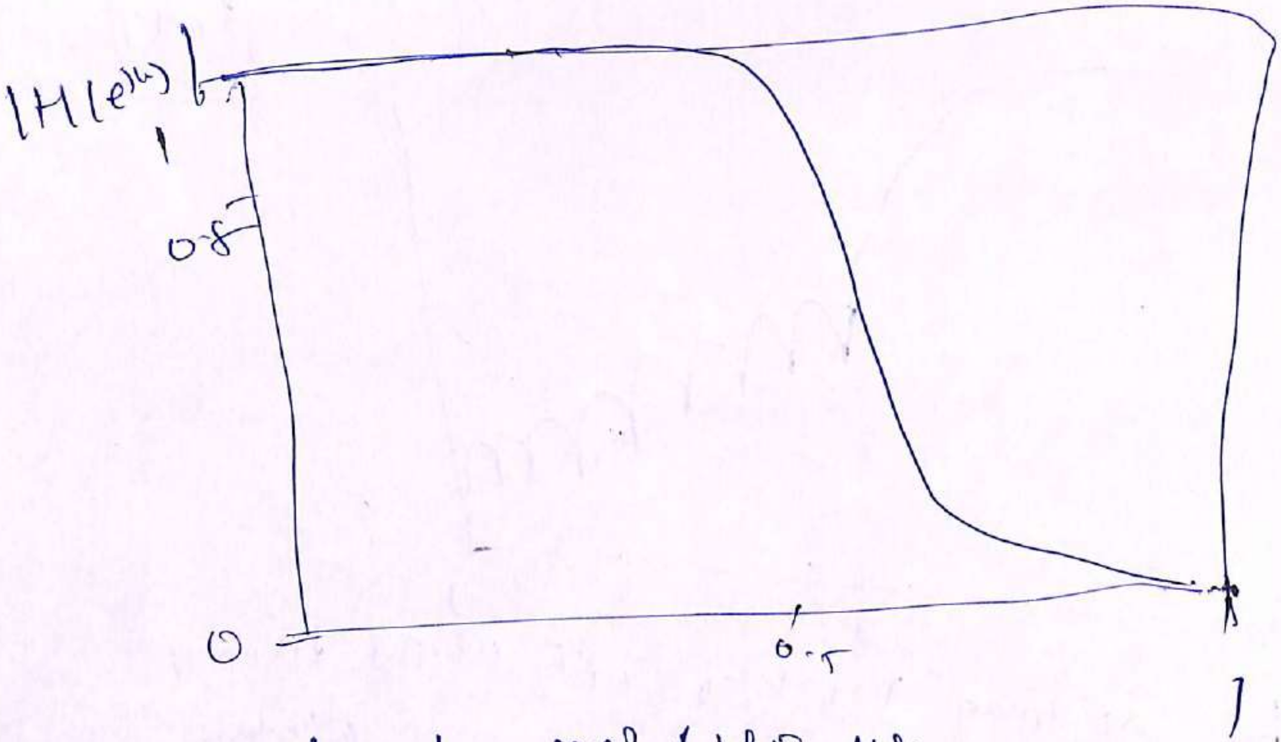


fig: freq. resp. of LPF using Hamming window

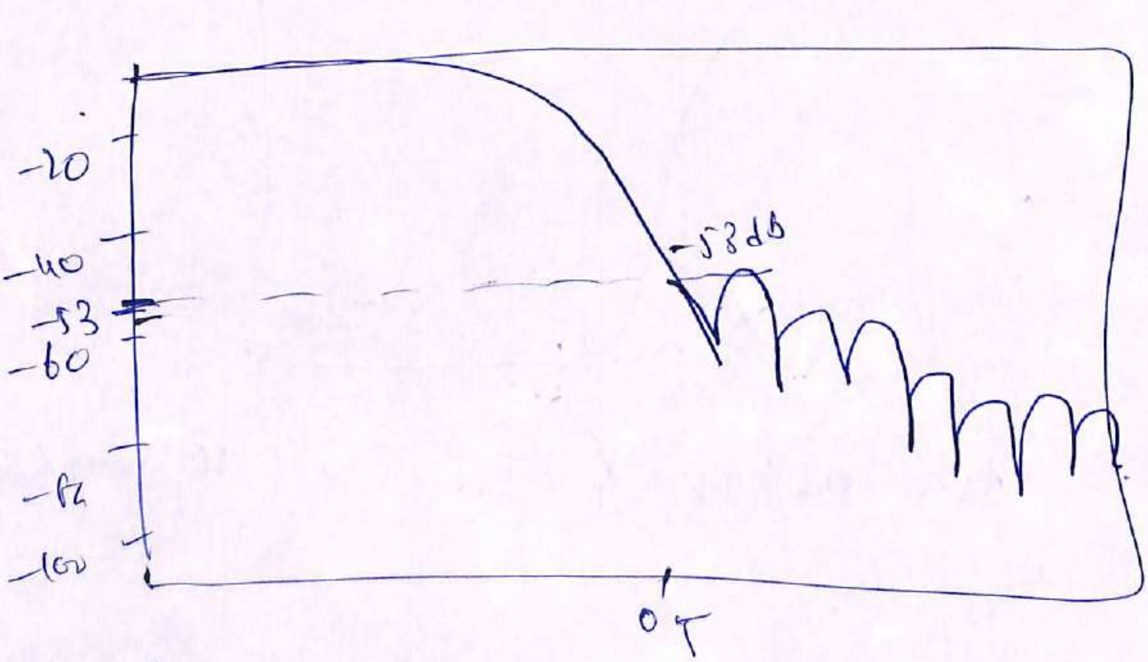


fig: Log. mag. freq. response of LPF using Hamming window

5, 7, 9, 11, 13, ~~15~~ 16, 17,
20, 3, 4, 7, 8, 9, 30, 28,
4, 8, 6, 9, 40, 1, 2,
5, 6, 0, 1, 2, 3, 4, 5

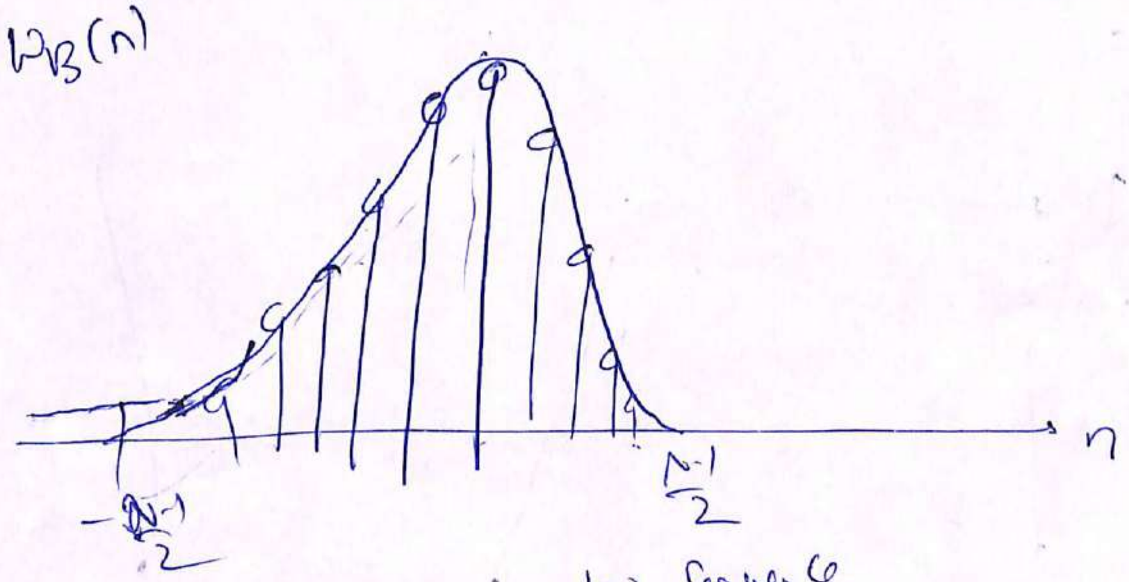
(6) Blackman window:-

The Blackman window seq. is given by

$$w_B(n) = 0.42 + 0.5\cos\frac{2\pi n}{N-1} + 0.08\cos\frac{4\pi n}{N-1}$$

for $-\left(\frac{N-1}{2}\right) \le n \le \frac{N-1}{2}$

= 0 o.w



Blackman window sequence.

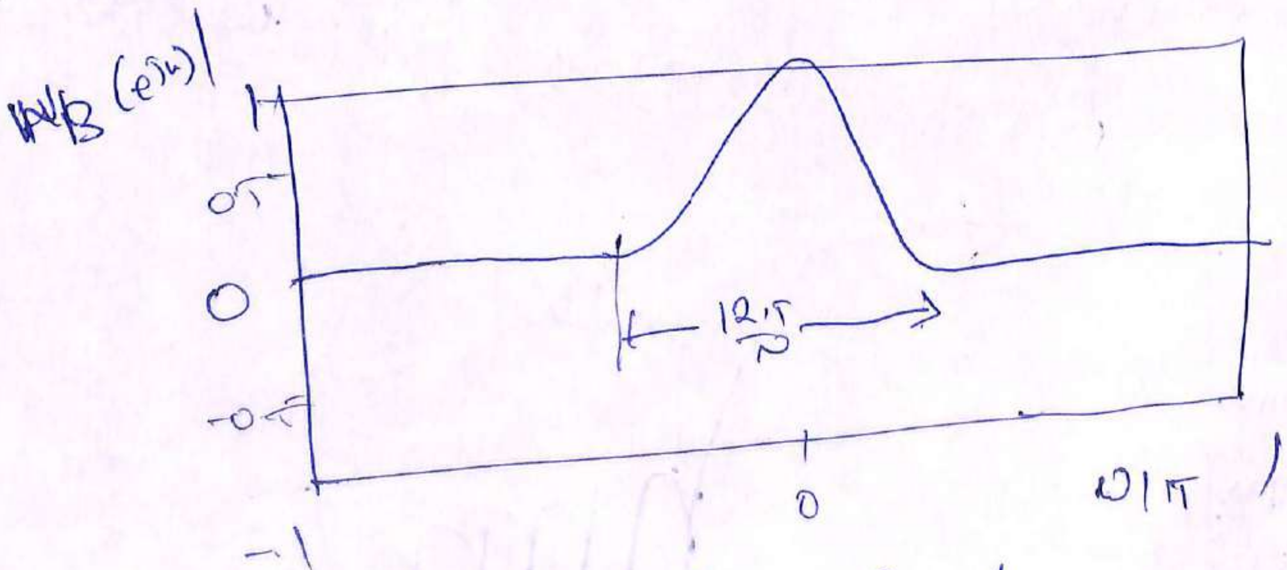


fig.- freq. response of $w_B(n)$

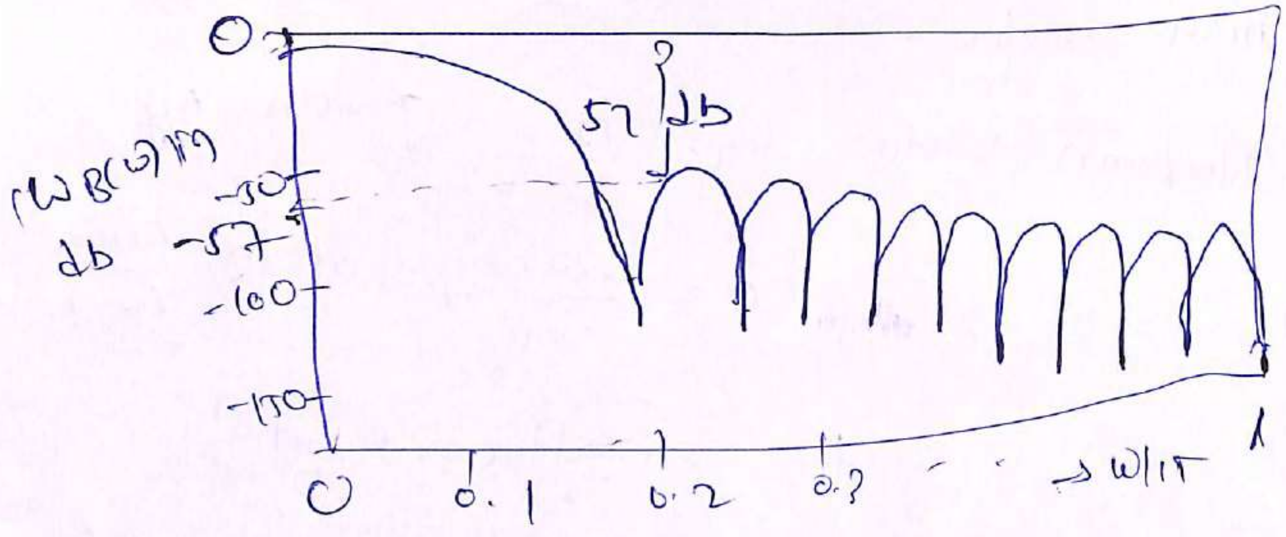


fig: Log mag. response of ωB(n).

fig: Freq. response of LPF using Blackman window

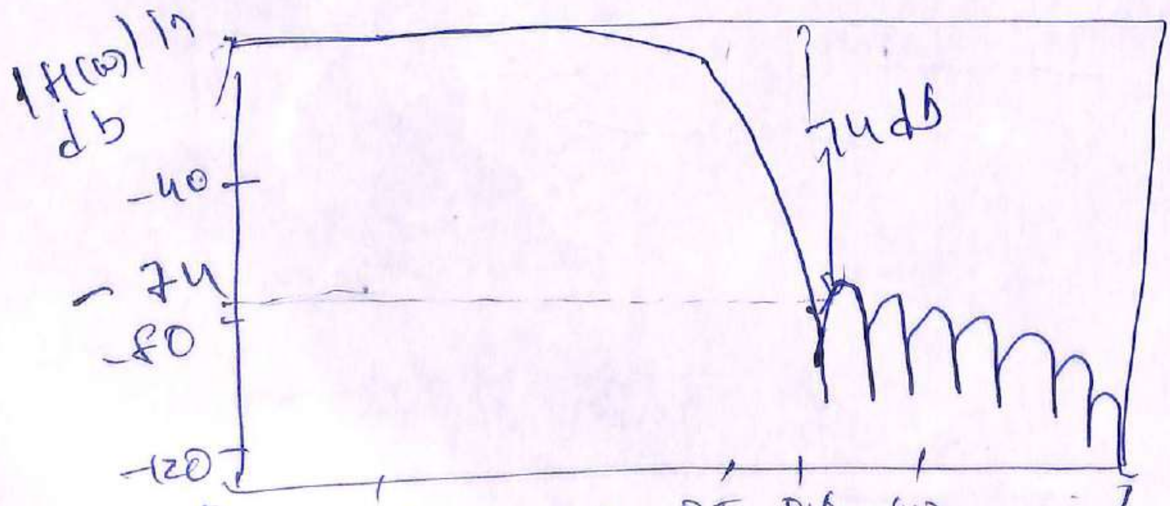


fig: log mag. response of LPF using ωB(n)

Window	Peak amp. of Side lobe (dB)	Main lobe width	Min. stop band attenuation
Rectangular	-13	$\frac{4\pi}{N}$	-21
Bartlett or triangular	-25	$\frac{8\pi}{N}$	-25
Hanning	-31	$\frac{8\pi}{N}$	-44
Hamming	-41	$\frac{8\pi}{N}$	-53
Blackman	-57	$\frac{12\pi}{N}$	-74

(7) Kaiser Window:-

→ To acheive Prescribed minimum stop band attenuation & Passband ripple, the designer must find a window with an appropriate side lobe level & then choose 'N' to acheive a Prescribed transition width.

→ To over come this problem, kaiser has chosen a class of windows based on the Prolate spherodial functions.

→ The kaiser window is given by,

$$w_k(n) = \begin{cases} I_0\left[\alpha\sqrt{1-\left(\frac{2n}{N-1}\right)^2}\right], & \text{for } |n| \le \frac{N-1}{2} \\ 0, & \text{o.w} \end{cases}$$

where

$\alpha \rightarrow$ adjustable Parameter

$I_0(x) \rightarrow$ modified zeroth-order Bessel function of the 1st kind of order zero.

$$I_0(x) = 1 + \sum_{k=1}^{\infty}\left[\frac{1}{k!}\left(\frac{x}{2}\right)^k\right]^2$$

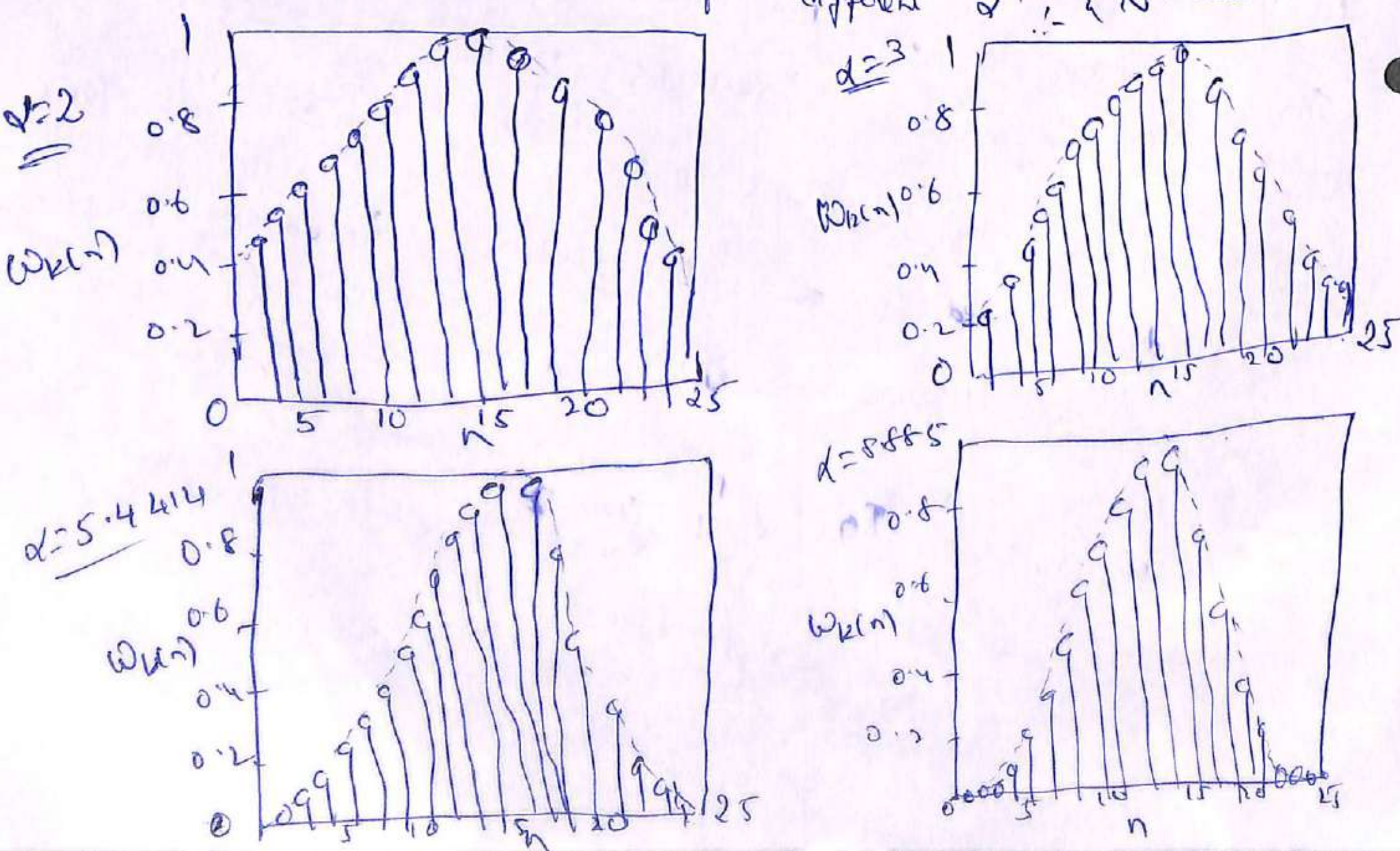
$$= 1 + \frac{0.25x^2}{(1!)^2} + \frac{(0.25x^2)^2}{(2!)^2} + \cdots$$

→ For most practical purposes, the summation upto 25 terms is sufficient.

→ The freq. response of the Kaiser window is given by

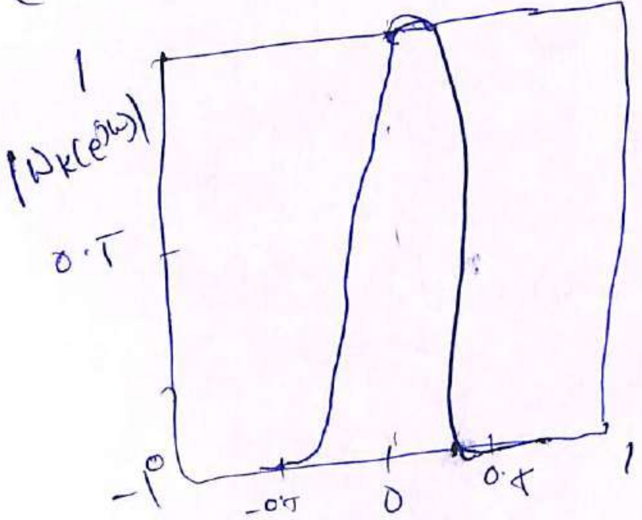
$$W_K(e^{j\omega}) \simeq \frac{2}{I_0(\alpha)} \frac{\sin\left\{\frac{(N-1)}{2}\left[\omega^2 - \left(\frac{2\alpha}{N-1}\right)^2\right]^{1/2}\right\}}{\left[\omega^2 - \left(\frac{2\alpha}{N-1}\right)^2\right]^{1/2}}$$

→ Kaiser window Sequences for different 'α' & $N = 25$

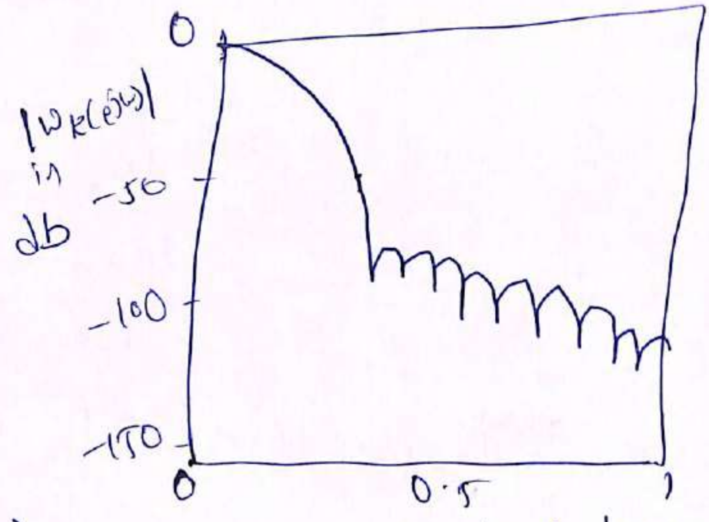


$\alpha = 11$

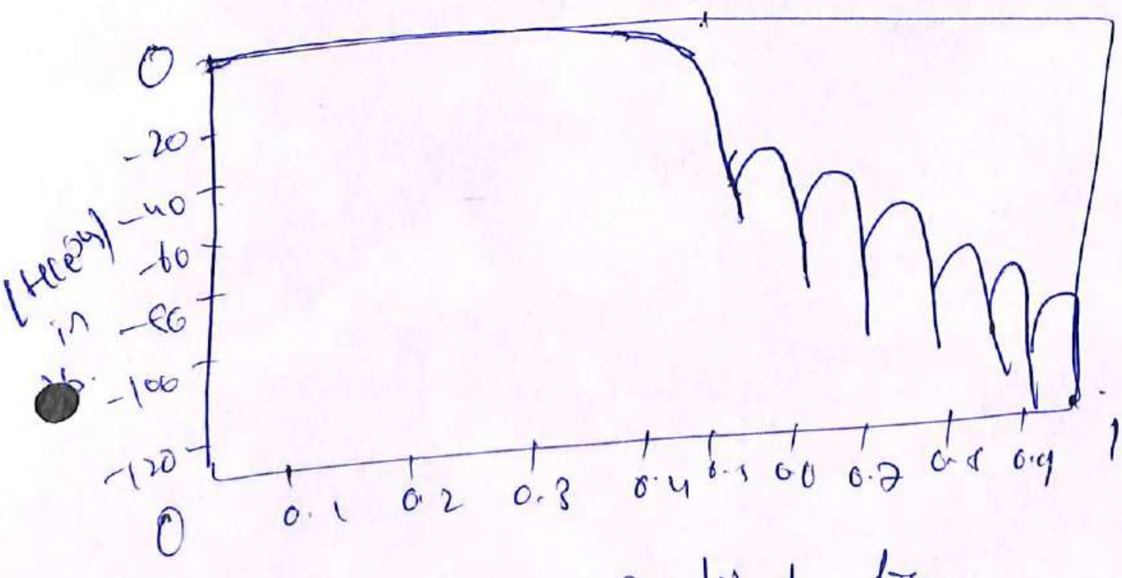
(1) Freq. response of $w_K(n)$



(2) Log mag response of $W_K(n)$



(3) Log mag. response of LPF with $\alpha = 11$ in kaiser window



→ When the parameter α is varied, both the transition B.W & peak ripple in the side lobe changes.

→ for $\alpha = 1$, kaiser window becomes Rectangular Window

→ for $\alpha = 5.4414$, " Hamming "

→ $\alpha = 8.885$, " Blackman "

> Design a Low Pass FIR filter with five stage. [Given: Sampling time 1 ms; $f_c = 200\,Hz$]. Also find the freq. response of the filter. Using (1) rectangular window (2) ... window

Sol]
$$f_c = 200\,Hz, \quad f_s = \frac{1}{1\,ms} = 1\,kHz$$

$$\omega_c = \frac{2\pi f_c}{f_c} = \frac{2\pi \times 200}{1000} = 0.4\pi \text{ rad/sec}$$

① → The desired Low pass filter T.F. is

$$H_d(\omega) = \begin{cases} 1, & -\omega_c \le \omega \le \omega_c \\ 0, & \text{for } -\pi \le \omega \le -\omega_c \\ 0, & \omega_c \le \omega \le \pi. \end{cases}$$

② → The desired impulse response of the filter is

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) \cdot e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-0.4\pi}^{0.4\pi} 1 \cdot e^{j\omega n}\, d\omega$$

$$h_d(n) = \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{-0.4\pi}^{0.4\pi} = \frac{\sin 0.4 n\pi}{n\pi}, \quad n \ne 0$$

for $n = 0$, L'hospital rule, $h_d(0) = \lim_{n \to 0} \frac{\sin 0.4 n\pi}{n\pi} = 0.4$

Given $N = 5$.

$h_d(0) = 0.4$, $h_d(1) = 0.3027 = h_d(-1)$, $h_d(2) = 0.0935 = h_d(-2)$

4, 7, 22, 9, 5, 9, 34, 6, 46, 9, 51, 6-5

→ Consrectangular window.

$$W_R(n) = \begin{cases} 1 & \text{for } -\left(\frac{N-1}{2}\right) \le n \le \frac{N-1}{2} \\ 0 & \text{o.w.} \end{cases}$$

$\therefore h(n) = h_d(n) \cdot W_R(n)$ for $-\left(\frac{N-1}{2}\right) \le n \le \frac{N-1}{2}$

ie $-2 \le n \le 2$.

$h(-2) = 0.0935$

$h(-1) = 0.3027$

$h(0) = 0.4$

$h(1) = 0.3027$, $h(2) = 0.0935$.

→ The above coefficients correspond to a non-causual filter which is not realizable since we get +ve powers of z.

The transfer fn. of the realizable digital filter is

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} \left[\sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} h(n) \cdot z^{-n} \right]$$

$$= z^{-\left(\frac{N-1}{2}\right)} \left[h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left(z^{n} + z^{-n}\right) \right]$$

$$= z^{-2} \left[h(0) + \sum_{n=1}^{2} h(n) \cdot \left[z^{-n} + z^{n} \right] \right].$$

$$= z^{-2}\left[0.4 + 0.3027\left[z + z^{-1}\right] + 0.0935\left[z^{2} + z^{-2}\right]\right]$$

T.F

$$H(z) = 0.0935 + 0.3027 z^{-1} + 0.4 z^{-2} + 0.3027 z^{-3} + 0.0935 z^{-4}$$

$\therefore$ filter coefficients ~~of~~, realizable digital filter are:

$h(0) = 0.0935,\ h(1) = 0.3027,\ h(2) = 0.4,\ h(3) = 0.3027,$

$h(4) = 0.0935.$

→ The freq. response of filter is

$$\boxed{H(e^{j\omega}) = 0.0935 + 0.3027\, e^{-j\omega} + 0.4\, e^{-2j\omega} + 0.3027\, e^{-3j\omega} + 0.0935\, e^{-4j\omega}}$$

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right)\cdot \cos\omega n\right]$$

$$= e^{-2j\omega}\left[0.4 + 0.6054\cos\omega + 0.1870\cos 2\omega\right]$$

$$|H(\omega)| = 0.4 + 0.6054\cos\omega + 0.1870\cos 2\omega,$$

$$\angle H(\omega) = -2\omega = \alpha\omega.$$

4, 24, 7, 34, 44, 6,
51 le-5

Q; Design a filter with 14-22

$$H_d(e^{j\omega}) = \begin{cases} e^{-3j\omega} & \text{for } -\frac{\pi}{4} \le \omega \le \frac{\pi}{4} \\ 0 & \text{o.w} \end{cases}$$

Using a Hamming window with N=7.

Sol

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{-3j\omega} \cdot e^{j\omega n}\, d\omega$$

$$= \frac{\sin \frac{\pi}{4}(n-3)}{\pi(n-3)}$$

for $N=7$

$$W_H(n) = 0.5 + 0.5 \cos \frac{2\pi n}{N-1} \quad \text{for } -\frac{N-1}{2} \le n \le \frac{N-1}{2}$$

$$= 0 \quad \text{o.w}$$

$N=7$, $-\frac{7-1}{2} \le n \le \frac{7-1}{2} \Rightarrow -3 \le n \le 3$

$$W_H(-3) = 0.5 + 0.5 \cdot \cos\left(\frac{2\pi \times -3}{6}\right) = 0 = W_H(3)$$

$$W_H(-2) = 0.25 = W_H(2)$$

$$W_H(-1) = 0.75 = W_H(1)$$

$$W_H(0) = 0.5 + 0.5 = 1$$

$$\therefore h(n) = h_d(n) \cdot W_H(n), \quad -\frac{(N-1)}{2} \le n \le \frac{N-1}{2}$$

$h(-3) = h_d(-3) \cdot \omega_H(-3) =$

$h(-2) = h_d(-2) \cdot \omega_H(-2) =$

$h(-1) = h_d(-1) \cdot \omega_H(-1) =$

$h(0) = h_d(0) \cdot \omega_H(0) =$

→ T.F of the filter is

$$H(z) = \sum_{n=-3}^{3} h(n) \cdot z^{-n}$$

→ T.F of realizable filter is

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} \left[h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2\, h\left(\frac{N-1}{2}\right) \right.$$

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} \left[\sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} h(n) \cdot z^{-n} \right]$$

$$= z^{-\left(\frac{N-1}{2}\right)} \left\{ h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) \left[z^{n} + z^{-n} \right] \right.$$

$$= 0.006 + 0.049 z^{-1} + 0.173 z^{-2} + 0.25 z^{-3} + 0.173 z^{-4} + 0.049 z^{-5} + 0.006 z^{-6}.$$

H.W Design a filter with

$$H_d(e^{j\omega}) = \begin{cases} e^{-j5\omega} & \text{for } -\frac{\pi}{2} \le \omega \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\omega| \le \pi \end{cases}$$

using Blackmann window with $N = 11$.

Fourier Series Method of Designing FIR filters! —

→ The freq. response $H(e^{j\omega})$ of a system is periodic in 2π.

→ From Fourier series analysis, we k.T any periodic function can be expressed as a linear combination of complex exponentials.

→ The desired freq. response of an FIR filter can be represented by the Fourier Series,

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n) \cdot e^{-j\omega n}$$

$h_d(n) \rightarrow$ fourier coefficients & are desired impulse response.

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) \cdot e^{j\omega n} \, d\omega$$

→ $h_d(n)$ is of infinite duration seq. for FIR filter T.F, the series can be trunctuated by

$$h(n) = h_d(n) \quad \text{for } |n| \leq \frac{N-1}{2}$$
$$= 0 \quad \text{o.w}$$

$$H(z) = \sum_{n=-\frac{(N-1)}{2}}^{\frac{N-1}{2}} h(n) \cdot z^{-n}$$

$$h\left(-\left(\tfrac{N-1}{2}\right)\right) \cdot z^{\frac{N-1}{2}} + \cdots + h(-1) \cdot z^{1} + h(0) + h(1) \cdot z^{-1} + \cdots + h\left(\tfrac{N-1}{2}\right) \cdot z^{-(N-1)/2}$$

$$= h(0) + \sum_{n=1}^{\frac{N-1}{2}} \left[h(n) \cdot z^{-n} + h(-n) \cdot z^{n} \right]$$

→ For a symmetrical impulse response having symmetry at $n = 0$,

$$h(-n) = h(n)$$

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) \left[z^{n} + z^{-n} \right]$$

→ The above T.F is not physically realizable.
∵ it is having +ve powers of z.

→ ∴ To make it realizable, multiply above eqn. by $z^{-(N-1)/2}$

$$H'(z) = z^{-(N-1)/2} \cdot H(z)$$

$$\boxed{H'(z) = z^{-\left(\frac{N-1}{2}\right)} \left[h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) \left[z^{n} + z^{-n} \right] \right]}$$

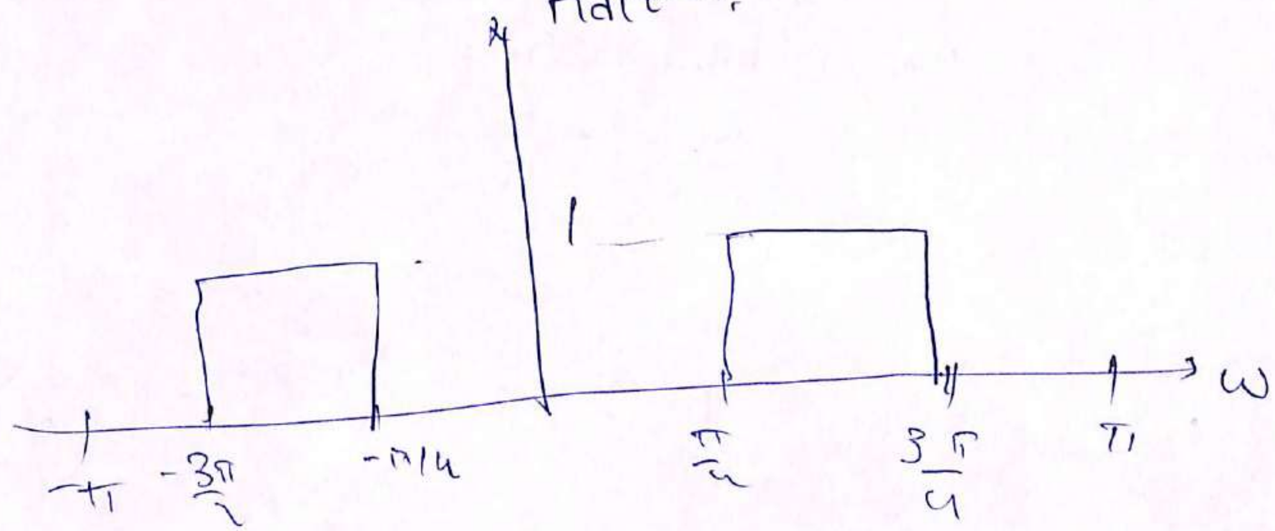
4-24

→ Design an Ideal BPF with a freq. response

$$H_d(e^{j\omega}) = 1 \quad \text{for} \quad \frac{\pi}{4} \le |\omega| \le \frac{3\pi}{4}$$
$$= 0 \quad \text{o.w}$$

Find the values of $h(n)$ for $N=11$. & plot the freq. response.

Sol



$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) \cdot e^{j\omega n}\, d\omega.$$

$$= \frac{1}{2\pi}\left[\int_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} 1 \cdot e^{j\omega n}\, d\omega + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 1 \cdot e^{j\omega n}\, d\omega\right]$$

$$h_d(n) = \frac{\sin\frac{3\pi}{4}n - \sin\frac{\pi}{4}n}{\pi n}, \quad -\infty \le n \le \infty.$$

$$h(n) = h_d(n) \quad \text{for} \quad -\left(\frac{N-1}{2}\right) \le n \le \frac{N-1}{2}$$
$$= 0 \quad \text{o.w}$$

$N = 11$, $\frac{N-1}{2} = \frac{11-1}{2} = 5$.

$h(n) = h_d(n)$ for $-5 \le n \le 5$
$= 0$ o.w.

$n=0$, $h(0) = \frac{0}{0}$ form
use L hospitalls rule

$$= \frac{\frac{3\pi}{4} - \frac{\pi}{4}}{\pi} = \frac{2\pi/4}{\pi} = \frac{1}{2} = 0.5$$

$n=1$, $h(1) = h(-1) = \frac{\sin\frac{3\pi}{4} - \sin\frac{\pi}{4}}{\pi} = 0$.

$h(2) = h(-2) = -0.3183$

$h(3) = h(-3) = 0$

$h(4) = h(-4) = 0$

$h(5) = h(-5) = 0$

$\therefore$ T.F of the filter is

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) [z^n + z^{-n}]$$

$$H(z) = 0.5 - 0.3183 (z^2 + z^{-2})$$

→ The T.F of real zable filter are

$$H'(z) = z^{-5} [0.5 - 0.3183 (z^2 + z^{-2})]$$
$$= -0.3183 z^{-3} + 0.5 z^{-5} - 0.3183 z^{-7}$$

→ The filter coefficients of the causal filter are

$h(0) = h(10) = h(1) = h(9) = h(2) = h(8) = h(4) = h(6) = 0$

$h(3) = h(7) = -0.3183, \quad h(5) = 0.5$

→ The freq. response of symmetrical impulse response & N is odd is

$$H(e^{j\omega}) = e^{-j\left(\frac{N-1}{2}\right)\omega}\left[h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} 2h(n)\cdot \cos\left(\omega\left(\frac{N-1}{2}-n\right)\right)\right]$$

$$|H(e^{j\omega})| = 0.5 - 0.6366 \cos 2\omega$$

12, 17, 26, 32, 4,
41, 2, 4, 5, 10-5

Frequency Sampling method of designing FIR filters:-

→ Let $h(n)$ be the filter coefficients of an FIR filter & $H(K)$ is the DFT of $h(n)$. Then,

$$h(n) = \frac{1}{N}\sum_{K=0}^{N-1} H(K)\cdot e^{\frac{j2\pi nK}{N}}, \quad n=0,1,\ldots N-1 \quad -(1)$$

&

$$H(K) = \sum_{K=0}^{N-1} h(n)\cdot e^{\frac{-j2\pi nK}{N}}, \quad K=0,1,\ldots,N-1 \quad -(2)$$

→ The DFT samples $H(K)$ for an FIR sequence can be regarded as samples of the filter Z-T evaluated at N points equally spaced around the unit cle.

$$H(K) = H(Z)\Big|_{Z=e^{j2\pi K/N}} \quad -(3)$$

→ The T.F of FIR filter is

$$H(Z) = \sum_{n=0}^{N-1} h(n)\cdot Z^{-n} \quad -(4)$$

Sub $h(n)$ in $H(Z)$ i.e (1) in (4).

$$H(Z) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{K=0}^{N-1} H(K)\cdot e^{\frac{j2\pi nK}{N}}\right]\cdot Z^{-n}$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} H(K)\cdot\sum_{n=0}^{N-1}\left(e^{\frac{j2\pi K}{N}}\cdot Z^{-1}\right)^{n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} H(k) \cdot \frac{1 - \left(e^{j\frac{2\pi n k}{N}} z^{-1}\right)^{N}}{1 - e^{\frac{2\pi k}{N}} z^{-1}}$$

$$\boxed{H(z) = \frac{1 - z^{-N}}{N} \cdot \sum_{k=0}^{N-1} \frac{H(k)}{1 - e^{\frac{2\pi k}{N}} z^{-1}}}$$

→ we k.T, $H(k) = H(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}}$.

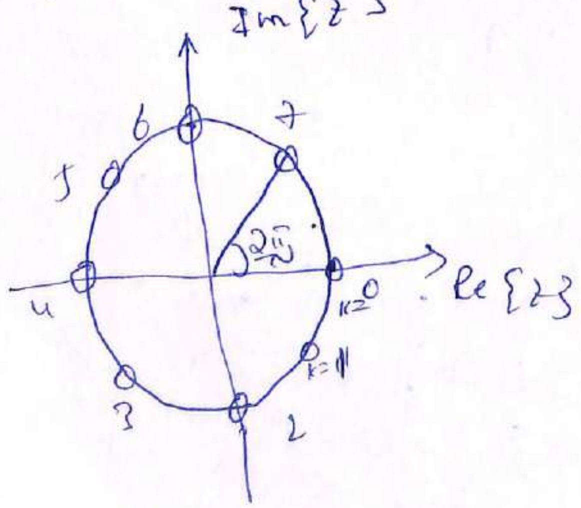
→ Design:-

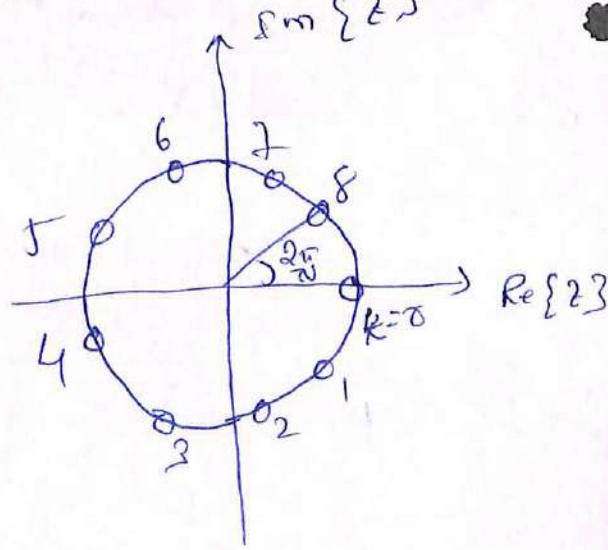
Type. I design:-

→ In this type of design, the freq. samples of the desired response $H_d(e^{j\omega})$ by using.

$$H(k) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k} \quad k = 0, 1, \ldots, N-1$$

$N = 8$ (even) $\qquad\qquad$ $N = 9$ (odd)





$H(\frac{N}{2}) = 0$.

fig: location of DFT samples on unit ○.

→ The freq. samples can be expressed as

$$H(k) = |H(k)| \cdot e^{j\theta(k)}$$

→ For linear phase,

$$\theta(k) = -\alpha\omega \Big|_{\omega = \frac{2\pi}{N}k}, \quad k = 0, 1, \ldots N-1$$

$$= -\left(\frac{N-1}{2}\right) \cdot \frac{2\pi}{N} k$$

$$\theta(k) = -\left(\frac{N-1}{N}\right) \cdot \pi k, \quad k = 0, 1, \ldots N-1$$

→ $$h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H(k) \cdot e^{\frac{j2\pi nk}{N}}, \quad n = 0, 1, \ldots N-1.$$

→ we K.T $H(k)$ satisfies the symmetry property

$$H(N-k) = H^*(k), \quad k = 0, 1, \ldots N-1$$

N even, $H(\frac{N}{2}) = 0$.

→ $$|H(k)| = |H(N-k)|, \quad k = 0, 1, \ldots N-1$$

phase is an odd fn.

→ $$\theta(k) = -\theta(N-k), \quad k = 0, 1, \ldots N-1$$

replace k by N-k in $\theta(k)$.

$$\theta(N-k) = -\left(\frac{N-1}{N}\right)\pi(N-k)$$

$$= (N-1)\pi +$$

$$\theta(N-k) = -\left(\frac{N-1}{N}\right)\pi(N-k)$$

$$\theta(N-k) = -(N-1)\pi + \left(\frac{N-1}{N}\right)\pi k$$

N odd

$$\theta(k) = -\theta(N-k)$$

$$\therefore \theta(k) = (N-1)\pi - \left(\frac{N-1}{N}\right)\pi k.$$

$$\theta(k) = \begin{cases} -\left(\frac{N-1}{N}\right)\pi k, & k = 0, 1, \dots \frac{N-1}{2} \\ (N-1)\pi - \left(\frac{N-1}{N}\right)\pi k, & k = \frac{N+1}{2}, \dots, N-1 \end{cases}$$

N even:-

$$\theta(k) = -\left(\frac{N-1}{N}\right)\pi k, \quad k = 0, 1, 2, \dots, \frac{N}{2}-1$$

$$= (N-1)\pi - \left(\frac{N-1}{N}\right)\pi k, \quad k = \frac{N}{2}+1, \dots N-1$$

$$= 0, \quad k = \frac{N}{2}$$

→ we get for N odd.

$$H(k) = |H(k)| \cdot e^{-j\frac{(N-1)\pi k}{N}}, \quad k = 0, 1, \dots, \frac{N-1}{2}$$

$$= |H(k)| \cdot e^{j\left[(N-1)\pi - \frac{(N-1)\pi k}{N}\right]}, \quad k = \frac{N+1}{2}, \dots, N-1$$

→ we get for N even

$$H(k) = |H(k)| \cdot e^{-j(N-1)\pi k/N}, \quad k = 0, 1, \dots \frac{N}{2}-1$$

$$= |H(k)| \cdot e^{j\left[(N-1)\pi - \frac{(N-1)\pi k}{N}\right]}, \quad k = \frac{N}{2}+1, \dots N-1$$

$$= 0, \quad k = \frac{N}{2}$$

4-28

→ If the filter is to have linear phase, then $h(n)$ must also satisfy the symmetry condition

$$h(n) = h(N-1-n).$$

→ The filter coefficients can be written as,

N odd

$$h(n) = \frac{1}{N}\left\{H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} \mathrm{Re}\left[H(k).\,e^{\frac{j2\pi nk}{N}}\right]\right\},$$

N even

$$h(n) = \frac{1}{N}\left\{H(0) + 2\sum_{k=1}^{\frac{N}{2}-1} \mathrm{Re}\left[H(k).\,e^{\frac{j2\pi nk}{N}}\right]\right\}$$

→ then, system fn. is

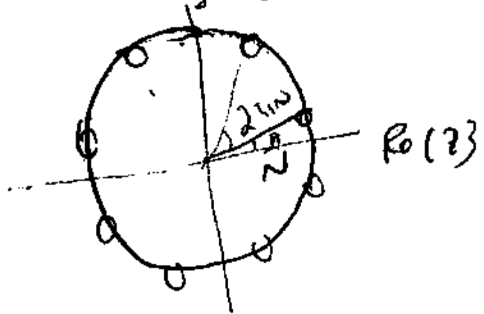
$$H(z) = \sum_{n=0}^{N-1} h(n).\,z^{-n}.$$

Type - II design:—

$$\rightarrow\ H(k) = H_d(e^{j\omega})\Big]_{\omega = \frac{2\pi}{N}(k+\frac{1}{2})},\ k = 0,1,..N-1$$

→ The initial point is located at $\frac{\pi}{N}$, & space b/w 2 points is $\frac{2\pi}{N}$.

N even



1, 6, 12, 17, 18, 24, 4, 32, 4, 5, 6, 46, 50, le-4.5

→ The filter coefficients are given by,

'N' odd
$$h(n) = \frac{2}{N} \sum_{k=0}^{\frac{N-3}{2}} Re\left[H(k) \cdot e^{jn\pi(2k+1)/N}\right]$$

'N' even
$$h(n) = \frac{2}{N} \sum_{k=0}^{\frac{N}{2}-1} Re\left[H(k) \cdot e^{jn\pi(2k+1)/N}\right]$$

→ Determine the filter coefficients h(n) obtained by sampling

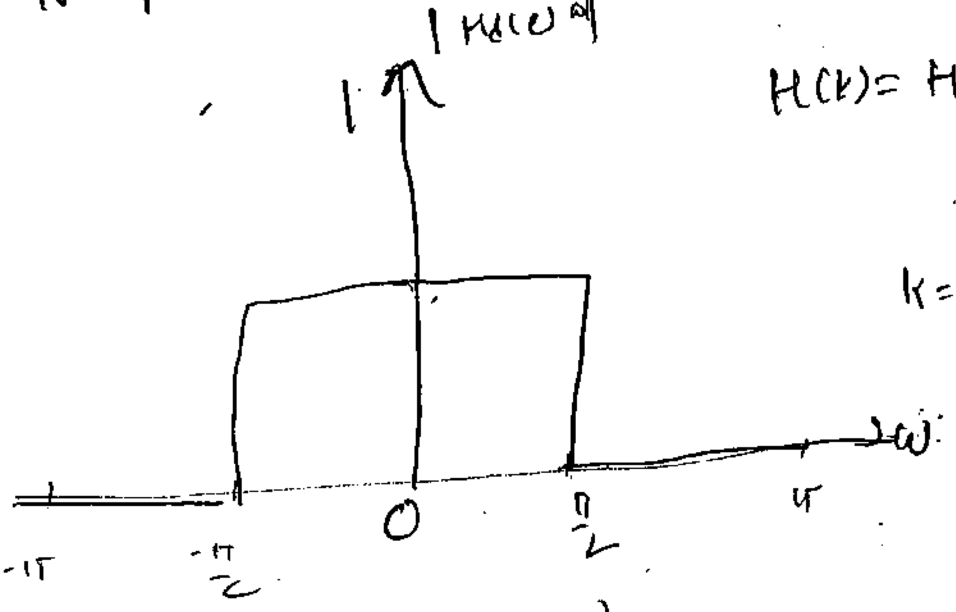
$$H_d(e^{j\omega}) = \begin{cases} e^{-j(N-1)\omega/2} & , \ 0 \le |\omega| \le \frac{\pi}{2} \\ 0 & , \ \frac{\pi}{2} \le |\omega| \le \pi. \end{cases}$$

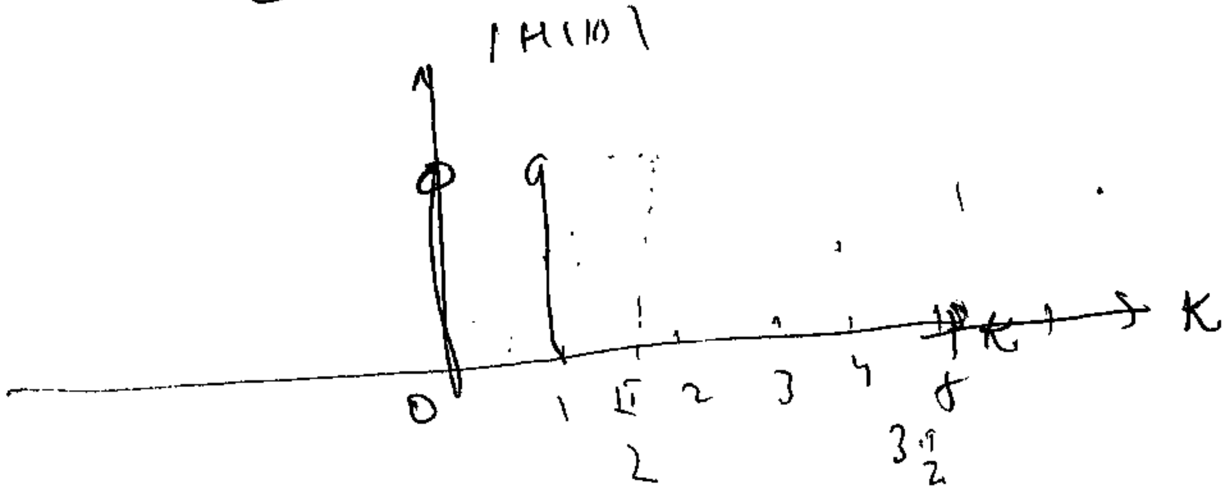
for N = 7.

Sol

$$H(k) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k}$$

k = 0, 1, 2, 3, 4, 5, 6.





$$H(k) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{7}}, \quad k = 0, 1, \dots 6$$

$$|H(k)| = \begin{cases} 1 & \text{for } k = 0, 1, 6 \\ 0 & \text{for } k = 2, 3, 4, 5 \end{cases}$$

Note

$\theta(k) = -$

$$H(k) = \begin{cases} e^{-j6\pi k/7}, & k = 0, 1 \\ e^{j6\pi(?)/6}, & k = 2, 3, 4, 5 \\ & k = 6 \end{cases}$$

$$h(n) = \frac{1}{N}\left\{H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left[H(k) \cdot e^{\frac{j2\pi nk}{N}}\right]\right\}$$

$$= \frac{1}{7}\left\{1 + 2\sum_{k=1}^{3} Re\left\{e^{-\frac{6\pi k}{7}} e^{\frac{j2\pi nk}{7}}\right\}\right\}$$

$$= \frac{1}{7}\left[1 + 2\sum_{k=1}^{3} Re\left\{e^{j\frac{2\pi}{7}k(n-3)}\right\}\right]$$

$$h(n) = \frac{1}{7}\left[1 + 2\sum_{k=1}^{3} \cos\frac{2\pi k}{7}(n-3)\right]$$

$h(0) = h(6) = -0.11456$

$h(1) = h(5) = \frac{1}{7}(1 + 2\cos\frac{4\pi}{7}) = 0.07928$

$h(2) = h(4) = 0.321$

$h(3) = 0.48257.$

0.15, 0.16, 0.17, 26, 32, 4, 5, 6
9, 42, 44, 8, 79 le-5

→ Using freq. sampling method, design a BPF with the following specifications.

Sampling freq. $F = 8000$ Hz

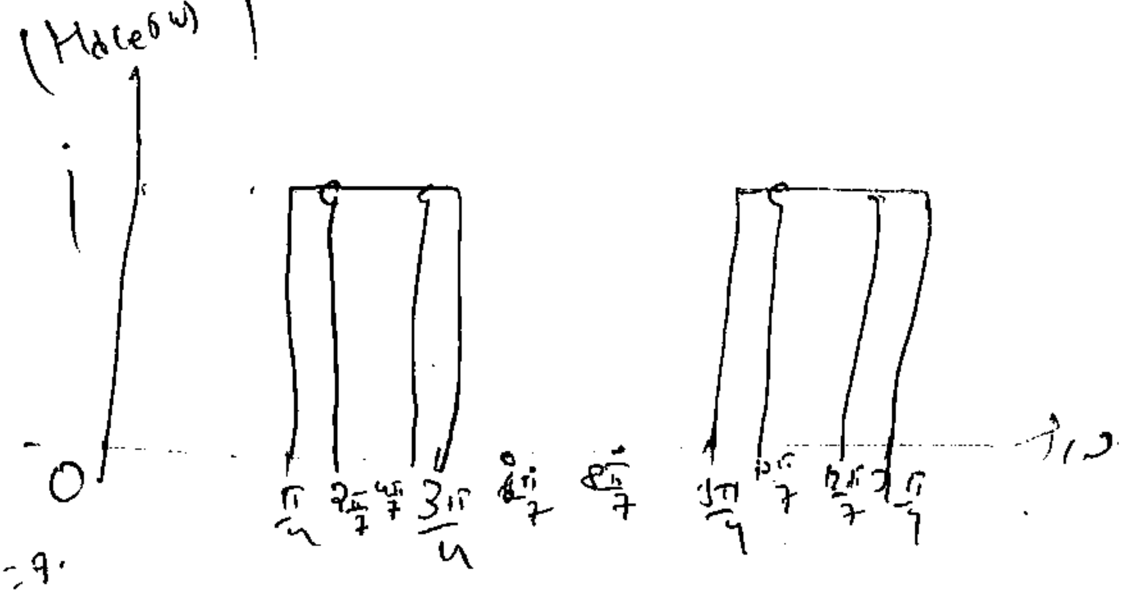
Cut-off freq. $f_{c1} = 1000$ Hz

$f_{c2} = 3000$ Hz

Determine the filter coeff. for $N=7$.

Sol $\omega_{c1} = \dfrac{2\pi f_{c1}}{F} = \dfrac{2\pi \times 1000}{8000} = \dfrac{\pi}{4}$

$\omega_{c2} = \dfrac{3\pi}{4}$



$N=7$.

$\omega_k = \frac{2\pi}{7}$; 0, 2, 4 ;

$H(k) = \begin{cases} 1, & k = 1, 2, 5, 6 \\ 0, & k = 0, 3 \end{cases}$

3, 4, 5, 7, 12, 13, 14, 17, 22, 3, 5, 6, 7, 9, 30, 4, 3, 4, 6, 7, 8, 12, 3, 4, 5, 7, 9, 5, 1, k = 2, 3, 5

$h(n) = \frac{2}{7}\left[\cos \frac{2\pi}{7}(3-n) + \cos \frac{4\pi}{7}(3-n)\right]$

$h(0) = h(6) = -0.07928$

$h(1) = h(5) = -0.321$, $h(2) = h(4) = 0.11456$

$h(3) = 0.57$

Designing of filters using Kaiser window:-

LPF:-

Pass band ripple $\rightarrow \alpha_p = 20 \log \frac{1+\delta}{1-\delta}$

stop band ripple $\rightarrow \alpha_s = -20 \log \delta$

transition width.

$\delta = \min(\delta_1, \delta_2)$, $B = \omega_s - \omega_p$

$\delta_1 = 10^{-0.05 \alpha_s'}$, $\delta_2 = \dfrac{10^{0.05 \alpha_p'} - 1}{10^{0.05 \alpha_p'} + 1}$

1, Determine $h_d(n)$ using Fourier series method for an ideal freq. response

$$H(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| \le \omega_c \\ 0 & \text{for } \omega_c \le |\omega| \le \omega_{sf}. \end{cases}$$

where $\omega_c = \frac{1}{2}[\omega_p + \omega_s]$

2, Choose δ.

3, Cal. $\alpha_s = -20 \log_{10} \delta$

4, Determine the parameter α from.

~~α~~ 0

1, 4, 6, 11, 25, 7,
34, 5, 9, 44, 7,
1c-3

$$\alpha = 0 \longrightarrow \text{for } \alpha_s \le 21$$

$$= 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21) \rightarrow 21 \le \alpha_s \le 50$$

$$= 0.1102(\alpha_s - 8.7) \longrightarrow \alpha_s > 50$$

5. Choose the parameter D as.

$$D = \begin{cases} 0.9222 & \text{for } \alpha_s \le 21 \\ \dfrac{\alpha_s - 7.95}{14.36} & \text{for } \alpha_s > 21 \end{cases}$$

6. Choose the 'N'

$$N \ge \frac{\omega_{sf} D}{B} + 1$$

7. Compute window seq.

8.

$$h(n) = \omega_k(n) \cdot h_d(n)$$

9. T.F

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} \left[h(0) + 2 \sum_{n=0}^{\frac{N-1}{2}} h(n) (z^n + z^{-n}) \right]$$

10 Mag. res.

→ for HPF,

$$B = \omega_p - \omega_s \;\&$$

$$H(e^{j\omega}) = \begin{cases} 0 & \text{for } |\omega| < \omega_c \\ 1 & \text{for } \omega_c \leq |\omega| \leq \frac{\omega_{sf}}{2} \end{cases}$$

$$\omega_c = \frac{1}{2}[\omega_s + \omega_p]$$

→ for BPF,

$$H(e^{j\omega}) = \begin{cases} 0 & \text{for } 0 \leq |\omega| \leq \omega_{c_1} \;\&\; \omega_{c_2} < |\omega| \leq \frac{\omega_{s_f}}{2} \\ 1 & \text{for } \omega_{c_1} \leq |\omega| \leq \omega_{c_2} \end{cases}$$

$$\omega_{c_1} = \omega_{p_1} - \frac{B}{2}, \quad \omega_{c_2} = \omega_{p_2} + \frac{B}{2}.$$

$$B = \min\left[(\omega_{p_1} - \omega_{s_1}), (\omega_{s_2} - \omega_{p_2})\right]$$

→ for BR F,

$$B = \min\left[(\omega_{s_1} - \omega_{p_1}), (\omega_{p_2} - \omega_{s_2})\right]$$

$$H(e^{j\omega}) = \begin{cases} 1 & \text{for } 0 \leq |\omega| \leq \omega_{c_1} \;\&\; \omega_{c_2} < |\omega| \leq \frac{\omega_{sf}}{2} \\ = 0 & \text{for } \omega_{c_1} \leq |\omega| \leq \omega_{c_2} \end{cases}$$

$$\omega_{c_1} = \omega_{p_1} + \frac{B}{2}, \quad \omega_{c_2} = \omega_{p_2} - B/2.$$

→ Design an FIR LP filter satisfying the following specifications: using Kaiser window

$\alpha_p \leq 0.1$ db, $\alpha_s \geq 44$ db

$\omega_p = 20$ rad/sec $\omega_s = 30$ rad/sec, $\omega_{sf} = 100$ rad

Sol

$$B = \omega_s - \omega_p = 10 \text{ rad/sec}$$

$$\omega_c = \frac{1}{2}(\omega_p + \omega_s) = 25 \text{ rad/sec}$$

$$\omega_c = \omega_c T = \omega_c \cdot \frac{2\pi}{\omega_{sf}} = 25\left(\frac{2\pi}{100}\right) = \frac{\pi}{2}$$

1)

$$H(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| \leq \frac{\pi}{2} \\ 0 & \text{otw.} \end{cases}$$

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} 1 \cdot H(\omega) \cdot d\omega = \frac{\sin \frac{\pi}{2} n}{\pi n}$$

2)

$$\delta_1 = 10^{-0.05 \alpha_s'} = 10^{-0.05(44)} = 6.3093 \times 10^{-3}$$

$$\delta_2 = \frac{10^{0.05 \alpha_p'} - 1}{10^{0.05 \alpha_p'} - 1} = 5.7563 \times 10^{-3}$$

$$\delta = \min(\delta_1, \delta_2) = 5.7563 \times 10^{-3}$$

Multirate Digital Signal processing :-

I. Introduction :-

→ In many practical applications of DSP, we have to change the sampling rate of a signal, either increasing it or decreasing it by some amount. Eg: telecommunication

→ There is a requirement to process the various signals at different rates with different B.W's.

● Def: The process of converting a signal from a given rate to a different rate is called "sampling rate conversion".

→ Def: Systems that employ multiple sampling rates in the processing of digital signals are called "multirate DSP systems".

→ Sampling rate conversion of a digital signal can be
● accomplished by

i) pass digital signal to D/A converter & then resample & then pass to ADC. Drawback :- more signal distortion b'cos of DAC & ADC.

ii) In digital domain. { Decimation (↓D), Interpolation (↑I).

→ Sampling :- A continuous-time signal $x(t)$ can be converted into a discrete time signal $x(nT)$ by sampling it at regular intervals of time with sampling period 'T'.

$$x(nT) = x(t)|_{t=nT}, \quad -\infty < n < \infty$$

→ **Sampling Theorem:-** It states that a band limited signal $x(t)$ having finite energy, which has no spectral components higher that f_m Hz can be completely reconstructed from its samples taken at a rate of $2f_m$ or more samples / sec.

$$\text{Nyquist rate} = 2f_m$$

$$\text{" Period} = 1/2f_m.$$

II. Down Sampling & Decimation:-

Down Sampling:-

→ Reducing the sampling rate of a discrete-time signal is called down sampling.

→ The sampling rate of the discrete-time signal can be reduced by a factor 'D' by taking every Dth value of the signal.

→ Mathematically, down sampling is represented by.

$$y(n) = x(Dn).$$

$$x(n) \longrightarrow \boxed{\downarrow D} \xrightarrow{y(n) = x(Dn)}$$

fig: down sampling.

Eg:- $x(n) = \{1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots\}$

$x(2n) = \{1, 3, 2, 1, 3, \ldots\}$

$x(3n) = \{1, 1, 1, 1, \ldots\}$

S-2

→ **Decimation:-** Decimation (Sampling rate Compression) is the process of decreasing the sampling rate by an integer factor D by keeping every Dth sample & removing $D-1$ samples.

→

fig: Block diagram of decimator.

Spectrum of down sampled Signal:-

→ Let us consider the i/p signal $x(n)$.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

→ The down sampled signal $y(n)$ is obtained by multiplying the sequence $x(n)$ with a periodic train of impulses $p(n)$ with a period of D & then leaving $D-1$ zeros b/w each sample.

→ The periodic train of impulses is given by

$$p(n) = \begin{cases} 1, & n = 0, \pm D, \pm 2D, \ldots \\ 0 & \text{o.w} \end{cases}$$

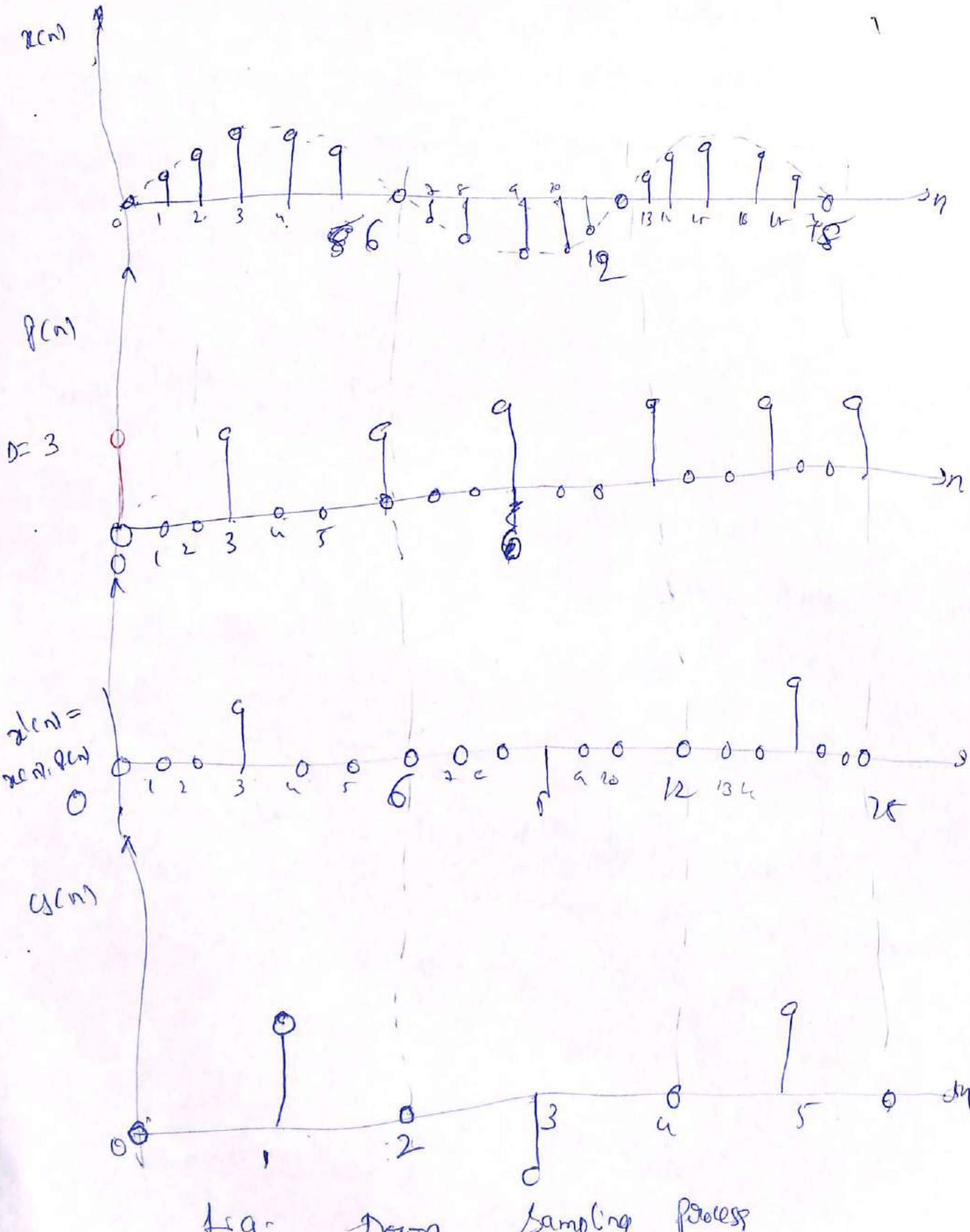


Fig: Down Sampling Process

→ The DFS rep. of p(n) is

$$p(n) = \frac{1}{D} \sum_{k=0}^{D-1} e^{j2\pi nk/D}$$

$x'(n) = x(n)\,P(n).$

→

$$x'(n) = \begin{cases} x(n), & n = 0, \pm D, \pm 2D, \ldots \\ 0 & o.w \end{cases}$$

→ leave D-1 zeros b/w each sample,

$$y(n) = x'(nD)$$

$$= x(nD)\,P(nD)$$

$$= x(nD)$$

→

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n)\,z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x'(nD)\cdot z^{-n}$$

$nD = P.$

$$Y(z) = \sum_{P=-\infty}^{\infty} x'(P)\cdot z^{-P/D}$$

Replace P by n

$$Y(z) = \sum_{n=-\infty}^{\infty} x'(n)\cdot z^{-n/D}.$$

$$Y(z) = \sum_{n=-\infty}^{\infty} x(n)\cdot P(n)\cdot z^{-n/D}$$

$$Y(z) = \sum_{n=-\infty}^{\infty} x(n) \left[\frac{1}{D} \sum_{K=0}^{D-1} e^{j2\pi nk/D}\right] \cdot z^{-n/D}$$

$$Y(z) = \frac{1}{D} \sum_{k=0}^{D-1} \sum_{n=-\infty}^{\infty} x(n) \cdot \left(e^{-j\frac{2\pi k}{D}} \cdot z^{1/D}\right)^{-n}$$

We KT $X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$

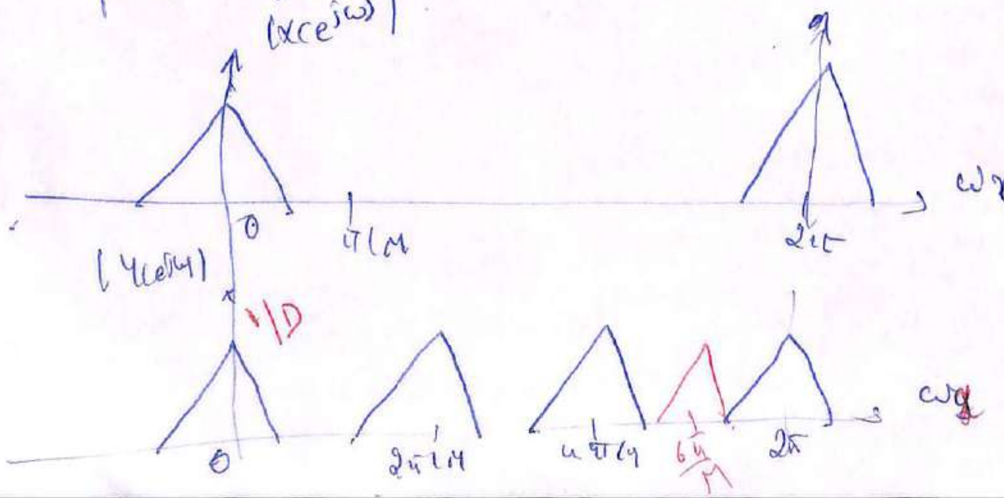
$$Y(z) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(e^{-j\frac{2\pi k}{D}} z^{1/D}\right)$$

Sub $z = e^{j\omega}$

$$Y(e^{j\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(e^{-j\frac{2\pi k}{D}} e^{j\omega/D}\right)$$

$$Y(e^{j\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(e^{j(\omega - 2\pi k)/D}\right)$$

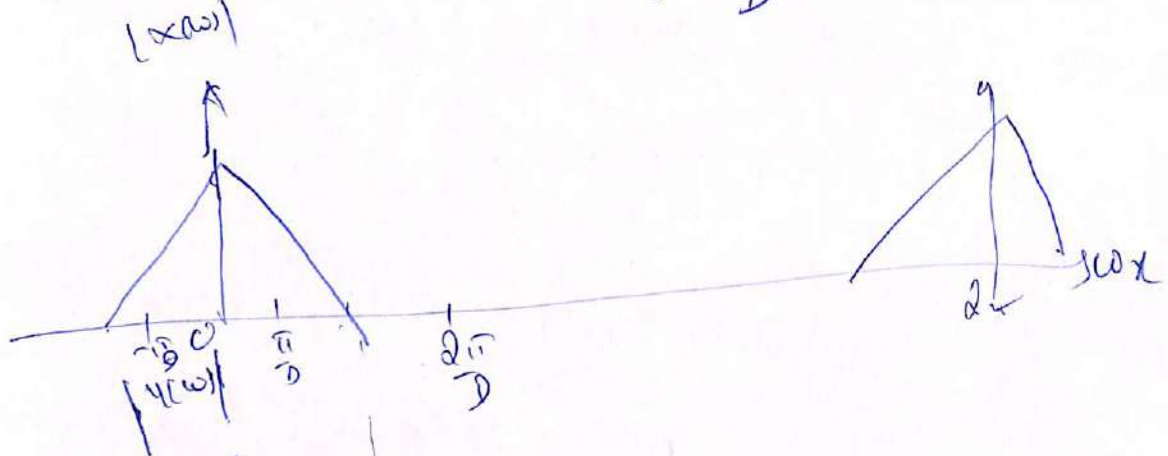
∴ F.T of O/P signal is a sum of D uniformly shifted & stretched versions of $X(e^{j\omega})$ scaled by a factor $1/D$.

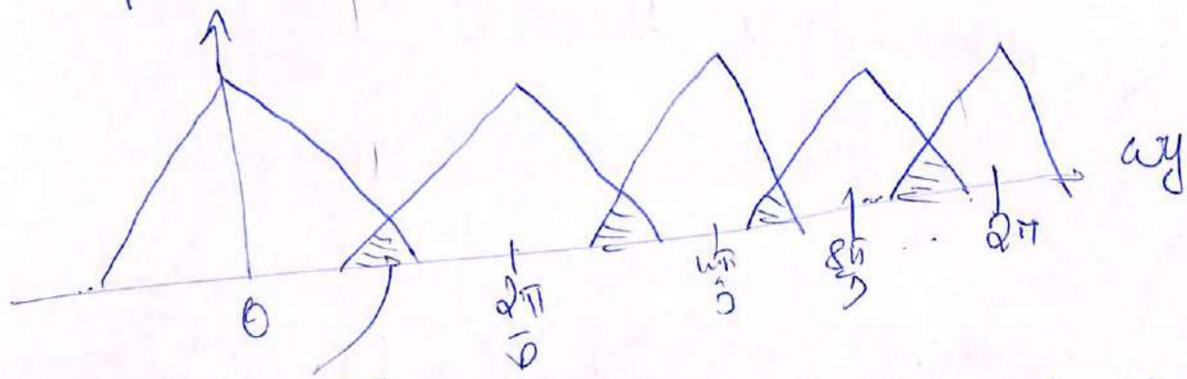


→ Aliasing effect & Anti-aliasing filter :-

→ If the original signal spectrum is not bandlimited to $\omega = \frac{\pi}{D}$, then the spectrum obtained after down sampling will overlap. This overlapping of spectra is called aliasing.

$F_y = \frac{F_x}{D}$ ~~$F_y = D \cdot F_x$~~





→ To avoid this, the signal x(n) must be band limited to $\pm \pi/D$. So, a LPF with cut-off freq. $\frac{\pi}{D}$ is used prior to down sampling.

→ This LPF which is connected before downsampler to prevent the effect of aliasing by band limiting the i/p signal is called the anti-aliasing filter.

Up Sampling & Interpolation :-

→ Increasing the sampling rate of a discrete-time signal is called upsampling.

→ The sampling rate of a discrete-time signal can be increased by a factor 'I' by placing I-1 equally spaced zeros b/w each pair of samples. or by interpolating I-1 new samples b/w successive values of the signal.

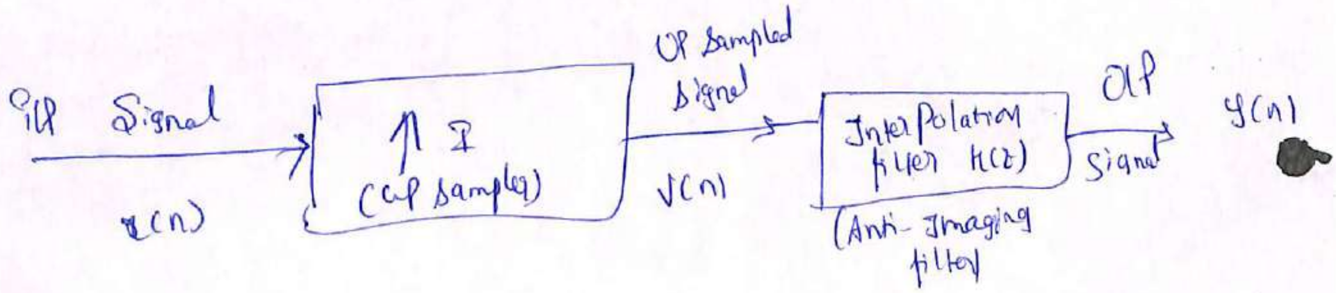
→

Eg:- $x(n) = \{1, 2, 3, 1, 2, 3\}$

$\underline{\underline{I=2}}$ $\quad y(n) = x\left(\frac{n}{2}\right) = \{1, 0, 2, 0, 3, 0, 1, 0, 2, 0, 3, 0\}$

$$v(n) = \begin{cases} x\left(\frac{n}{I}\right), & n = 0, \pm I, \pm 2I, \dots \\ 0, & \text{o.w.} \end{cases}$$

→



$$V(z) = \sum_{n=-\infty}^{\infty} v(n)\, z^{-n}$$

$$V(z) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{I}\right) \cdot z^{-n}$$

let $\frac{n}{I} = m.$ $\quad m \to -\infty$ to ∞

$\Rightarrow n = mI.$

$$V(z) = \sum_{m=-\infty}^{\infty} x(m) \cdot z^{-mI}$$

$$= \sum_{m=-\infty}^{\infty} x(m)(z^{I})^{-m}$$

$$V(z) = X(z^{I})$$

→ for freq. response, sub $z = e^{j\omega}$

$$V(e^{j\omega}) = X((e^{j\omega})^{I})$$

$$V(e^{j\omega}) = X(e^{j\omega I})$$

or $V(\omega_y) = X(\omega_y I)$.

→ ω_y → freq. variable relative to new sampling rate F_y. $\omega_y = \frac{2\pi f}{F_y}$

→ Relation b/w original sampling rate F_x & new sampling rate is

$F_y = F_x \cdot I$, $\omega_x = \frac{2\pi f}{F_x}$

$$\boxed{\begin{array}{l} \cancel{\omega_y = \omega} \\ \frac{2\pi f}{F_y} = \frac{\cancel{2\pi f}}{F_y} \cdot F_x \cdot I \end{array}}$$

$$\frac{1}{F_y} = \frac{1}{F_x \cdot I}$$

Mult with $2\pi f$

$$\frac{2\pi f}{F_y} = \frac{2\pi f}{F_x} \cdot \frac{1}{I} \Rightarrow \omega_y = \frac{\omega_x}{I}.$$

→ We observe that the sampling rate increase, obtained by the addition of $I-1$ zero samples b/w successive values of $x(n)$, results in a signal $v(n)$ whose spectrum $V(\omega_y)$ is an I-fold periodic repetition of the i/p signal spectrum.

→ Since only the freq. components of $x(n)$ in the range $0 \le \omega_y \le \frac{\pi}{I}$ are unique, the images of $X(\omega)$ above $\omega_y = \frac{\pi}{I}$ should be rejected by passing the sequence $v(n)$ through a LPF with freq. response $H_I(\omega_y)$ as

$$H_I(\omega_y) = \begin{cases} C, & 0 \le |\omega_y| \le \frac{\pi}{I} \\ 0, & o.w \end{cases}$$

$C \rightarrow$ scale factor required to properly normalize the o/p sequence.

→ $C = I$ is the desired normalization factor.

$$y(n) = v(n) * h(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(m-k) \cdot v(k)$$

take Z.T

$$Y(z) = Z.T[v(n) * h(n)]$$

$$Y(z) = V(z) \cdot H(z)$$

To get freq. response

$\boxed{5\text{-}6}$

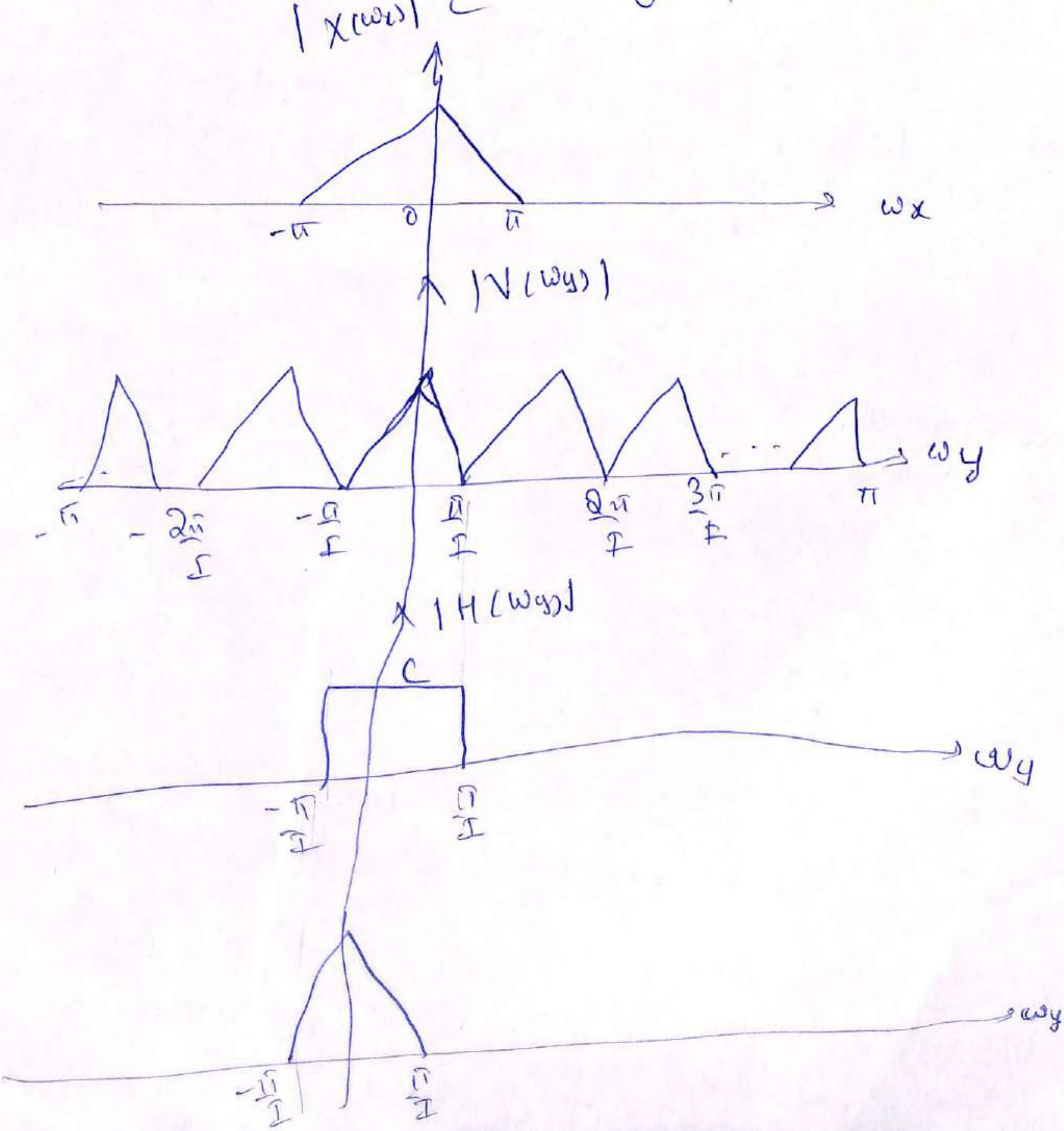
$z = e^{j\omega_y}$

$Y(e^{j\omega_y}) = H_I(e^{j\omega_y}) \cdot V(e^{j\omega_y})$

$Y(\omega_y) = H(\omega_y) \cdot V(\omega_y)$

$Y(\omega_y) = H(\omega_y) \cdot X(I \cdot \omega_y)$

$\Rightarrow Y(\omega_y) = \begin{cases} C \cdot X(I \cdot \omega_y), & 0 \le |\omega_y| \le \frac{\pi}{I} \\ 0, & \text{o.w} \end{cases}$



iii. Sampling Rate Conversion by a Rational Factor I/D:-

→ A sampling rate conversion by the rational factor I/D is accomplished by cascading an interpolator with a decimator.

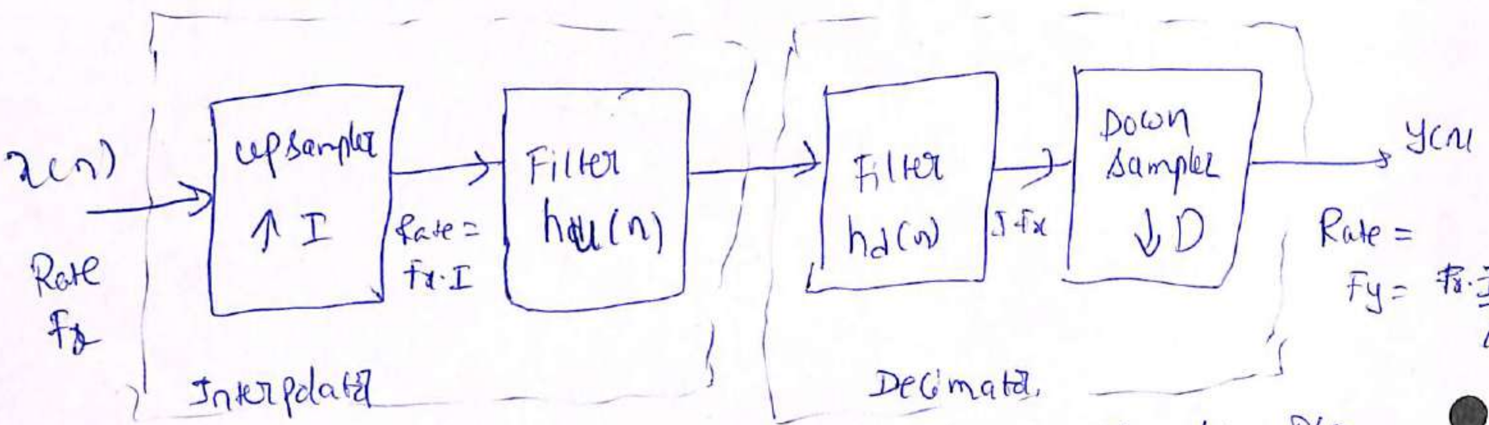


fig: Method for sampling rate conversion by I/D.

→ The importance of performing the interpolation first & the decimation second is to preserve the desired spectral characteristics of x(n).

→ The two filters with impulse response hu(n) & hd(n) are operated at the same rate, i.e I Fx & hence can be combined into a single LPF with impulse res. h(n).

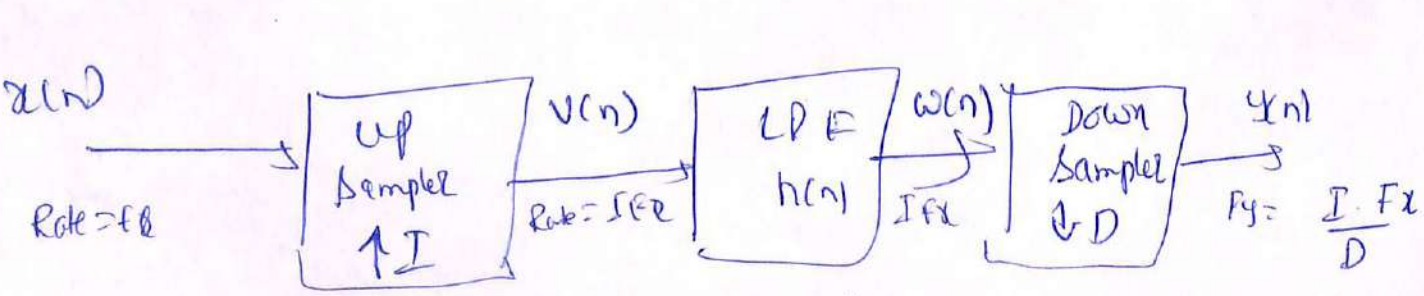


fig: Method for Sampling rate conversion by I/D.

→ The freq. response $H(\omega_y)$ of the combined filter must incorporate the filtering operations for both interpolation & decimation & hence it should have ideally the freq. response characteristic.

9, 27, 9, 36, 44,
7, 8, 9, 10-3,

(5-7)

$$H(\omega_y) = \begin{cases} I, & 0 \le |\omega_y| \le \min\left(\frac{\pi}{D}, \frac{\pi}{I}\right). \\ 0, & o.w \end{cases}$$

$$\omega_y = \frac{2\pi f}{F_y} = \frac{2\pi f}{I F_x} = \frac{\omega_x}{I}.$$

→ Time domain:–

O/P after Interpolator → $v(n) = \begin{cases} x(n/I), & n = 0, \pm I, \pm 2I, \ldots \\ 0, & o.w \end{cases}$

O/P after LPF → $w(n) = v(n) * h(n)$

$$= \sum_{k=-\infty}^{\infty} v(n-k).\, h(k) = \sum_{k=-\infty}^{\infty} v(k).\, h(n-k).$$

$$w(n) = \sum_{k=-\infty}^{\infty} x(k).\, h(n-kI).$$

O/P (aft decimator) $y(n) = w(n \cdot D)$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k).\, h(nD-kI)$$

→ Freq. Domain:–

$V(\omega_y) = X(I\omega_y)$

$W(\omega_y) = H(\omega_y) \cdot X(\omega_y \cdot I)$

$$= \begin{cases} I \cdot X(\omega_y \cdot I), & 0 \le |\omega_y| \le \min\left(\frac{\pi}{D}, \frac{\pi}{I}\right) \\ 0, & o.w \end{cases}$$

$$\rightarrow Y(\omega_y) = \frac{1}{D}\sum_{k=0}^{D-1} W\left(\frac{\omega_y - 2\pi k}{D}\right)$$

Since the linear filter prevents aliasing

$$Y(\omega_y) = \begin{cases} \frac{I}{D} \cdot X\left(\frac{\omega_y}{D}\right), & 0 \le |\omega_y| \le \min\left(\pi, \frac{\pi D}{I}\right) \\ 0, & o.w \end{cases}$$

IV. <u>Sampling Rate Conversion of Bandpass Signals</u>:-

→ Suppose, we want to decimate an integer positioned band pass signal by a factor D with spectrum confined to the bands as:

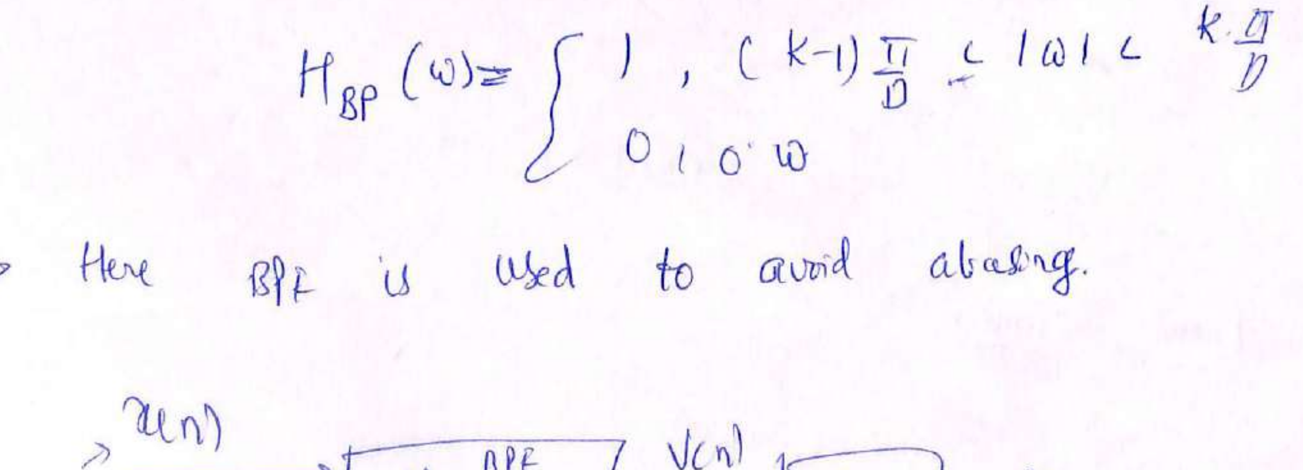
$$(k-1)\frac{\pi}{D} < |\omega| < k\frac{\pi}{D}$$

$k \rightarrow$ +ve integer.

→ A BPF is defined by,

$$H_{BP}(\omega) = \begin{cases} 1, & (k-1)\frac{\pi}{D} < |\omega| < k\frac{\pi}{D} \\ 0, & o.w \end{cases}$$

→ Here BPF is used to avoid aliasing.



K=3 5-8

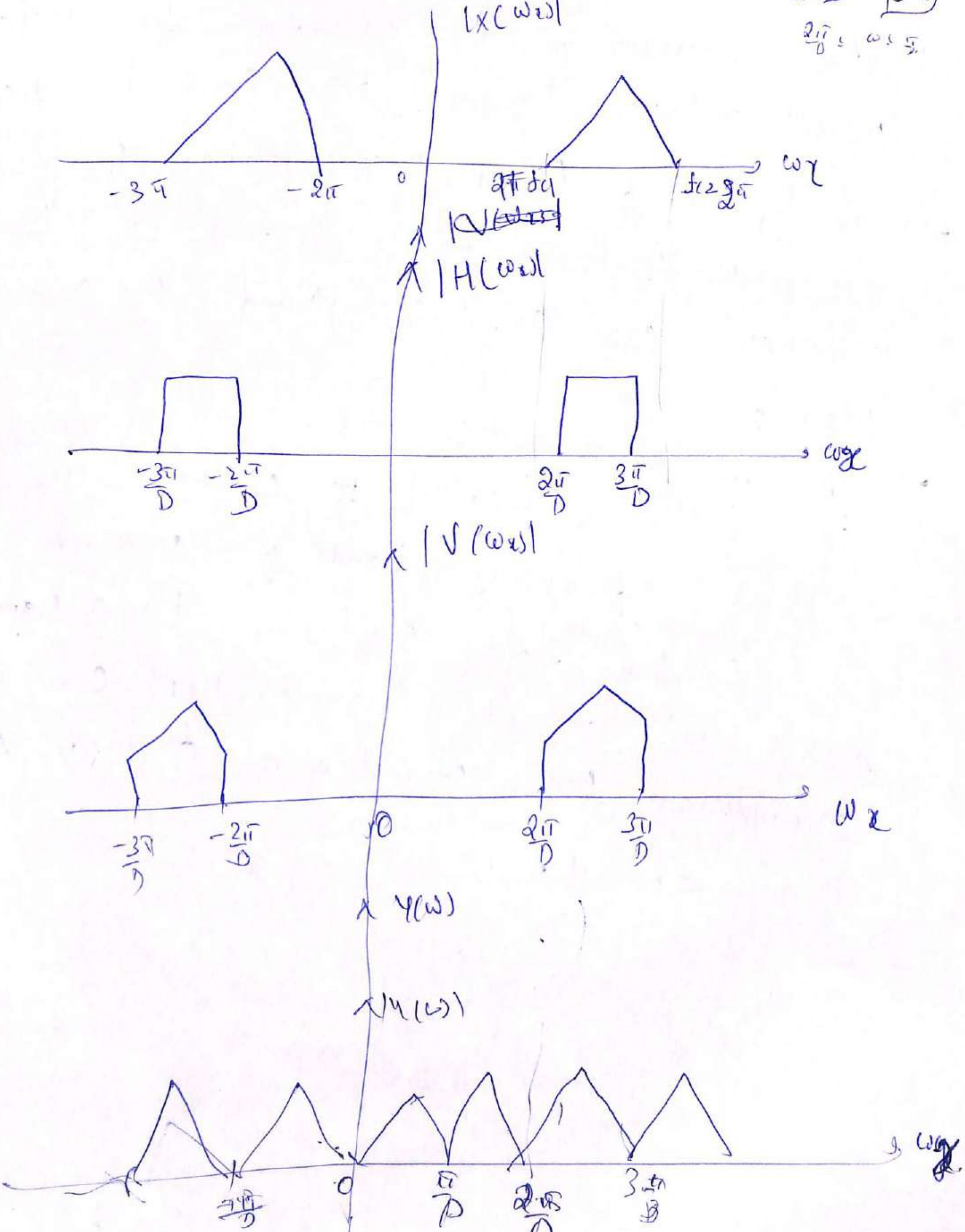


fig: Spectral interpretation of bandpass decimation for integer band positioning.

→ The Spectrum of the decimated Signal $y(n)$ is obtained by scaling the freq, axis $\omega_y = D\omega_x$.

→ The process of bandpass Interpolation by an Integer I is the inverse of that of bandpass decimation & can be accomplished in a similar manner.

→ A BPF proceeding the sampling converter is usually required to isolate the signal frequency bands of interest.

A sampling rate conversion for a Bandpass signal by a rational factor I/D can be accomplished by cascading a decimator with an interpolator.

Applications of Multi-rate Signal Processing :-

→ There are many practical applications. Some of them are

1) Design of phase shifters
2) Interfacing of Digital Systems with different sampling rates
3) Implementation of narrow band LPF
4) Subband coding of Speech Signals.

1) Design of phase shifters :-

→ Suppose, we wish to design a n/w that delays the signal $x(n)$ by a 'k' fraction of a sample.

→ In the freq. domain, the delay corresponds to a linear phase shift of the form,

$$\theta(\omega) = -\frac{K \cdot \omega}{I}$$

→ The design of an all-pass linear phase filter is relatively difficult.

→ So, we can use sampling rate conversion to achieve a delay of $\left(\frac{K}{I}\right) \cdot T_x$ exactly w/o introducing any distortion in the signal.

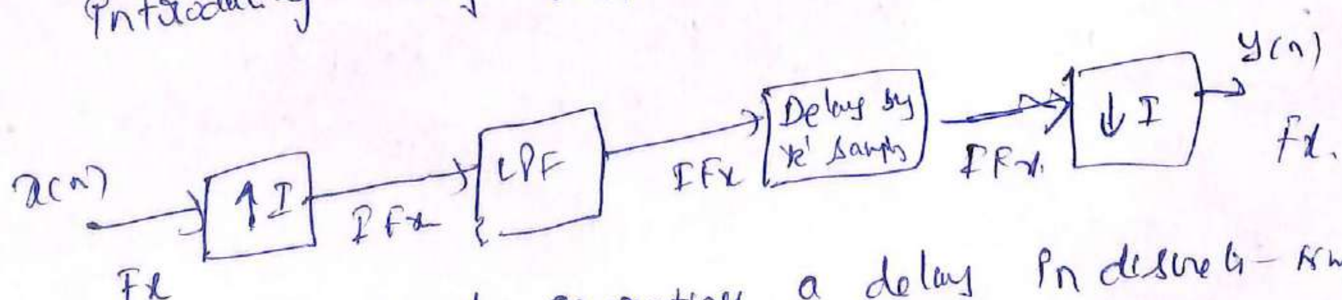


fig: Method of generating a delay in discrete-time signals.

2) <u>Interfacing of Digital</u> Systems with different Sampling <u>rates</u> :-

→ In practice, we get the problem of interfacing two digital systems that are controlled independently operating clocks.

→ However, a simpler approach is one where the interfacing is done by a digital method using the sampling rate conversion.

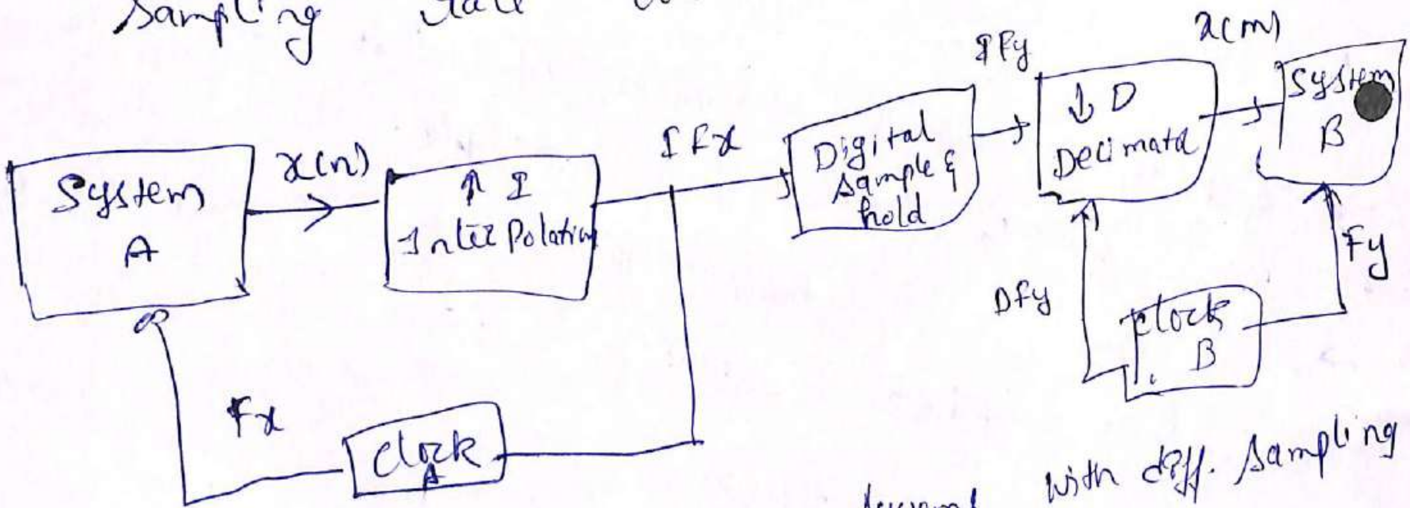


fig: Interfacing of two digital systems with diff. sampling rates

→ The O/P of system A at rate Fx is fed to an interpolator which increases the sampling rate by I
→ The O/P of the interpolator is fed at the rate IFx to a digital sample & hold which serves as the interface to system B at the high sampling rates IFx.
→ Signals from the digital sample & hold are read out into system B at the clock rate D Fy of system B.

3) **Implementation of narrowband LPF :-**

→ A narrow band LPF requires a very large no of coefficients. So, requires a large no of computations & memory locations.

→ To overcome these problems, multirate approach is used in implementing narrow band LPF.

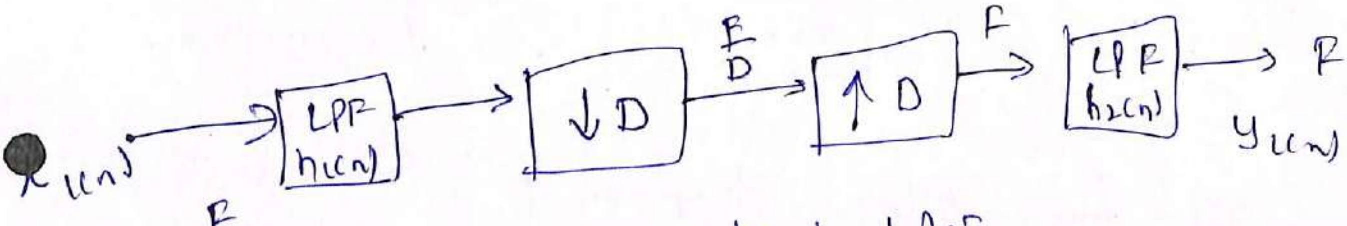


fig 1 A narrow band LPF.

→ To meet the desired specifications of a narrow band LPF, the filters $h_1(n)$ & $h_2(n)$ should be identical with the same pass band & stop band signals.

4) **Sub band Coding of Speech Signals :-**

→ Most of the speech energy is contained in the lower frequencies. So, we would like to encode the lower - freq. band with more bits than the high freq. band.

→ Subband coding is a method where the speech signal is subdivided into several freq. bands & each band is digitally encoded separately.

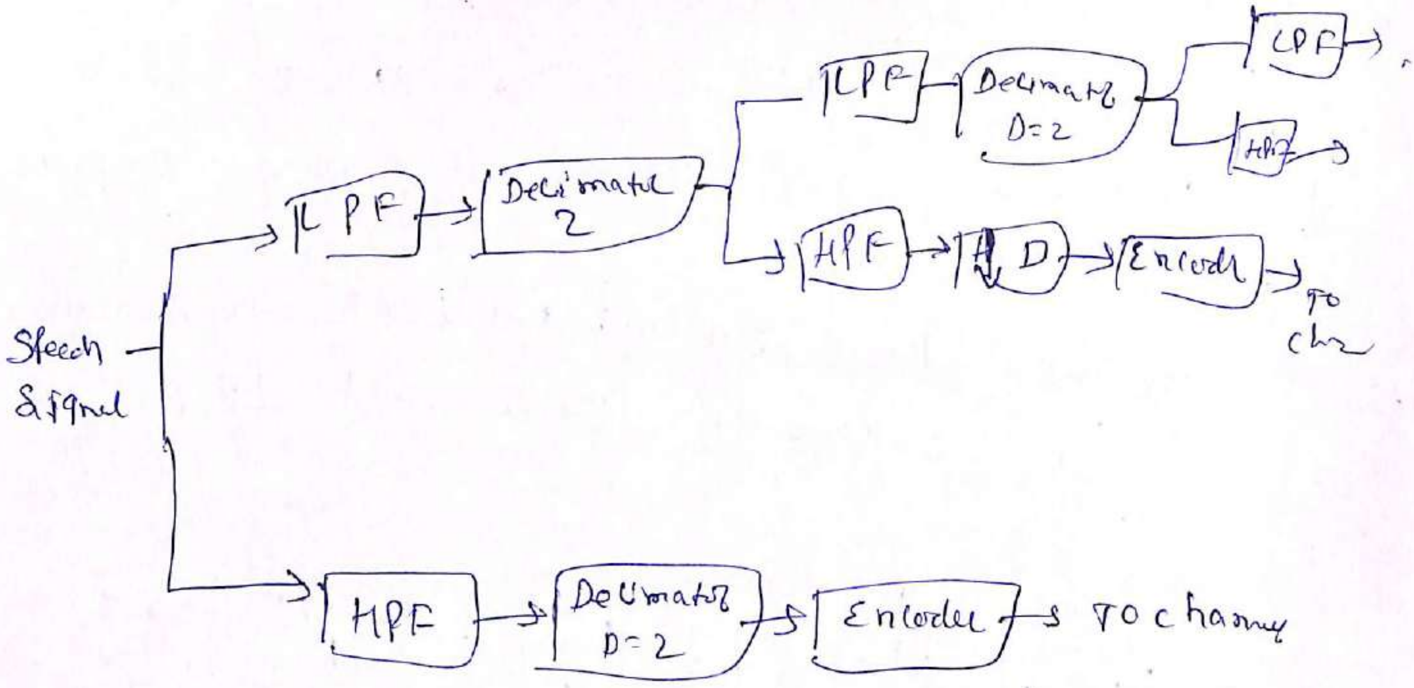


fig: Block diagram of a Subband Speech Encoder

<u>Problems:-</u>

(i) Consider a Signal $x(n) = \sin \pi n \cdot u(n)$.
  i) obtain a signal with a decimation factor 2.
  ii) " Interpolation " "

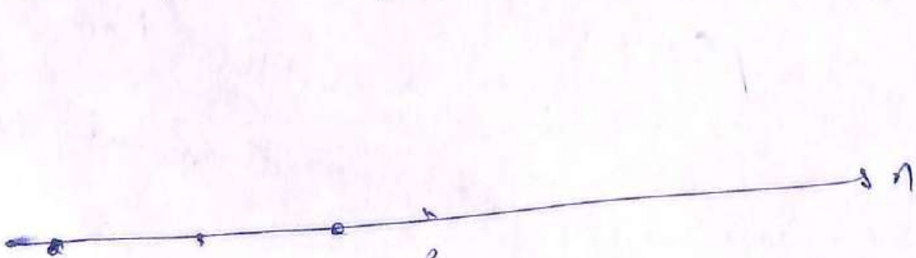
i) $x(n) = \sin \pi n \cdot u(n)$

$y(n) = x(2n) = \sin 2\pi n \cdot u(2n)$.

2, 5, 8, 9, 11, 15, 18, 19, 20, 3, 7, 9, 34, 6, 9, 40, 1, 6, 18, 1e-5

$x(n)$

$y(n) = x(2n)$



1st Stage:-

Pass band : $0 \le F \le 90$ Hz

transition band : $90 \text{ Hz} \le F \le F_1 - F_s$

$\Rightarrow 90 \text{ Hz} \le F \le 5\text{ kHz} - 100$

$90 \text{ Hz} \le F \le 4.91 \text{ kHz}$

$$\Delta f_1 = \frac{4.91 \text{ kHz} - 90}{10 \text{ kHz}} = 0.482$$

$$\delta_p' = \frac{\delta_p}{2} = \frac{10^{-2}}{2} = 0.005$$

$$N_1 = \frac{-10 \log(\delta_p' \cdot \delta_s) - 13}{14.6 \times 0.482} + 1$$

$$N_1 \cong 7$$

2nd Stage:-

Pass band : $0 \le F \le 90$ Hz

transition Band : $90 \text{ Hz} \le F \le 500 - 100$ Hz

$90 \text{ Hz} \le F \le 400$ Hz

$$\Delta f_2 = \frac{400 - 90}{10 \text{ kHz}} = 0.031$$

$$\delta_s' = \frac{\delta_s}{2} = \frac{10^{-3}}{2} = 0.5 \times 10^{-3}$$

$$N_2 = \frac{-10\log(\delta_p \cdot \delta_s') - 13}{14.6\ \Delta f_2} + 1$$

$$N_2 = 90.$$

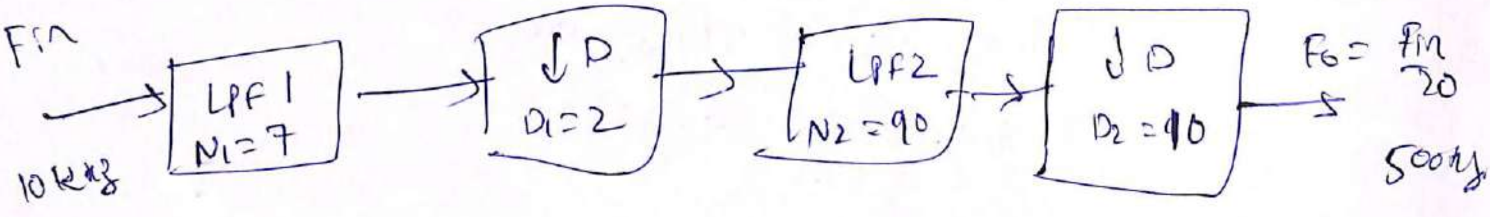


fig: two-Stage decimator

Transpose of above gives 2-stage Interpolator

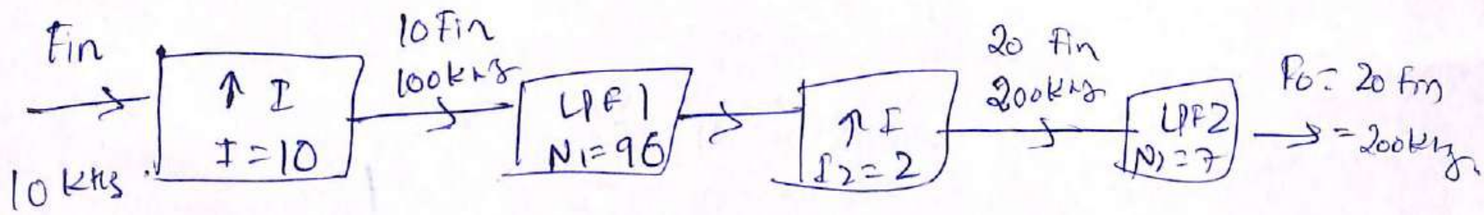


fig: two Stage Interpolator

2, 5, 8, 9, 15, 18, 19,
20, 1, 3, 7, 9, 34, 6,
40, 16, 8, 6-4, 5

Part-B:- FINITE WORD LENGTH EFFECTS IN DIGITAL FILTERS:-

Introduction:-

→ In DSP, the numbers & coefficients are stored in finite length registers. ∴ coefficients & numbers are quantized by truncation or rounding off when they are used.

→ The following errors arise due to quantization of no's:

1) I/P quantization error
2) Product quantization error
3) Coefficient quantization error

1) I/P quantization error:- The conversion of a continuous-time i/p signal into digital value produce an error, which is known as i/p quantization error. This error arises due to the representation of the i/p signal by a fixed no of digits in A/D conversion process.

2) Product quantization error:- This arises at the o/p of a multiplier. Multiplication of a 'b' bit data with a 'b' bit coefficients results a product having 2b bits. Since a 'b' bit register is used, the multiplier o/p must be rounded or truncated to b bits which produces an error.

3) Coefficient Quantization Error:- The filter coefficients are computed to infinite precision in theory. But if they are quantized, the freq. response of the resulting filter may differ from the the desired response leading to instability.

FINITE WORD LENGTH EFFECTS

5-14

1) Discuss in detail the errors resulting from rounding & truncation. or Errors introduced by quantization.

A) Mostly the i/p signal is continuous in time. This signal is to be converted into digital by using ADC.

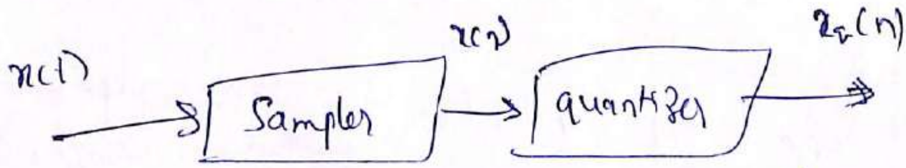


fig: Block diag of ADC

→ The numeric equivalent of each sample x(n) is expressed by a finite no of bits gives $x_q(n)$.

→ The difference between $x_q(n)$ & x(n) is called quantization noise.

$$e(n) = x_q(n) - x(n)$$

→ The common methods of quantization are

1) Truncation

2) Rounding

<u>Truncation</u>:- Truncation is the process of discarding all bits less significant than ~~least~~ least significant bit that is retained.

→ Eg: truncate binary no from 8 bits to 4 bits.

(8 bits) 0.00110011 to 0.0011 (4 bits)

(") 1.01001001 to 1.0100 (")

→ When we truncate the no, the signal value is approximated by the highest quantization level that is not greater than the signal.

Rounding :-

→ Rounding of a no of 'b' bits is accomplished by choosing the rounded result as the 'b' bit no the unrounded is closest to the original no.

Eg:- 0.11010 rounded 3 bits → 0.110 & 0.111

Error due to truncation & rounding :-

Error due to truncation :-

Fixed pt

→ If the quantization method is truncation, the no is approximated by the nearest level that does not exceed the original level.

→ In this case, the error, $x_T - x$ is negative or zero.

→ The error made by truncating a no to b bits which satisfies the inequality.

$$0 \geq x_T - x \geq -2^{-b} \quad -(1)$$

→ Eg:- decimal no 0.12890625 → binary 0.00100001

truncate to 4 bits 0.125 → 0.0010

$x_T - x = -0.00390625 > -2^{-4} = -0.0625$

→ If $x > 0$, the above eqn. satisfies the inequality of eq-(i) for sign magnitude, 1's complement & 2's complement.

→ $x < 0$:–

i) 2's Complement representation:–

→ the magnitude of the negative no using 2's Comp. is

$$x = 1 - \sum_{i=1}^{b} c_i 2^{-i}.$$

→ truncate the no to N bits, then,

$$x_T = 1 - \sum_{i=1}^{N} c_i 2^{-i}.$$

Error,

$$x_T - x = \sum_{i=1}^{b} c_i 2^{-i} - \sum_{i=1}^{N} c_i 2^{-i}.$$

$$x_T - x = \sum_{i=N}^{b} c_i 2^{-i} \geq 0$$

→ ∴ truncation increases the mag. & so error is –ve.

$$\therefore \quad 0 \geq x_T - x \geq -2^{-b}.$$

$$\boxed{0 \geq e \geq -2^{-b}}$$

1's complement :-

→ $$x = 1 - \sum_{i=1}^{b} c_i 2^{-i} - 2^{-b}$$

→ when the no is tranctuated to N bits,

$$x_T = 1 - \sum_{i=1}^{N} c_i 2^{-i} - 2^{-N}.$$

→ $$x_T - x = \sum_{i=N}^{b} c_i 2^{-i} - (2^{-N} - 2^{-b})$$

< 0

$$\therefore \boxed{0 \le x_T - x < 2^{-b}}, \quad x < 0$$

Floating Point ~~Re~~presentation :-

→ In floating point, the effect of truncation only in mantissa.

if $x = 2^c \cdot M$.

then $x_T = 2^c \cdot M_T$.

$$e = x_T - x = 2^c (M_T - M)$$

2's complement rep. of mantissa:-

$$0 \geq M_T - M \geq -2^{-b}$$

$$0 \geq \frac{x_T - x}{2^c} \geq -2^{-b}$$

$e = x_T - x = 2^c(M_T - M)$

$$0 \geq e \geq -2^c \cdot 2^{-b}$$

relative error, $\varepsilon = \frac{x_T - x}{x} = \frac{e}{x}$

$$0 \geq \varepsilon x \geq -2^c \cdot 2^{-b}$$

$$0 \geq \varepsilon \cdot 2^c \cdot M \geq -2^c \cdot 2^{-b}$$

If $M = \frac{1}{2}$, the error is max

$$\therefore 0 \geq \varepsilon \geq -2 \times 2^{-b}, \quad M>0$$

If $M = -\frac{1}{2}$,

$$0 \leq \varepsilon \leq -2 \times 2^{-b}$$

1's complement rep of mantissa:-

$$0 \geq M_T - M \geq -2^{-b}$$

or $0 \geq e \geq -2^{-b} 2^c$

$e = \varepsilon \cdot 2^c M.$

→ Error due to Rounding :- The error due to rounding a number to 'b' bits produces an error,

$e = x_T - x$. which satisfies inequality

Fixed pt:

$$-\frac{2^{-b}}{2} \leq x_T - x \leq \frac{2^{-b}}{2}.$$

Floating pt:-

$$x = 2^c \cdot M$$

$$x_T = 2^c M_T$$

$$e = x_T - x = (M_T - M) \cdot 2^c$$

But for rounding

$$M_T - M$$

$$-\frac{2^{-b}}{2} \leq \cancel{(x_T - x)} \leq \frac{2^{-b}}{2}$$

$$-\frac{2^{-b}}{2} \leq \frac{x_T - x}{2^c} \leq \frac{2^{-b}}{2}$$

$$\varepsilon = \frac{x_T - x}{x}$$

$$-2^c \cdot \frac{2^{-b}}{2} \leq x_T - x \leq 2^c \cdot \frac{2^{-b}}{2}$$

$$-2^c \frac{2^{-b}}{2} \leq \varepsilon x \leq 2^c \cdot \frac{2^{-b}}{2}.$$

$$-2^c \frac{2^{-b}}{2} \leq \varepsilon \, 2^c M \leq 2^c \frac{2^{-b}}{2}$$

1, 2, 3, 4, 5, 6, 7,

$2^1, 2^2, 2^4, 2^8$

$$-\frac{2^{-b}}{2} \leq \varepsilon M \leq \frac{2^{-b}}{2}$$

If $M = \frac{1}{2}$,

$$\boxed{-2^{-b} \leq \varepsilon \leq 2^{-b}}$$

Expression for quantization error, steady state i/p noise power, SNR, o/p noise power:-

→ The quantization error is given by,

$$e(n) = x_q(n) - x(n)$$

$x_q(n)$ → quantized signal after sampling

$x(n)$ → sampled signal.

→ If rounding of a no. is used for quantization to get $x_q(n)$, then error signal satisfies

$$-\frac{q}{2} \le e(n) \le \frac{q}{2}, \qquad q = 2^{-b}.$$

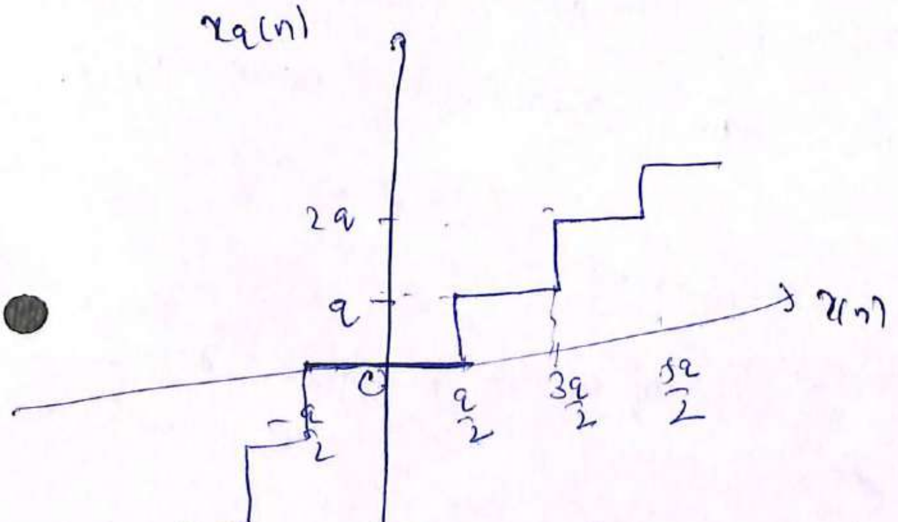


fig: quantization characteristic due to round off

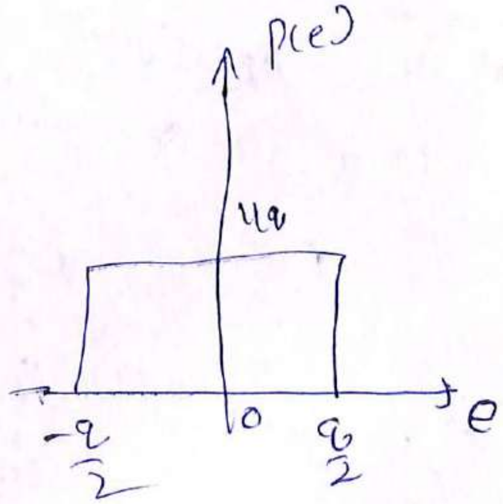
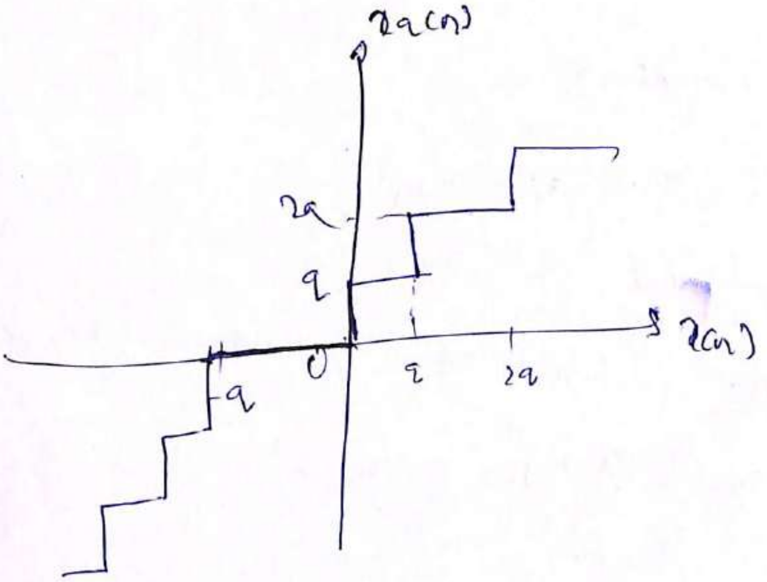


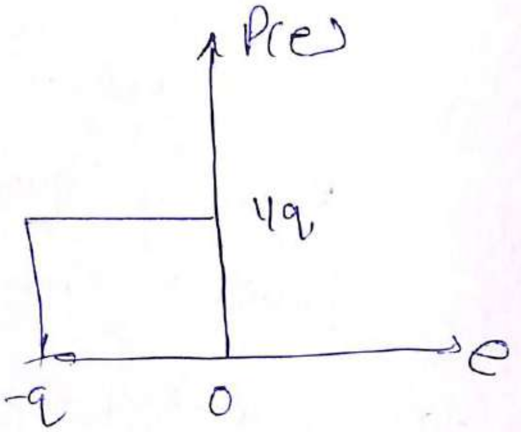
fig: Probability density fn. for round off error.

→ The other type of quantization can be obtained by truncation.

→ Since in truncation, the error $e(n)$ is always negative, & satisfies the inequality,

$$-q \le e(n) < 0$$





<u>I/P noise power:-</u>

→ In digital processing of analog signals, the quantization error is viewed as an additional noise signal.

i.e. $$x_q(n) = x(n) + e(n).$$

→ ∴ ADC O/P is the sum of the i/p signal $x(n)$ & error signal $e(n)$.

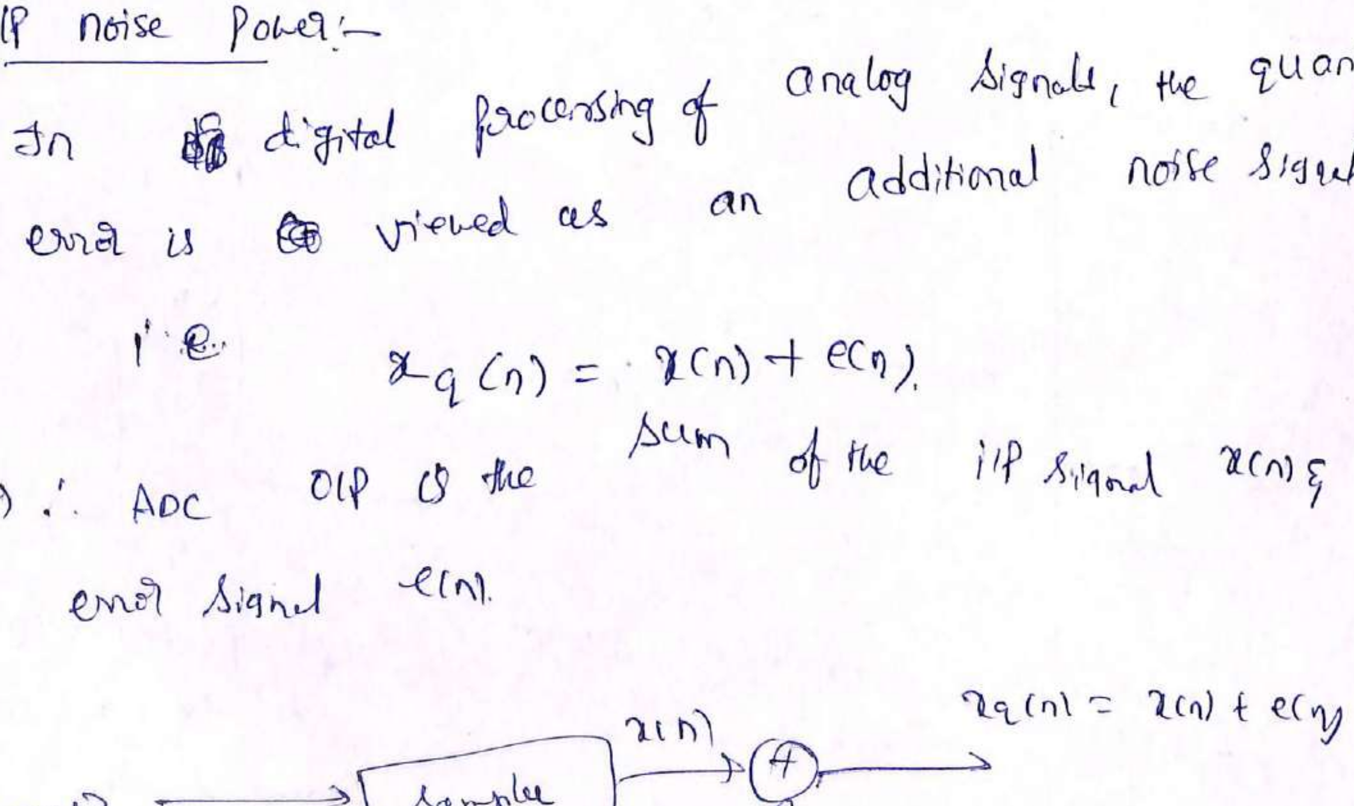


fig: Quantization noise model

→ Round-off is used for quantization:-

$$e(n) = x_q(n) - x(n) \text{ where}$$

$$-\frac{q}{2} \le e(n) \le \frac{q}{2}.$$

→ The variance or power of the error signal e(n) is given by

$$\sigma_e^2 = E[e^2(n)] - (E[e(n)])^2.$$

$E[e^2(n)] \rightarrow$ Mean square value

$E[e(n)] \rightarrow$ Mean of e(n).

$$E[e^2(n)] = \int_{-\infty}^{\infty} e^2(n) \cdot P(e) \cdot de$$

$$= \int_{-q/2}^{q/2} e^2(n) \cdot \frac{1}{q} \cdot de.$$

$$= \frac{1}{q} \cdot \left[\frac{e^3}{3}\right]_{-q/2}^{q/2}$$

$$= \frac{1}{q} \cdot \frac{1}{3} \left[(q/2)^3 - \left(-\frac{q}{2}\right)^3\right]$$

$$E[e^2(n)] = \frac{q^2}{4 \times 3} = \frac{q^2}{12}$$

$$E[e(n)] = \int_{-\infty}^{\infty} e(n) \cdot P(e) \cdot de = \frac{1}{q} \int_{-q/2}^{q/2} e(n)\, de = \frac{1}{q}\left[\frac{e^2}{2}\right]_{-q/2}^{q/2}$$

$$= 0.$$

$$\therefore \sigma_e^2 = \frac{q^2}{12} - 0 = \frac{q^2}{12}.$$

→ we K.T quantization level q for a b bit no is $q = 2^{-b}$.

$$\sigma_e^2 = \frac{(2^{-b})^2}{12} = \frac{2^{-2b}}{12}.$$

Truncation is used for quantization:-

$$p(e) = \begin{cases} \frac{1}{q}, & -q \le e(n) \le 0 \\ 0, & o.w \end{cases}$$

$$\sigma_e^2 = \frac{1}{q} \int_{-q}^{0} e^2(n)\, de - \left[-\frac{q}{2}\right]^2$$

$$= \frac{q^2}{12}$$

$$\therefore \sigma_e^2 = \frac{q^2}{12} = \frac{2^{-2b}}{12}.$$

→ In both case, $\sigma_e^2 = \frac{2^{-2b}}{12}$ i.e It is called as quantization steady state noise power due to i/p

SNR:-

i/p signal $x(n)$ → variance or power $= \sigma_x^2$

noise signal $e(n)$ → " or Pwr $= \sigma_e^2$

$$SNR = \frac{\sigma_x^2}{\sigma_e^2} = \frac{\sigma_x^2}{2^{-2b}/12} = 12(2^{2b}\sigma_x^2)$$

In db,

$$SNR(dB) = 10 \log \frac{\sigma_x^2}{\sigma_e^2} = 10 \log (12 (2^{2b} \sigma_x^2))$$

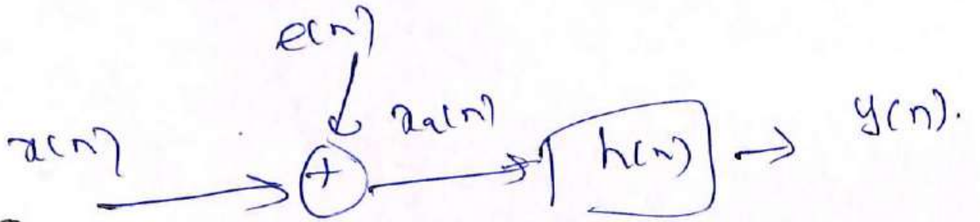
$$= 10 \log \sigma_x^2 + 10.79 + 6.02b.$$

→ SNR increases approximately 6 db for each bit added to reg length.

O/P Noise Power:-

→ The I/P quantization noise propagates to the O/P of the digital filter.

→ The quantized i/p to a digital system with $h(n)$ can represented as, $y(n) = x_q(n) * h(n)$



→ $y(n) = x_q(n) * h(n) = (x(n) + e(n)) * h(n)$

$$= x(n) * h(n) + \underbrace{e(n) * h(n)}_{\varepsilon(n)}$$

$\varepsilon(n)$ → O/P noise due to quantization of i/p

$$\varepsilon(n) = e(n) * h(n).$$

$$= \sum_{k=0}^{n} h(k) \cdot e(n-k).$$