LECTURE NOTES

ON

OPERATIONS RESEARCH (AME021)

IV B.Tech. VI - SEM

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UNIT-I INTRODUCTION ANDALLOCATION

TERMINOLOGY

The What is Operation Research

- Definition
- Scope

Operations Research is the science of rational decision-making and the study, design and integration of complex situations and systems with the goal of predicting system behavior and improving or optimizing system performance.

The formal activities of operation research were initiated in England during World War II to make decisions regarding the best utilization of war material. After the war the ideas advanced in military operations were adapted to improve efficiency and productivity in the civilian sector. That developed to today's dominant and indispensable decision-making tool, Operations research. It encompasses managerial decision making, mathematical and computer modeling and the use of information technology for informed decision-making.

The concepts and methods of Operations Research are pervasive. Students and graduates advise the public and private sectors on energy policy; design and operation of urban emergency systems; defense; health care; water resource planning; the criminal justice system; transportation issues. They also address a wide variety of design and operational issues in communication and data networks; computer operations; marketing; finance; inventory planning; manufacturing; and many areas designed to improve business productivity and efficiency. The subject impacts biology, the internet, the airline system, international banking and finance. It is a subject of beauty, depth, infinite breadth and applicability.

The Meaning of Operations Research

From the historical and philosophical summary just presented, it should be apparent that the term “operations research” has a number of quite distinct variations of meaning. To some, OR is that certain body of problems, techniques, and solutions that has been accumulated under the name of OR over the past 30 years, and we apply OR when we recognize a problem of that certain genre. To others, it is an activity or process-something we do, rather than know-which by its very nature is applied.

Perhaps in time the meaning will stabilize, but at this point it would be premature to exclude any of these interpretations. It would also be counterproductive to attempt to make distinctions between “operations research” and the “systems approach.” While these terms are sometimes viewed as distinct, they are often conceptualized in such a manner as to defy separation. Any attempt to draw boundaries between them would in practice be arbitrary.

The Operational Research Society of Great Britain has adopted the following definition:

Operational research is the application of the methods of science to complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government, and defense. The distinctive approach is to develop a scientific model of the system, incorporating measurements of factors such as chance and risk, with which to predict and compare the outcomes of alternative decisions, strategies or controls. The purpose is to help management determine its policy and actions scientifically.

Operations research is concerned with scientifically deciding how to best design and operate man-machine systems, usually under conditions requiring the allocation of scarce resources.

Although both of these definitions leave something to be desired, they are about as specific as one would want to be in defining such a broad area. It is noteworthy that both definitions emphasize the motivation for the work; namely, to aid decision makers in dealing with complex real-world problems. Even when the methods seem to become so abstract as to lose real-world relevance, the student may take some comfort in the fact that the ultimate goal is always some useful application.

Both definitions also mention methodology, describing it only very generally as “scientific.” That term is perhaps a bit too general, inasmuch as the methods of science are so diverse and varied. A more precise description of the OR methodology would indicate its reliance on “models.” Of course, that term would itself require further elaboration, and it is to that task that we now turn our attention.

Operations Research has been defined so far in various ways and still not been defined in an authoritative way. Some important and interesting opinions about the definition of OR which have been changed according to the development of the subject been given below:

OR is a scientific method of providing executive departments with a quantitative basis for decision regarding the operations under their control. -Morse and Kimbal(9164)

OR is a scientific method of providing executive with an analytical and objective basis for decisions. - P.M.S.Blacket(1948)

OR is the application of scientific methods, techniques and tools to problems involving the operations of systems so as to provide these in control of the operations with optimum solutions to the problem."

-Churchman, Acoff, Arnoff (1957)

OR is the art of giving bad answers to problems to which otherwise worse answers are given.

-T. L Saaty (1958)

OR is a management activity pursued in two complementary ways-one half by the free and bold

Exercise of commonsense untrammeled by any routine, and other half by the application of a repertoire of well-established

MODELS IN OPERATIONS RESEARCH

Modeling in Operations Research:

The essence of the operations research activity lies in the construction and use of models. Although modeling must be learned from individual experimentation, we will attempt here to discuss it in broad, almost philosophical terms. This overview is worth having, and setting a proper orientation in advance may help to avoid misconceptions later.

Definition. A model in the sense used in 0 R is defined as a representation of an actual object or situation. It shows the relationships (direct or indirect) and inter-relationships of action and reaction in terms of cause and effect.

Since a model is an abstraction of reality, it thus appears to be less complete than reality itself. For a model to be complete, it must be a representative of those aspects of reality that are being investigated.

The main objective of a model is to provide means for analyzing the behaviour of the system for the purpose of improving its performance. Or, if a system is not in existence, then a model defines the ideal structure of this future system indicating the functional relationships among its elements. The reliability of the solution obtained from a model depends on the validity of the model in representing the real systems. A model permits to 'examine the behavior of a system without interfering with ongoing operations.

Models can be Classified According to Following Characteristics:

1. Classification by Structure

   i. Iconic models. Iconic models represent the system as it is by scaling it up or down (i.e., by enlarging or reducing the size). In other words, it is an image.

For example, a toy airplane is an iconic model of a real one. Other common examples of it are: photographs, drawings, maps etc. A model of an atom is scaled up so as to make it visible to the naked eye. In a globe, the diameter of the earth is scaled down, but the globe has approximately the same shape as the earth, and the relative sizes of continents, seas, etc., are approximately correct. The iconic model is usually the simplest to conceive and the most specific and concrete. Its function is generally descriptive rather than explanatory. Accordingly, it cannot be easily used to determine or predict what effects many important changes on the actual system.

ii. Analogue models. The models, in which one set of properties is used to represent another set of properties, are called analogue models. After the problem is solved, the solution is reinterpreted in terms of the original system.

For example, graphs are very simple analogues because distance is used to represent the properties such as: time, number, percent, age, weight, and many other properties. Contour lines on a map represent the rise and fall of the heights. In general, analogues are less specific, less concrete but easier to manipulate than are iconic models.

iii. Symbolic (Mathematical) models. The symbolic or mathematical model is one which employs a set of mathematical symbols (i.e., letters, numbers, etc.) to represent the decision variables of the system. These variables are related together by means of a mathematical equation or a set of equations to describe the behavior (or properties) of the system. The solution of the problem is then obtained by applying well-developed mathematical techniques to the model.

The symbolic model is usually the easiest to manipulate experimentally and it is most general and abstract. Its function is more often explanatory rather than descriptive.

2. Classification by Purpose

Models can also be classified by purpose of its utility. The purpose of a model may be descriptive predictive or prescriptive.

i. Descriptive models. A descriptive model simply describes some aspects of a situation based on observations, survey. Questionnaire results or other available data. The result of an opinion poll represents a descriptive model.

ii.Predictive models. Such models can answer ‘what if’ type of questions, i.e. they can make predictions regarding certain events. For example, based on the survey results, television networks such models attempt to explain and predict the election results before all the votes are actually counted.

iii.Prescriptive models. Finally, when a predictive model has been repeatedly successful, it can be used to prescribe a source of action. For example, linear programming is a prescriptive (or normative) model because it prescribes what the managers ought to do.

3. Classification by Nature of Environment

These are mainly of two types:

i. Deterministic models. Such models assume conditions of complete certainty and perfect knowledge.

For example, linear programming, transportation and assignment models are deterministic type of models.

Ii .Probabilistic (or Stochastic) models. These types of models usually handle such situations in which the consequences or payoff of managerial actions cannot be predicted with certainty. However, it is possible to forecast a pattern of events, based on which managerial decisions can be made. For example, insurance companies are willing to insure against risk of fire, accidents, sickness and so on, because the pattern of events have been compiled in the form of probability distributions.

4.Classification by Behavior

i. Static models. These models do not consider the impact of changes that takes place during the planning horizon, i.e. they are independent of time. Also, in astatic model only one decision is needed for the duration of a given time period.

ii.Dynamic models. In these models, time is considered as one of the important variables and admits the impact of changes generated by time. Also, in dynamic models, not only one but a series of interdependent' decisions is required during the planning horizon.

5. Classification by Method of Solution

i. Analytical models. These models have a specific mathematical structure-and thus can be solved by known analytical or mathematical techniques. For example, general linear programming models as well as the specially structured transportation and assignment models are analytical models. .

ii. Simulation models. They also have a mathematical structure but they cannot be solved by purely using the 'tools' and 'techniques' of mathematics. A simulation model is essentially computer-assisted experimentation on a mathematical structure of a real time structure in order to study the system under a variety of assumptions.

Simulation modeling has the advantage of being more flexible than mathematical modeling and hence can be used to represent complex systems which otherwise cannot be formulated mathematically. On the other hand, simulation has the disadvantage of not providing general solutions like those obtained from successful mathematical models.

6. Classification by use of Digital Computers The development of the digital computer has led to the introduction of the following types of modeling in OR.

i. Analogue and Mathematical models combined. Sometimes analogue models are also expressed in terms of mathematical symbols. Such models may belong to both the types (ii) and (iii) in classification 1 above.

For example, Simulation model is of analogue type but mathematical formulae are also used in it. Managers very frequently use this model to 'simulate' their decisions by summarizing the activities of industry in a scale-down period.

ii. Function models. Such models are grouped on the basis of the function being performed.

For example, a function may serve to acquaint to scientist with such things as-tables, carrying data, a blue-print of layouts, a program representing a sequence of operations (like' in computer programming).

(iii) Quantitative models. Such models are used to measure the observations.

For example, degree of temperature, yardstick, a unit of measurement of length value, etc.

Other examples of quantitative models are: (i) transformation models which are useful in converting a measurement of one scale to another (e.g., Centigrade vs. Fahrenheit conversion scale), and (ii) the test models that act as 'standards' against which measurements are compared (e.g., business dealings, a specified standard production control, the quality of a medicine).

iii. Heuristic models. These models are mainly used to explore alternative strategies (courses of action) that were overlooked previously, whereas mathematical models are used to represent systems possessing well- defined strategies. Heuristic models do not claim to find the best solution to the problem.

Principles of Modeling

Let us now outline general principles useful in guiding to formulate the models within the context of 0 R. The model building and their users both should be consciously aware of the following Ten principles:

1. Do not build up a complicated model when simple one will suffice. Building the strongest possible model is a common guiding principle for mathematicians who are attempting to extend the theory or to develop techniques that have wide applications. However, in the actual practice of building models for specific purposes, the best advice is to “keep it simple”.

2. Beware of molding the problem to fit the technique. For example, an expert on linear programming techniques may tend to view every problem he encounters as required in a linear programming solutions. In fact, not all optimization problems involve only linear functions. Also, not all OR problems involve optimization. As a matter of fact, not all real-world problems call for operations research! Of course, everyone search reality in his own terms, so the field of OR is not unique in this regard. Being human we rely on the methods we are most comfortable in using and have been most successful within the past. We are certainly not able to use techniques in which we have no competence, and we cannot hope to be competent in all techniques. We must divide OR experts into three main categories:

(i)Technique developers. (ii) Teacher and (iii) Problem solvers.

In particular one should be ready to tolerate the behavior “I have found a cure but I am trying to search a disease to fit it” among technique developers and teachers.

3.The deduction phase of modeling must be conducted rigorously. The reason for requiring rigorous deduction is that one wants to be sure that if model conclusions are inconsistent with reality, then the defect lie in the assumptions. One application of this principle is that one must be extremely careful when programming computers. Hidden “bugs” are especially dangerous when they do not prevent the program from running but simply produce results, which are not consistent with the intention of the model.

4. Models should be validated prior to implementation. For example, if a model is constructed to forecast the monthly sales of a particular commodity, it could be tested using historical data to compare the forecasts it would have produced to the actual sales. In case, if the model cannot be validated prior to its implementation, then it can be implemented in phases for validation. For example a new model for inventory control may be implemented for a certain selected

group of items while the older system is retained for the majority of remaining items. If the model proves successful, more items can be placed within its range. It is also worth noting that real things change in time. A highly satisfactory model may very well degrade with age. So periodic re-evaluation is necessary.

5. A model should never be taken too literally. For example, suppose that one has to construct an

elaborate computer model of Indian economy with many competent researchers spending a great deal of time and money in getting all kinds of complicated interactions and relationships. Under such circumstances, it can be easily believed as if the model duplicates itself the real system. This danger continues to increase as the models become larger and more sophisticated, as they must deal with increasingly complicated problems.

6.A model should neither be pressed to do, nor criticized for failing to do that for which it was never intended. One example of this error would be the use of forecasting model to predict so far into the future that the data on which the forecasts are based have no relevance. Another example is the use of certain network methods to describe the activities involved in a complex project. A model should not be stretched beyond its capabilities.

7. Beware of over-selling a model. This principle is of particular importance for the OR professional because most non- technical benefactors of an operations researcher's work are not likely to understand his methods. The increased technicality of one's methods also increases the burden of responsibility on the OR professional to distinguish clearly between his role as model manipulator and model interpreter. In those cases where the assumptions can be challenged, it would be dishonest to use the model.

8. Some of the primary benefits of modeling are associated with the process of developing the model. It is true in general that a model is never as useful to anyone else as it is to those who are involved in building it up. The model itself never contains the full knowledge and understanding of the real system that the builder must acquire in order to

Successfully model it, and there is no practical way to convey this knowledge and understanding properly. In some cases the sole benefits may occur while the model is being developed. In such cases, the model may have no further value once it is completed, An example of this case might occur when a small group of people attempts to develop a formal plan for some subject. The plan is the final model, but the real problem may be to agree on 'what the objectives ought to be'.

9. A model cannot be any better than the information that goes into it. Like a computer program, a model can only manipulate the data provided to it; it cannot recognize and correct for deficiencies in input. Models may condense data or convert it to more useful forms, but they do not have the capacity to generate it. In some situations it is always better to gather more information about the system instead of exerting more efforts on modern constructions.

10. Models cannot replace decision makers. The purpose of OR models should not be supposed to provide "Optimal solutions" free from human subjectivity and error. OR models can aid decision makers and thereby permit better decisions to be made. However they do not make the

job of decision making easier. Definitely, the role of experience, intuition and judgment in decision-making is undiminished.

GENERAL METHODS FOR SOLVING ‘OR’ MODLES

In OR, we do not have a single general technique that solves all mathematical models that arise in practice. Instead, the type and complexity of the mathematical model dictate the nature of the solution method. For example, in Section 1.1 the solution of the tickets problem requires simple ranking of the alternatives based on the total purchasing price, whereas the solution of the rectangle problem utilizes differential calculus to determine the maximum area.

The most prominent OR technique is linear programming. It is designed for models with strict linear objective and constraint functions. Other techniques include integer programming (in which the variables assume integer values), dynamic programming (in which the original model can be decomposed into smaller sub. problems), network programming (in which the problem can be modeled as a network), and nonlinear programming (in which the functions of the model are non. linear). The cited techniques are but a partial list of the large number of available OR tools.

A peculiarity of most OR techniques is that solutions are not generally obtained in (formula-like) closed forms. Instead, they are determined by algorithms. An algorithm provides fixed computational rules that are applied repetitively to the problem with each repetition (called iteration) moving the solution closer to the optimum. Because the computations associated with each iteration are typically tedious and voluminous, it is imperative that these algorithms be executed on the computer. .

Some mathematical models may be so complex that it is impossible to solve them by any of the available optimization algorithms. In such cases, it may be necessary to abandon the search for the optimal solution and simply seek a good solution using heuristics or rules of thumb.

Generally three types of methods are used for solving OR models.

Analytic Method. If the OR model is solved by using all the tools of classical mathematics such as: differential calculus and finite differences available for this task, then such type of solutions are called analytic solutions. Solutions of various inventory models are obtained by adopting the so-called analytic procedure.

Iterative Method. If classical methods fail because of complexity of the constraints or of the number of variables, then we are usually forced to adopt an iterative method. Such a procedure starts with a trial solution and a set of rules for improving it.

The trial solution is then replaced by the improved solution, and the process is repeated until either no further improvement is possible or the cost of further calculation cannot be justified.

Iterative method can be divided into three groups:

a. After a finite number of repetitions, no further improvement will be possible.

b. Although successive iterations improve the solutions, we are only guaranteed the solution as a limit of an infinite process.

c. Finally we include the trial and error method, which, however, is likely to be lengthy, tedious, and costly even if electronic computers are used.

The Monte-Carlo Method: The basis of so-called Monte-Carlo technique is random sampling of variable's values from a Distribution of that variable. Monte-Carlo refers to the use of sampling methods to estimate the value of non-stochastic variables. The following are the main steps of Monte-Carlo method:

Step 1. In order to have a general idea of the system, we first draw a flow diagram of the system.

Step 2. Then we take correct sample observations to select some suitable model for the system. In this step we compute the probability distributions for the variables of our interest.

Step 3. We, then, convert the probability distributions to a

cumulative distribution function.

Step 4. A sequence of random numbers is now selected with the help of random number tables.

Step 5. Next we determine the sequence of values of variables of interest with the sequence of random numbers obtained instep 4.

Step 6. Finally we construct some standard mathematical function to the values obtained in step 5.Step 3. Step 4. Step 5.

The British/Europeans refer to "operational research", the Americans to "operations research" - but both are often shortened to just "OR" (which is the term we will use). Another term which is used for this field is "management science" ("MS"). The Americans sometimes combine the terms OR and MS together and say "***OR/MS***" or"ORMS".

Yet other terms sometimes used are "industrial engineering" ("IE"), "decision science"

("DS"), and “problemsolving”. In recent years there has been a move towards a standardization upon a single term for the field, namely the term "OR".

“Operations Research (Management Science) is a scientific approach to decision making that seeks to best design and operate a system, usually under conditions requiring the allocation of scarce resources.”

A system is an organization of interdependent components that work together to accomplish the goal of the system.

THE METHODOLOGY OF OR

When OR is used to solve a problem of an organization, the following seven step procedure should be followed:

<u>Step 1. Formulate the Problem</u>

OR analyst first defines the organization's problem. Defining the problem includes specifying the organization's objectives and the parts of the organization (or system) that must be studied before the problem can be solved.

<u>Step 2. Observe the System</u>

Next, the analyst collects data to estimate the values of parameters that affect the organization's problem. These estimates are used to develop (in Step 3) and evaluate (in Step 4) a mathematical model of the organization's problem.

<u>Step 3. Formulate a Mathematical Model of the Problem</u>

The analyst, then, develops a mathematical model (in other words an idealized representation) of the problem. In this class, we describe many mathematical techniques that can be used to model systems.

<u>Step 4. Verify the Model and Use the Model for Prediction</u>

The analyst now tries to determine if the mathematical model developed in Step 3 is an accurate representation of reality. To determine how well the model fits reality, one determines how valid the model is for the current situation.

<u>Step 5. Select a Suitable Alternative</u>

Given a model and a set of alternatives, the analyst chooses the alternative (if there is one) that best meets the organization's objectives.

Sometimes the set of alternatives is subject to certain restrictions and constraints. In many situations, the best alternative may be impossible or too costly to determine.

<u>Step 6. Present the Results and Conclusions of the Study</u>

In this step, the analyst presents the model and the recommendations from Step 5 to the decision making individual or group. In some situations, one might present several alternatives and let the organization choose the decision maker(s) choose the one that best meets her/his/their needs.

After presenting the results of the OR study to the decision maker(s), the analyst may find that s/he does not (or they do not) approve of the recommendations. This may result from incorrect definition of the problem on hand or from failure to involve decision maker(s) from the start of the project. In this case, the analyst should return to Step 1, 2, or3.

Step 7. Implement and Evaluate Recommendation

If the decision maker(s) has accepted the study, the analyst aids in implementing the recommendations. The system must be constantly monitored (and updated dynamically as the environment changes) to ensure that the recommendations are enabling decision maker(s) to meet her/his/their objectives.

HISTORY OF OR

(Prof. Beasley‟s lecture notes)

OR is a relatively new discipline. Whereas 70 years ago it would have been possible to study mathematics, physics or engineering (for example) at university it would not have been possible to study OR, indeed the term OR did not exist then. It was only really in the late 1930's that operational research began in a systematic fashion, and it started in the UK.

Early in 1936 the British Air Ministry established Bawdsey Research Station, on the east coast, near Felixstowe, Suffolk, as the centre where all pre-war radar experiments for both the Air Force and the Army would be carried out. Experimental radar equipment was brought up to a high state of reliability and ranges of over 100 miles on aircraft were obtained.

It was also in 1936 that Royal Air Force (RAF) Fighter Command, charged specifically with the air defense of Britain, was first created. It lacked however any effective fighter aircraft - no Hurricanes or Spitfires had come into service - and no radar data was yet fed into its very elementary warning and controlsystem.

It had become clear that radar would create a whole new series of problems in fighter direction and control so in late 1936 some experiments started at Biggin Hill in Kent into the effective use of such data. This early work, attempting to integrate radar data with ground based observer data for fighter interception, was the start ofOR.

The first of three major pre-war air-defense exercises was carried out in the summer of 1937. The

Although scientists had (plainly) been involved in the hardware side of warfare (designing better planes, bombs, tanks, etc) scientific analysis of the operational use of military resources had never taken place in a systematic fashion before the Second World War. Military personnel, often by no means stupid, were simply not trained to undertake such analysis.

These early OR workers came from many different disciplines, one group consisted of a physicist, two physiologists, two mathematical physicists and a surveyor. What such people brought to their work were "scientifically trained" minds, used to querying assumptions, logic, exploring hypotheses, devising experiments, collecting data, analyzing numbers, etc. Many too were of high intellectual caliber (at leastfourwartime OR personnel were later to win Nobel prizes when they returned to their peacetime disciplines). By the end of the war OR was well established in the armed services both in the UK and in the USA. OR started just before World War II in Britain with the establishment of teams of scientists to study the strategic and tactical problems involved in military operations. The objective was to find the most effective utilization of limited military resources by the use of quantitative techniques. Following the end of the war OR spread, although it spread in different ways in the UK and USA. You should be clear that the growth of OR since it began (and especially in the last 30 years) is, to a large extent, the result of the increasing power and widespread availability of computers. Most (though not all) OR involves carrying out a large number of numeric calculations. Without computers this would simply not bepossible.

1. BASIC ORCONCEPTS

"OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing."

We can also define a mathematical model as consisting of:

¬*Decision variables*, which are the unknowns to be determined by the solution to the model.

¬*Constraints* to represent the physical limitations of the system

¬ An *objective* function

¬ An *optimal solution* to the model is the identification of a set of variable values which are feasible (satisfy all the constraints) and which lead to the optimal value of the objective function.

An optimization model seeks to find values of the decision variables that optimize (maximize or minimize) an objective function among the set of all values for the decision variables that satisfy the given constraints.

Two Mines Example

The Two Mines Company own two different mines that produce an ore which, after being crushed, is graded into three classes: high, medium and low-grade. The company has contracted to provide a smelting plant with 12 tons of high-grade, 8 tons of medium-grade and 24 tons of low-grade ore per week. The two mines have different operating characteristics as detailed below.

Consider that mines cannot be operated in the weekend. How many days per week should each mine be operated to fulfill the smelting plant contract?

Guessing

To explore the Two Mines problem further we might simply guess (i.e. use our judgment) how many days per week to work and see how they turn out.

- work one day a week on *X*, one day a week on*Y*

This does not seem like a good guess as it results in only 7 tones a day of high- grade, insufficient to meet the contract requirement for 12 tones of high-grade a day. We say that such a solution is *infeasible*.

- work 4 days a week on *X*, 3 days a week on*Y*

This seems like a better guess as it results in sufficient ore to meet the contract. We say that such a solution is *feasible*. However it is quite expensive (costly).

We would like a solution which supplies what is necessary under the contract at minimum cost. Logically such a minimum cost solution to this decision problem must exist. However even if we keep guessing we can never be sure whether we have found this minimum cost solution or not. Fortunately our structured approach will enable us to find the minimum cost solution.

Solution

What we have is a verbal description of the Two Mines problem. What we need to do is to translate that verbal description into an *equivalent* mathematical description.

In dealing with problems of this kind we often do best to consider them in the order:

- Variables
- Constraints
- Objective

This process is often called *formulating* the problem (or more strictly formulating a mathematical representation of the problem).

<u>Variables</u>

These represent the "decisions that have to be made" or the "unknowns". We have two decision variables in this problem:

x = number of days per week mine *X* is operated

y = number of days per week mine Y is operated Note
here that $x \geq 0$ and $y \geq 0$.

Constraint
It is best to first put each constraint into words and then express it in a mathematical form.

ore production constraints - balance the amount produced with the quantity required under the smelting plantcontract

Ore	
High	$6x + 1y \geq 12$
Medium	$3x + 1y \geq 8$
Low	$4x + 6y \geq 24$

days per week constraint - we cannot work more than a certain maximum number of days a week e.g. for a 5 day week we have

$x \leq 5$

$y \leq 5$

Inequality constraints

Note we have an inequality here rather than an equality. This implies that we may produce more of some grade of ore than we need. In fact we have the general rule: given a choice between an equality and an inequality choose the inequality

For example - if we choose an equality for the ore production constraints we have the three equations $6x+y=12$, $3x+y=8$ and $4x+6y=24$ and there are no values of x and y which satisfy all three equations (the problem is therefore said to be "over- constrained"). For example the values of x and y which satisfy $6x+y=12$ and $3x+y=8$ are $x=4/3$ and $y=4$, but these values do not satisfy $4x+6y=24$.

The reason for this general rule is that choosing an inequality rather than an equality gives us more flexibility in optimizing (maximizing or minimizing) the objective (deciding values for the decision variables that optimize the objective).

Implicit constraints

Constraints such as days per week constraint are often called implicit constraints because they are implicit in the definition of the variables.

Objective

Again in words our objective is (presumably) to minimize cost which is given by $180x + 160y$

Since we have the ***complete mathematical representation*** of the problem:

Minimize

$180x + 160y$

subject to

$x + y \geq 12$ $3x$
$+ y \geq 8$$4x$
$+ 6y \geq 24$ $x \leq$
5
$y \leq 5$
$x, y \geq 0$

Discussion

This problem was a *decision problem.*

We have taken a real-world situation and constructed an equivalent mathematical representation - such a representation is often called a mathematical *model* of the real-world situation (and the process by which the model is obtained is called *formulating* the model).

Just to confuse things the mathematical model of the problem is sometimes called the *formulation* of the problem.

Having obtained our mathematical model we (hopefully) have some quantitative method which will enable us to numerically solve the model (i.e. obtain a numerical solution) - such a quantitative method is often called an *algorithm* for solving the model.

Essentially an algorithm (for a particular model) is a set of instructions which, when followed in a step-by-step fashion, will produce a numerical solution to that model.

Our model has an *objective*, that is something which we are trying to *optimize*. Having obtained the numerical solution of our model we have to translate that solution back into the real-world situation.

"OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing."

2. LINEAR PROGRAMMING

It can be recalled from the Two Mines example that the conditions for a mathematical model to be a linear program (LP)were:

- all variables continuous (i.e. can take fractional values)
- a single objective (minimize or maximize)
- the objective and constraints are linear i.e. any term is either a constant or a constant multiplied by an unknown.

LP's are important - this is because:

- many practical problems can be formulated as LP's
- there exists an algorithm (called the *simplex* algorithm) which enables us to solve LP's numerically relatively easily

We will return later to the simplex algorithm for solving LP's but for the moment we will concentrate upon formulating LP's.

Some of the major application areas to which LP can be applied are:

- Work scheduling
- Production planning & Production process
- Capital budgeting
- Financial planning
- Blending (e.g. Oil refinery management)
- Farm planning
- Distribution
- Multi-period decision problems
    - Inventory model
    - Financial models
    - Work scheduling

Note that the key to formulating LP's is practice. However a useful hint is that common objectives for LP's are maximize profit/minimize cost.

There are four basic assumptions in LP:

- Proportionality
    - The contribution to the objective function from each decision variable is proportional to the value of the decision variable (The contribution to the objective function from making four soldiers (4 × \$3=\$12) is exactly four times the contribution to the objective function from making one soldier(\$3))

- The contribution of each decision variable to the LHS of each constraint is proportional to the value of the decision variable (It takes exactly three times as many finishing hours (2hrs□ 3=6hrs) to manufacture three soldiers as it takes to manufacture one soldier (2hrs))

- Additivity
  - The contribution to the objective function for any decision variable is independent of the values of the other decision variables (No matter what the value of train (x_2), the manufacture of soldier (x_1) will always contribute $3x_1$ dollars to the objective function)
  - The contribution of a decision variable to LHS of each constraint is independent of the values of other decision variables (No matter what the value of x_1, the manufacture of x_2 uses x_2 finishing hours and x2 carpentry hours)
    - ♣ *1st implication*: The value of objective function is the sum of the contributions from each decision variables.
    - ♣ *2nd implication*: LHS of each constraint is the sum of the contributions from each decision variables.
- Divisibility
  - Each decision variable is allowed to assume fractional values. If we actually can not produce a fractional number of decision variables, we use IP (It is acceptable to produce 1.69trains)
- Certainty
  - Each parameter is known with certainty

FORMULATING LP

Giapetto Example

(Winston 3.1, p. 49)

Giapetto's wooden soldiers and trains. Each soldier sells for $27, uses $10 of raw materials and takes $14 of labor & overhead costs. Each train sells for $21, uses $9 of raw materials, and takes $10 of overhead costs. Each soldier needs 2 hours finishing and 1 hour carpentry; each train needs 1 hour finishing and 1 hour carpentry. Raw materials are unlimited, but only 100 hours of finishing and 80 hours of carpentry are available each week. Demand for trains is unlimited; but at most 40 soldiers can be sold each week. How many of each toy should be made each week to maximize profits?

Answer

Decision variables completely describe the decisions to be made (in this case, by Giapetto). Giapetto must decide how many soldiers and trains should be manufactured each week. With this in mind, we define:

x_1 = the number of soldiers produced per week

x_2 = the number of trains produced per week

Objective function is the function of the decision variables that the decision maker wants to maximize (revenue or profit) or minimize (costs). Giapetto can concentrate on maximizing the total weekly profit(z).

Here profit equals to (weekly revenues) – (raw material purchase cost) – (other variable costs). Hence Giapetto‟s objective functions:

Maximize $z = 3x_1 + 2x_2$

Constraints show the restrictions on the values of the decision variables. Without constraints Giapetto could make a large profit by choosing decision variables to be very large. Here there are three constraints:

Finishing time per week

Carpentry time per week

Weekly demand for soldiers

Sign restrictions are added if the decision variables can only assume nonnegative values (Giapetto cannot manufacture negative number of soldiers or trains!)

All these characteristics explored above give the following ***Linear Programming*** (LP) model

	$\max z = 3x_1+2x_2$	(The Objective function)
s.t.	$2x_1 + x_2 \le 100$	(Finishing constraint)
	$x_1 + x_2 \le 80$	(Carpentry constraint)
	$x_1 \le 40$	(Constraint on demand for soldiers)
	$x_1, x_2 \geq 0$	(Sign restrictions)

A value of (x_1, x_2) is in the ***feasible region*** if it satisfies all the constraints and sign restrictions.

Graphically and computationally we see the solution is (x_1, x_2) = (20, 60) at which z = 180. (***Optimal solution***)

Report

The maximum profit is $180 by making 20 soldiers and 60 trains each week. Profit is limited by the carpentry and finishing labor available. Profit could be increased by buying more labor.

Advertisement Example

(Winston 3.2, p.61)

Dorian makes luxury cars and jeeps for high-income men and women. It wishes to advertise with 1 minute spots in comedy shows and football games. Each comedy spot costs $50K and is seen by 7M high-income women and 2M high-income men. Each football spot costs $100K and is seen by 2M high-income women and 12M high-income men. How can Dorian reach 28M high-income women and 24M high- income men at the leastcost?

Answer

The decision variables are

x_1 = the number of comedy spots

x_2 = the number of football spots The

model of the problem:

$$\begin{aligned} \min z &= 50x_1 + 100x_2 \\ \text{st} \quad & 7x_1 + \quad 2x_2 = 28 \\ & 2x_1 + 12x_2 = 24 \\ & x_1, x_2 > 0 \end{aligned}$$

The graphical solution is z = 320 when (x_1, x_2) = (3.6, 1.4). From the graph, in this problem rounding up to (x_1, x_2) = (4, 2) gives the best *integer* solution.

Report

The minimum cost of reaching the target audience is $400K, with 4 comedy spots and 2 football slots. The model is dubious as it does not allow for saturation after repeated viewings.

Diet Example

(Winston 3.4., p. 70)

Ms. Fidan"s diet requires that all the food she eats come from one of the four "basic food groups". At present, the following four foods are available for consumption: brownies, chocolate ice cream, cola, and pineapple cheesecake. Each brownie costs 0.5$, each scoop of chocolate ice cream costs 0.2$, each bottle of cola costs 0.3$, and each pineapple cheesecake costs 0.8$. Each day, she must ingest at least 500 calories, 6 oz of chocolate, 10 oz of sugar, and 8 oz of fat. The nutritional content per unit of each food is shown in Table. Formulate an LP model that can be used to satisfy her daily nutritional requirements at minimum cost.

	Calories	Chocolate (ounces)	Sugar (ounces)	Fat (ounces)
Brownie	400	3	2	2
Choc. ice cream (1 scoop)	200	2	2	4
Cola (1 bottle)	150	0	4	1
Pineapple cheesecake (1 piece)	500	0	4	5

Answer

The decision variables:

x_1: number of brownies eaten daily

x_2: number of scoops of chocolate ice cream eaten daily

x_3: bottles of cola drunk daily

x_4: pieces of pineapple cheesecake eaten daily The

objective function (the total cost of the diet in cents):

$$\min w = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

Constraints:

$$\begin{array}{lllllll}
400x_1 + & 200x_2 + & 150x_3 + & 500x_4 & \geq 500 & & \text{(daily calorie intake)} \\
3x_1 + & 2x_2 & & & \geq 6 & & \text{(daily chocolate intake)} \\
2x_1 + & 2x_2 + & 4x_3 + & 4x_4 & \geq 10 & & \text{(daily sugar intake)}
\end{array}$$

The minimum cost diet incurs a daily cost of 90 cents by eating 3 scoops of chocolate and drinking 1 bottle of cola ($w = 90$, $x_2 = 3$, $x_3 = 1$)

Post Office Example

(Winston 3.5, p.74)

A PO requires different numbers of employees on different days of the week. Union rules state each employee must work 5 consecutive days and then receive two days off. Find the minimum number of employees needed.

	Mon	Tue	Wed	Thur	Fri	Sat	Sun
StaffNeeded	17	13	15	19	14	16	11

Answer

The decision variables are x_i (# of employees starting on day i)

Mathematically we must

$$\begin{array}{lccccccccc}
\min z = & x_1 & + x_2 & + x_3 & + x_4 & + x_5 & + x_6 & + x_7 & & \\
\text{s.t.} & x_1 & & & + x_4 & + x_5 & + x_6 & + x_7 & \geq & 17 \\
 & x_1 & + x_2 & & & + x_5 & + x_6 & + x_7 & \geq & 13 \\
 & x_1 & + x_2 & + x_3 & & & + x_6 & + x_7 & \geq & 15 \\
 & x_1 & + x_2 & + x_3 & + x_4 & & & + x_7 & \geq & 19 \\
 & x_1 & + x_2 & + x_3 & + x_4 & + x_5 & & & \geq & 14 \\
 & & + x_2 & + x_3 & + x_4 & + x_5 & + x_6 & & \geq & 16 \\
 & & & + x_3 & + x_4 & + x_5 & + x_6 & + x_7 & \geq & 11
\end{array}$$

$$x_t \geq 0, \forall t$$

The solution is $(x_i) = (4/3, 10/3, 2, 22/3, 0, 10/3, 5)$ giving $z = 67/3$.

We could round this up to $(x_i) = (2, 4, 2, 8, 0, 4, 5)$ giving $z = 25$ (may be wrong!).

However restricting the decision var.s to be integers and using Lindo again gives (x_i) = (4, 4, 2, 6, 0, 4, 3) giving z =23.

Sailco Example

(Winston 3.10, p. 99)

Sailco must determine how many sailboats to produce in the next 4 quarters. The demand is known to be 40, 60, 75, and 25 boats. Sailco must meet its demands. At the beginning of the 1st quarter Sailco starts with 10 boats in inventory. Sailco can produce up to 40 boats with regular time labor at \$400 per boat, or additional boats at \$450 with overtime labor. Boats made in a quarter can be used to meet that quarter's demand or held in inventory for the next quarter at an extra cost of \$20.00 per boat.

Answer

The decision variables are for t = 1,2,3,4

x_t= # of boats in quarter t built in regular time

y_t= # of boats in quarter t built in overtime For

convenience, introduce variables:

i_t= # of boats in inventory at the end quarter t

d_t= demand in quarter t

We aregiventhat $d_1 = 40,\ d_2 = 60,\ d_3 = 75,\ d_4 = 25,\ i_0=10$

$x_t \leq 40,\ \forall t$

Bylogic $i_t= i_{t-1}+ x_t+ y_t- d_t, \forall t.$

Demand ismetiff $i_t \geq 0, \forall t$

(Signrestrictions $x_t,\ y_t \geq 0, \forall t$)

We need to minimize total cost z subject to these three sets of conditions where

$$z = 400\,(x_1 + x_2 + x_3 + x_4) + 450\,(y_1 + y_2 + y_3 + y_4) + 20\,(i_1 + i_2 + i_3 + i_4)$$

Report:

Lindo reveals the solution to be (x_1, x_2, x_3, x_4) = (40, 40, 40, 25) and (y_1, y_2, y_3, y_4) = (0, 10, 35, 0) and the minimum cost of \$78450.00 is achieved by the schedule

		Q_1	Q_2	Q_3	Q_4
Regular time (x_t)		40	40	40	25
Overtime (y_t)		0	10	35	0
Inventory (i_t)	10	10	0	0	0
Demand (d_t)		40	60	75	25

Customer Service Level Example

(Winston 3.12, p. 108)

CSL services computers. Its demand (hours) for the time of skilled technicians in the next 5 months is

t	Jan	Feb	Mar	Apr	May
d_t	6000	7000	8000	9500	11000

It starts with 50 skilled technicians at the beginning of January. Each technician can work 160 hrs/month. To train a new technician they must be supervised for 50 hrs by an experienced technician for a period of one month time. Each experienced

technician is paid $2K/mth and a trainee is paid $1K/mth. Each month 5% of the skilled technicians leave. CSL needs to meet demand and minimize costs.

Answer

The decision variable is

x_t= # to be trained in month t

We must minimize the total cost. For convenience let

y_t= # experienced tech. at start of t^{th} month

d_t= demand during month t

Then we must

min $z = 2000\ (y_1+...+ y_5) + 1000\ (x_1 +...+ x_5)$

subject to

$160y_t - 50x_t \geq d_t$ for $t = 1,...,5$

$y_1 = 50,\ d_1 = 6000,\ d_2 = 7000,\ d_3 = 8000,\ d_4 = 9500,\ d_5 = 11000$

$y_t = .95y_{t-1}+x_{t-1}$ for $t = 2,3,4,5$

$x_t,\ y_t \geq 0$

SOLVING LP

LP Solutions: Four Cases

When an LP is solved, one of the following four cases will occur:

1. The LP has a **unique optimal solution**.
2. The LP has **alternative (multiple) optimal solutions**. It has more than one (actually an infinite number of) optimal solutions
3. The LP is **infeasible**. It has no feasible solutions (The feasible region contains no points).
4. The LP is **unbounded**. In the feasible region there are points with arbitrarily large

(in a max problem) objective function values.

The Graphical Solution

Any LP with only two variables can be solved graphically

Example 1. Giapetto

Since the Giapetto LP has two variables, it may be solved graphically.

Answer

The feasible region is the set of all points satisfying the constraints. max $z = 3x_1 + 2x_2$

$2x_1 + x_2 \leq 100$ (Finishing constraint)

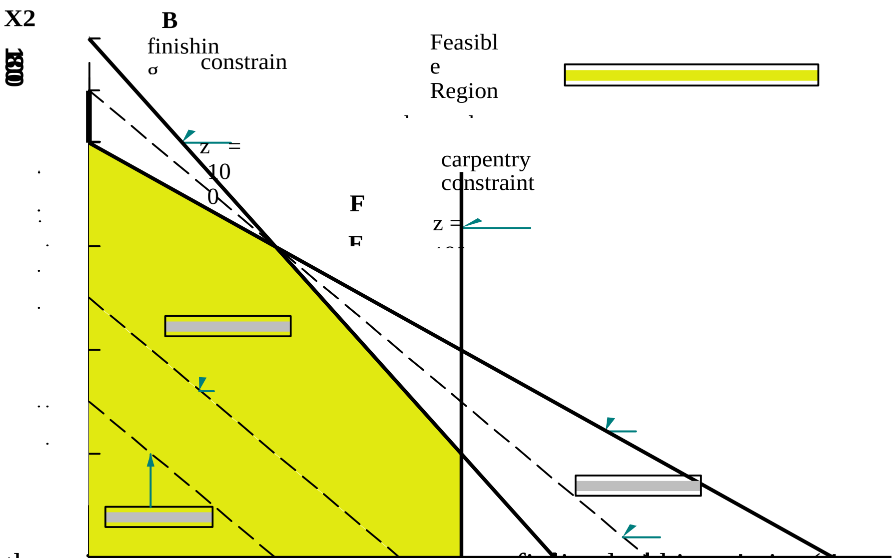
$x_1 + x_2 \leq 80$ (Carpentry constraint)

$x_1 \leq 40$ (Demand constraint)

$x_1, x_2 \geq 0$ (Sign restrictions)

The set of points satisfying the LP is bounded by the five sided polygon DGFEH. Any point ***on*** or ***in*** the interior of this polygon (the shade area) is in the ***feasible region***. Having identified the feasible region for the LP, a search can begin for the ***optimal solution*** which will be the point in the feasible region with the *largest* z-value (maximization problem).

To find the optimal solution, a line on which the points have the same z-value is graphed. In a max problem, such a line is called an ***isoprofit*** line while in a min problem, this is called the ***isocost*** line. (*The figure shows the isoprofit lines for $z = 60$, $z = 100$, and $z = 180$*).



In the unique optimal solution case, isoprofit line last hits a point (vertex - corner) before leaving the feasible region.

The optimal solution of this LP is point G where $(x_1, x_2) = (20, 60)$ giving $z = 180$.

Example 2. Advertisement

(*Winston 3.2, p. 61*)

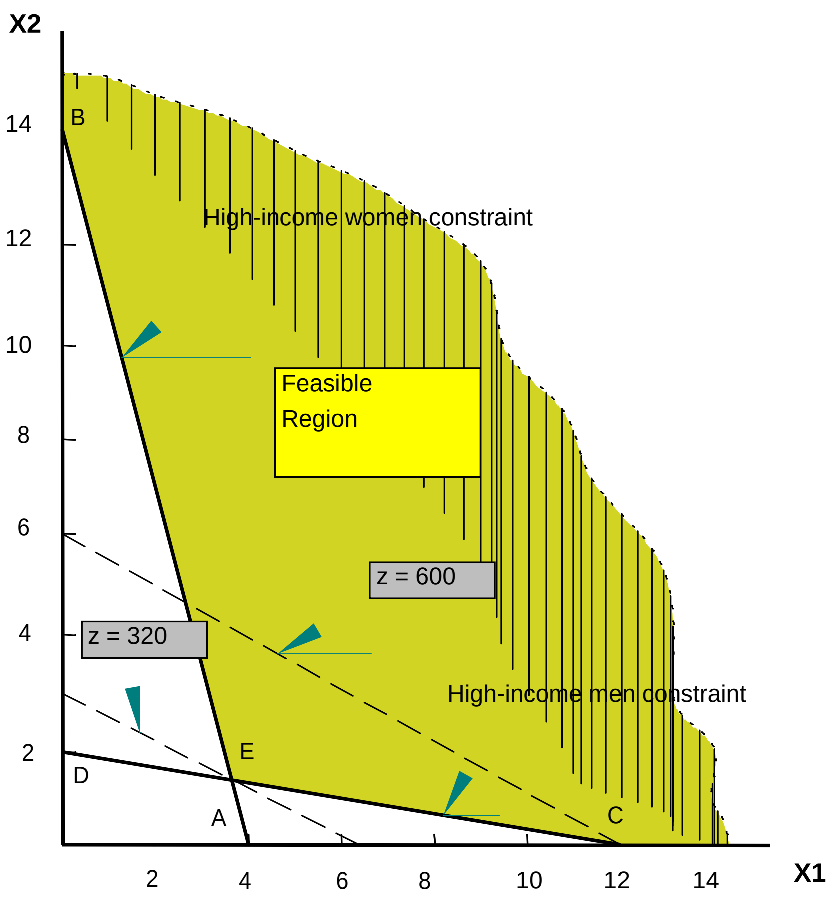
Since the Advertisement LP has two variables, it may be solved graphically.

Answer

The feasible region is the set of all points satisfying the constraints. min z $= 50x_1 + 100x_2$

s.t. $7x_1 + 2x_2 \geq 28$ (high income women)

$2x_1 + 12x_2 \geq 24$ (high income men) x_1, $x_2 \geq 0$



Since Dorian wants to minimize total advertising costs, the optimal solution to the problem is the point in the feasible region with the smallest z value.

An isocost line with the smallest z value passes through point E and is the optimal solution at $x_1 = 3.6$ and $x_2 = 1.4$ giving $z = 320$.

Both the high-income women and high-income men constraints are satisfied, both constraints are binding.

Example 3. Two Mines

min $180x + 160y$

st $6x + y \geq 12$

$3x + y \geq 8$ $4x$

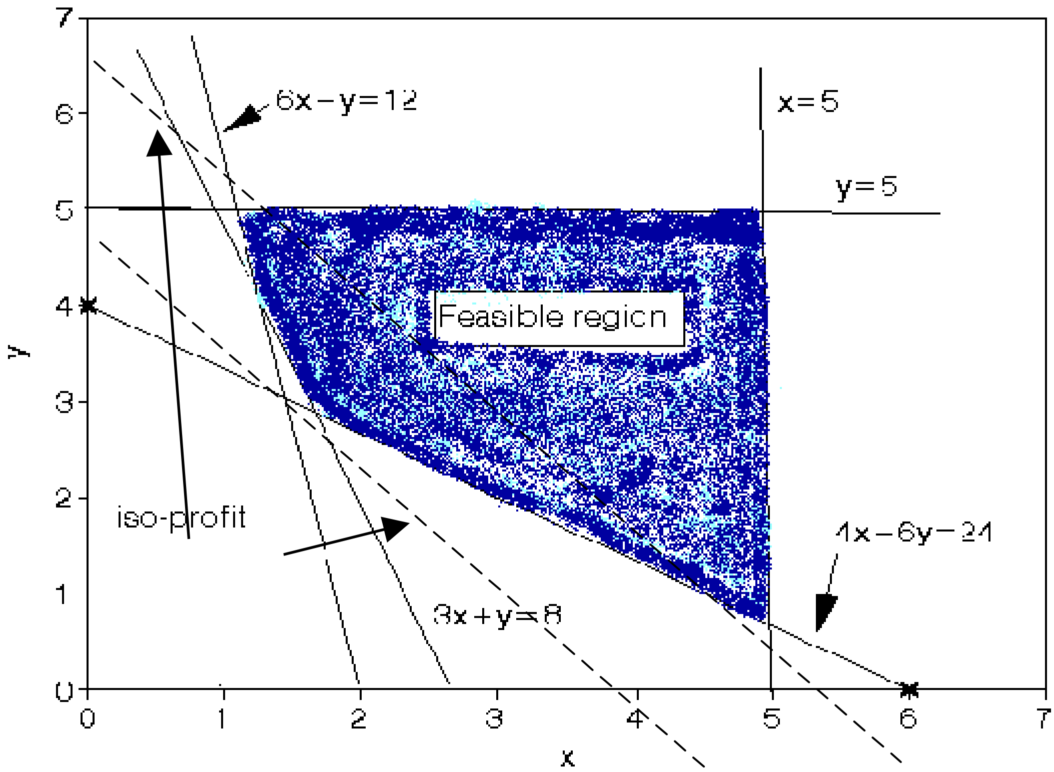
$+ 6y \geq 24$ $x \leq$

5

$y \leq 5$

$x, y \geq 0$

Answer



Optimal sol"n is 765.71. 1.71 days mine *X* and 2.86 days mine *Y* are operated.

Example 4. Modified Giapetto

max $z = 4x_1 + 2x_2$

s.t.	$2x_1 + x_2 \leq 100$	(Finishing constraint)
	$x_1 + x_2 \leq 80$	(Carpentry constraint)
	$x_1 \leq 40$	(Demand constraint)
	$x_1, x_2 \geq 0$	(Sign restrictions)

Answer

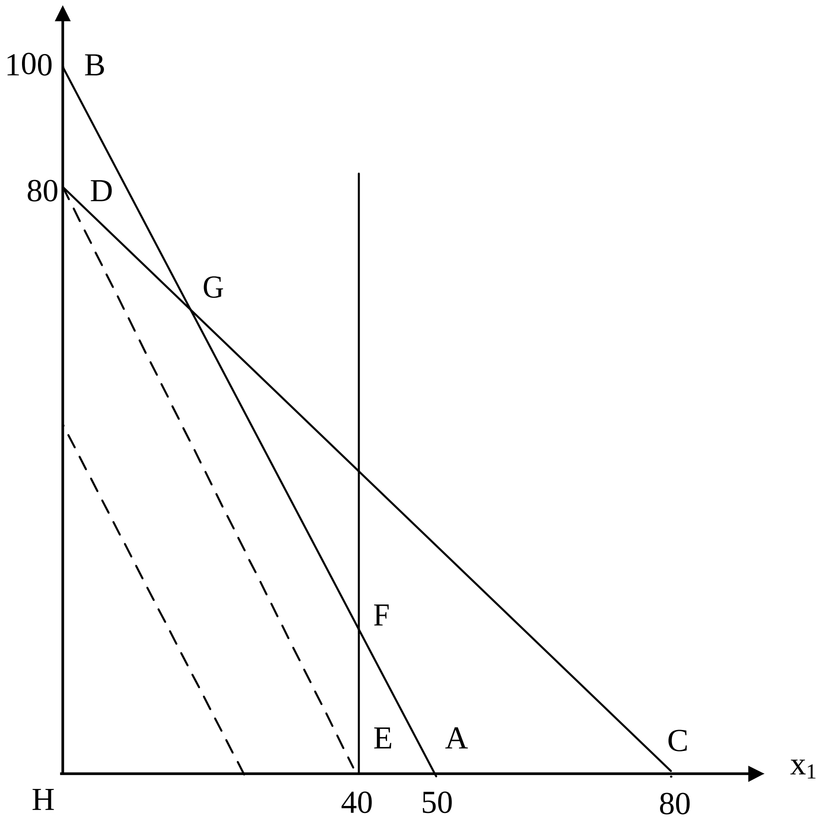
Points on the line between points G (20, 60) and F (40, 20) are the ***alternative optimal solutions*** (see figure below).

Thus, for $0 \leq c \leq 1$,

c [20 60] + (1 - c) [40 20] = **[40 - 20c, 20 + 40c]**

will be optimal

For all optimal solutions, the optimal objective function value is 200.



Example 5. Modified Giapetto (v. 2)

Add constraint $x_2 \geq 90$ (Constraint on demand for trains).

Answer

No feasible region: ***Infeasible LP***

Example 6. Modified Giapetto (v. 3)

Only use constraint $x_2 \geq 90$

Answer

Isoprofit line never lose contact with the feasible region: ***Unbounded LP***

The Simplex Algorithm

Note that in the examples considered at the graphical solution, the unique optimal solution to the LP occurred at a vertex (corner) of the feasible region. In fact it is true that for *any* LP the optimal solution occurs at a vertex of the feasible region. This fact is the key to the simplex algorithm for solving LP's.

Essentially the simplex algorithm starts at one vertex of the feasible region and moves (at each iteration) to another (adjacent) vertex, improving (or leaving unchanged) the objective function as it does so, until it reaches the vertex corresponding to the optimal LP solution.

The simplex algorithm for solving linear programs (LP's) was developed by Dantzig in the late 1940's and since then a number of different versions of the algorithm have been developed. One of these later versions, called the *revised simplex* algorithm (sometimes known as the "product form of the inverse" simplex algorithm) forms the basis of most modern computer packages for solvingLP's.

Steps

1. Convert the LP to standard form
2. Obtain a basic feasible solution (bfs) from the standard form
3. Determine whether the current bfs is optimal. If it is optimal, stop.
4. If the current bfs is not optimal, determine which non-basic variable should become a basic variable and which basic variable should become a non-basic variable to find a new bfs with a better objective function value
5. Go back to Step3.

Related concepts:

- Standard form: all constraints are equations and all variables are nonnegative
- bfs: any basic solution where all variables are nonnegative
- Non-basic variable: a chosen set of variables where variables equal to0
- Basic variable: the remaining variables that satisfy the system of equations at the standard form

Example 1. Dakota Furniture

(Winston 4.3, p. 134)

Dakota Furniture makes desks, tables, and chairs. Each product needs the limited resources of lumber, carpentry and finishing; as described in the table. At most 5 tables can be sold per week. Maximize weekly revenue.

Resource	Desk	Table	Chair	Max Avail.
Lumber (board ft.)	8	6	1	48
Finishing hours	4	2	1.5	20
Carpentry hours	2	1.5	.5	8
Max Demand	unlimited	5	unlimited	
Price ($)	60	30	20	

LP Model:

Let x_1, x_2, x_3 be the number of desks, tables and chairs produced.

Let the weekly profit be $z. Then, we must

$$\max z = 60x_1 + 30x_2 + 20x_3$$

$$\text{s.t.} \quad \begin{aligned} 8x_1 + 6x_2 + x_3 &\leq 48 \\ 4x_1 + 2x_2 + 1.5x_3 &\leq 20 \\ 2x_1 + 1.5x_2 + .5x_3 &\leq 8 \\ x_2 &\leq 5 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solution with Simplex Algorithm

<u>First introduce slack variables and convert the LP to the standard form and write a canonicalform</u>

$$\begin{array}{lllllllll} R_0 & z & -60x_1 & -30x_2 & -20x_3 & & & & = 0 \\ R_1 & & 8x_1 & +6x_2 & +x_3 & +s_1 & & & = 48 \\ R_2 & & 4x_1 & +2x_2 & +1.5x_3 & & +s_2 & & = 20 \\ R_3 & & 2x_1 & +1.5x_2 & +.5x_3 & & & +s_3 & = 8 \\ R_4 & & & x_2 & & & & +s_4 & = 5 \end{array}$$

$x_1, x_2, x_3, s_1, s_2, s_3, s_4 \geq 0$

<u>Obtain a starting bfs.</u>

As $(x_1, x_2, x_3) = 0$ is feasible for the original problem, the below given point where three of the variables equal 0 (the ***non-basic variables***) and the four other variables (the ***basic variables***) are determined by the four equalities is an obvious bfs:

$x_1 = x_2 = x_3 = 0$, $s_1 = 48$, $s_2 = 20$, $s_3 = 8$, $s_4 = 5$.

.

<u>Determine whether the current bfs is optimal.</u>

Determine whether there is any way that z can be increased by increasing some nonbasic variable.

If each nonbasic variable has a nonnegative coefficient in the objective function row (***row 0***), current bfs isoptimal.

However, here all nonbasic variables have negative coefficients: It is not optimal.

Find a new bfs

- z increases most rapidly when x_1 is made non-zero; i.e. x_1 is the ***enteringvariable***.
- Examining R_1, x_1 can be increased only to 6. More than 6 makes $s_1 < 0$. Similarly R_2, R_3, and R_4, give limits of 5, 4, and no limit for x_1 (***ratio test***). The smallest ratio is the largest value of the entering variable that will keep all the current basic variables nonnegative. Thus by R_3, x_1 can only increase to $x_1 = 4$ when s_3 becomes 0. We say s_3 is the ***leaving variable*** and $R_3$is the ***pivote quation***.
- Now we must rewrite the system so the values of the basic variables can be read off.

The new *pivot equation* ($R_3/2$)is

$$R_3'': x_1 + .75x_2 + .25x_3 + .5s_3 = 4$$

Then use R_3'' to eliminate x_1 in all the other rows.

$R0''=R0+60R3''$, $R1''=R1-8R3''$, $R2''=R2-4R3''$, $R4''=R4$

R_0''	z	$+15x_2$	$-5x_3$			$+30s_3$	$= 240$	$z = 240''$
R_1			$-x_3$	$+s_1$		$-4s_3$	$=16$	$s_1 = 16''$
R_2		$-x_2$	$+.5x_3$		$+s_2$	$-2s_3$	$= 4$	$s_2 = 4''$
R_3	x_1	$+.75x_2$	$+.25x_3$			$+.5s_3$	$= 4$	$x_1 = 4$
R_4''		x_2				$+s_4$	$= 5$	$s_4 = 5$

The new bfs is $x_2 = x_3 = s_3 = 0$, $x_1 = 4$, $s_1 = 16$, $s_2 = 4$, $s_4 = 5$ making $z = 240$.

Check optimality of current bfs. Repeat steps until an optimal solution is reached

- We increase z fastest by making x_3 non-zero (i.e. $x_3$enters).
- x_3 can be increased to at most $x_3 = 8$, when $s_2 = 0$ (i.e. $s_2$leaves.)

Rearranging the pivot equation gives

$$R_2'''' \quad -2x_2 + x_3 + 2s_2 - 4s_3 = 8 \qquad (R_2'' \times 2).$$

Row operations with R_2'''' eliminate x_3 to give the new system

$R_0''''=R_0''+5R_2''''$, $R_1''''=R_1''+R_2''''$, $R_3''''=R_3''-.5R_2''''$,$R_4''''=R_4''$

The bfs is now $x_2 = s_2 = s_3 = 0$, $x_1 = 2$, $x_3 = 8$, $s_1 = 24$, $s_4 = 5$ making $z = 280$. Each nonbasic variable has a nonnegative coefficient in row 0 ($5x_2$, $10s_2$, $10s_3$). THE CURRENT SOLUTION IS OPTIMAL

Report: Dakota furniture‟s optimum weekly profit would be 280$ if they produce 2 desks and 8 chairs.

This was once written as a tableau.

(Use table format for each operation in all HW and exams!!!)

$$\begin{aligned} \max z &= 60x_1 + 30x_2 + 20x_3 \\ \text{s.t.}\quad & 8x_1 + 6x_2 + x_3 \le 48 \\ & 4x_1 + 2x_2 + 1.5x_3 \le 20 \\ & 2x_1 + 1.5x_2 + .5x_3 \le 8 \\ & x_2 \le 5 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Initial tableau:

Z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV	Ratio
1	-60	-30	-20	0	0	0	0	0	z = 0	
0	8	6	1	1	0	0	0	48	s_1 = 48	6
0	4	2	1.5	0	1	0	0	20	s_2 = 20	5
0	2	1.5	0.5	0	0	1	0	8	s_3 = 8	4
0	0	1	0	0	0	0	1	5	s_4 = 5	-

First tableau:

Z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV	Ratio
1	0	15	-5	0	0	30	0	240	z = 240	
0	0	0	-1	1	0	-4	0	16	s_1 = 16	-
0	0	-1	0.5	0	1	-2	0	4	s_2 = 4	8
0	1	0.75	0.25	0	0	0.5	0	4	x_1 = 4	16
0	0	1	0	0	0	0	1	5	s_4 = 5	-

Second and optimal tableau:

Z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV	Ratio
1	0	5	0	0	10	10	0	280	z = 280	
0	0	-2	0	1	2	-8	0	24	s_1 = 24	
0	0	-2	1	0	2	-4	0	8	x_3 = 8	
0	1	1.25	0	0	-0.5	1.5	0	2	x_1 = 2	
0	0	1	0	0	0	0	1	5	s_4 = 5	

Example 2. Modified Dakota Furniture

Dakota example is modified: $35/table new

$z = 60\, x_1 + 35\, x_2 + 20\, x_3$

Second and optimal tableau for the modified problem:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV	Ratio
1	0	**0**	0	0	10	10	0	280	z=280	
0	0	-2	0	1	2	-8	0	24	s_1=24	-
0	0	-2	1	0	2	-4	0	8	x_3=8	-
0	1	1.25	0	0	-0.5	1.5	0	2	x_1=2	2/1.25
0	0	1	0	0	0	0	1	5	s_4=5	5/1

Another optimal tableau for the modified problem:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV
1	0	0	0	0	10	10	0	280	z=280
0	1.6	0	0	1	1.2	-5.6	0	27.2	s_1=27.2
0	1.6	0	1	0	1.2	-1.6	0	11.2	x_3=11.2
0	0.8	1	0	0	-0.4	1.2	0	1.6	x_2=1.6
0	-0.8	0	0	0	0.4	-1.2	1	3.4	s_4=3.4

Therefore the optimal solution is as follows:

$z = 280$ and for $0 \le c \le 1$

$$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = c \begin{vmatrix} 2 \\ 0 \\ 8 \end{vmatrix} + (1-c) \begin{vmatrix} 0 \\ 1.6 \\ 11.2 \end{vmatrix} = \begin{vmatrix} 2c \\ 1.6 - 1.6c \\ 11.2 - 3.2c \end{vmatrix}$$

Example 3. Unbounded LPs

z	x_1	x_2	x_3	s_1	s_2	z	RHS	BV	Ratio
1	0	2	**-9**	0	12	4	100	z=100	
0	0	1	-6	1	6	-1	20	x_4=20	None
0	1	1	-1	0	1	0	5	x_1=5	None

Since ratio test fails, the LP under consideration is an unbounded LP.

The Big M Method

If an LP has any ≥ or = constraints, a starting bfs may not be readily apparent. When a bfs is not readily apparent, the Big M method or the two-phase simplex method may be used to solve the problem.

The Big M method is a version of the Simplex Algorithm that first finds a bfs by adding "artificial" variables to the problem. The objective function of the original LP must, of course, be modified to ensure that the artificial variables are all equal to 0 at the conclusion of the simplex algorithm.

Steps

1. Modify the constraints so that the RHS of each constraint is nonnegative (This requires that each constraint with a negative RHS be multiplied by -1. Remember that if you multiply an inequality by any negative number, the direction of the inequality is reversed!). After modification, identify each constraint as a ≤, ≥ or =constraint.
2. Convert each inequality constraint to standard form (If constraint i is a ≤ constraint, we add a slack variable s_i; and if constraint i is a ≥ constraint, we subtract an excess variable e_i).
3. Add an artificial variable a_i to the constraints identified as ≥ or = constraints at the end of Step 1. Also add the sign restriction $a_i \geq 0$.
4. Let M denote a very large positive number. If the LP is a min problem, add (for each artificial variable) Ma_i to the objective function. If the LP is a max problem, add (for each artificial variable) $-Ma_i$t o the objective function.
5. Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex. Now solve the transformed problem by the simplex (In choosing the entering variable, remember that M is a very large positive number!).

If all artificial variables are equal to zero in the optimal solution, we have found the **optimal solution** to the original problem.

If any artificial variables are positive in the optimal solution, the original problem is **infeasible**!!!

Example 1. Oranj Juice

(Winston 4.10, p. 164)

Bevco manufactures an orange flavored soft drink called Oranj by combining orange soda and orange juice. Each ounce of orange soda contains 0.5 oz of sugar and 1 mg of vitamin C. Each ounce of orange juice contains 0.25 oz of sugar and 3 mg of vitamin C. It costs Bevco

2¢ to produce an ounce of orange soda and 3¢ to produce an ounce of orange juice. Marketing department has decided that each 10 oz bottle of Oranj must contain at least 20 mg of vitamin C and at most 4 oz of sugar. Use LP to determine how Bevco can meet marketing dept.‟s requirements at minimum cost. **LP Model:**

Let x_1 and x_2 be the quantity of ounces of orange soda and orange juice (respectively) in a bottle of Oranj.

$$\min z = 2x_1 + 3x_2$$

$$\begin{array}{llr}
\text{s.t.} & 0.5\, x_1 + 0.25\, x_2 \le 4 & \text{(sugar const.)} \\
& x_1 + 3\, x_2 \ge 20 & \text{(vit. Cconst.)} \\
& x_1 + x_2 = 10 & \text{(10 oz in bottle)} \\
& x_1, x_2 \ge 0 &
\end{array}$$

Solving Oranj Example with Big M Method

1. Modify the constraints so that the RHS of each constraint is nonnegative The RHS of each constraint is non negative

2. Convert each inequality constraint to standardform

$$\begin{aligned}
z - 2x_1 - 3x_2 &= 0 \\
0.5x_1 + 0.25x_2 + s_1 &= 4 \\
x_1 + 3x_2 - e_2 &= 20 \\
x_1 + x_2 &= 10
\end{aligned}$$

all variables nonnegative

3. Add a_i to the constraints identified as > or =const.s

$$\begin{array}{lr}
z - 2x_1 - 3x_2 = 0 & \text{Row} \\
0\ 0.5x_1 + 0.25x_2 + s_1 = 4 & \text{Row1} \\
x_1 + 3x_2 - e_2 + a_2 = 20 & \text{Row 2} \\
x_1 + x_2 + a_3 = 10 & \text{Row 3}
\end{array}$$

all variables nonnegative

4. Add Ma_i to the objective function (min problem)

$$\min z = 2x_1 + 3x_2 + Ma_2 + Ma_3$$

Row 0 will change to

$$z - 2x_1 - 3x_2 - Ma_2 - Ma_3 = 0$$

5. Since each artificial variable are in our starting bfs, they must be eliminated from row 0

_____ New Row 0 = Row 0 + M * Row 2 + M * Row 3

$z + (2M–2)\, x_1 + (4M–3)\, x_2 – Me_2 \qquad = 30M \qquad$ New Row0

Initial tableau:

z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	BV	Ratio
1	2M-2	**4M-3**	0	-M	0	0	30M	z=30M	
0	0.5	0.25	1	0	0	0	4	s_1=4	16
0	1	$\boxed{3}$	0	-1	1	0	20	a_2=20	**20/3***
0	1	1	0	0	0	1	10	a_3=10	10

In a min problem, entering variable is the variable that has the "most positive" coefficient in row0!

First tableau:

z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	BV	Ratio
1	**(2M-3)/3**	0	0	(M-3)/3	(3-4M)/3	0	20+3.3M	z	
0	5/12	0	1	1/12	-1/12	0	7/3	s_1	28/5
0	1/3	1	0	-1/3	1/3	0	20/3	x_2	20
0	$\boxed{2/3}$	0	0	1/3	-1/3	1	10/3	a_3	**5***

Optimal tableau:

z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	BV
1	0	0	0	-1/2	(1-2M)/2	(3-2M)/2	25	z=25
0	0	0	1	-1/8	1/8	-5/8	1/4	s_1=1/4
0	0	1	0	-1/2	1/2	-1/2	5	x_2=5
0	1	0	0	1/2	-1/2	3/2	5	x_1=5

Report:

In a bottle of Oranj, there should be 5 oz orange soda and 5 oz orange juice. In this case the cost would be 25¢.

Example 2. Modified Oranj Juice

Consider Bevco"s problem. It is modified so that 36 mg of vitamin C are required. Related LP model is given as follows:

Let x_1 and x_2 be the quantity of ounces of orange soda and orange juice (respectively) in a bottle of Oranj.

$$
\begin{aligned}
&\min z = 2x_1 + 3x_2 \\
&\text{s.t.} \quad 0.5\, x_1 + 0.25\, x_2 \leq 4 && \text{(sugarconst.)} \\
&\qquad x_1 + 3\, x_2 \geq 36 && \text{(vit. Cconst.)} \\
&\qquad x_1 + x_2 = 10 && \text{(10 oz inbottle)} \\
&\qquad x_1, x_2 \geq 0
\end{aligned}
$$

Solving with Big M method:

Initial tableau:

Z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	BV	Ratio
1	2M-2	**4M-3**	0	-M	0	0	46M	z=46M	
0	0.5	0.25	1	0	0	0	4	s_1=4	16
0	1	3	0	-1	1	0	36	a_2=36	36/3
0	1	$\boxed{\mathbf{1}}$	0	0	0	1	10	a_3=10	**10**

Optimal tableau:

Z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	BV
1	1-2M	0	0	-M	0	3-4M	30+6M	z=30+6M
0	¼	0	1	0	0	-1/4	3/2	s_1=3/2
0	-2	0	0	-1	1	-3	6	a_2=6
0	1	1	0	0	0	1	10	x_2=10

An artificial variable (a_2) is BV so the original LP has no feasible solution

DUALITY

Primal – Dual

Associated with any LP is another LP called the ***dual***. Knowledge of the dual provides interesting economic and sensitivity analysis insights. When taking the dual of any LP, the given LP is referred to as the ***primal***. If the primal is a max problem, the dual will be a min problem and vice versa

Finding the Dual of an LP

The dual of a ***normal max*** problem is a ***normal min*** problem.

Normal max problem is a problem in which all the variables are required to be nonnegative and all the constraints are ≤constraints.

Normal min problem is a problem in which all the variables are required to be nonnegative and all the constraints are ≥constraints.

Similarly, the dual of a normal min problem is a normal max problem.

Finding the Dual of a Normal Max Problem

PRIMAL

Max z= $c_1x_1 + c_2x_2 + \ldots + c_nx_n$

s.t. $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \quad \leq$

$b_1 a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \quad \leq b_2$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \; x_j$

≥ 0 (j = 1, 2, …, n)

DUAL

min$w$= $b_1y_1 + b_2y_2 + \ldots + b_my_m$

$_2 + \ldots + a_{m1}y_m \geq c_1 a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \geq$

c_2

… … … …

$a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \geq c_n$ y_i

≥ 0 (i = 1, 2, …, m)

Finding the Dual of a Normal Min Problem

PRIMAL

Min w= $b_1y_1 + b_2y_2 + \ldots + b_my_m$

s.t. $a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \geq$

$c_1 a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \geq c_2$

… … … …

$a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \geq c_n y_i$

≥ 0 (i = 1, 2, …, m)

DUAL

Max z = $c_1x_1 + c_2x_2 + \ldots + c_nx_n$

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n$ $\leq$

$b_1 a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n$ $\leq b_2$

… … … …

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m$ x_j

≥ 0 (j = 1, 2, …, n)

Finding the Dual of a Nonnormal Max Problem

- If the *i*th primal constraint is a ≥ constraint, the corresponding dual variable y_i must satisfy $y_i \leq 0$
- If the *i*th primal constraint is an equality constraint, the dual variable y_i is now unrestricted in sign(urs).
- If the *i*th primal variable is urs, the *i*th dual constraint will be an equality constraint

Finding the Dual of a Nonnormal Min Problem

- If the *i*th primal constraint is a ≤ constraint, the corresponding dualvariable x_i must satisfy $x_i \leq 0$
- If the *i*th primal constraint is an equality constraint, the dual variable x_i is now urs.
- If the *i*th primal variable is urs, the *i*th dual constraint will be an equality constraint

The Dual Theorem

The primal and dual have equal optimal objective function values (if the problems have optimal solutions).

Weak duality implies that if for any feasible solution to the primal and an feasible solution to the dual, the *w*-value for the feasible dual solution will be at least as large as the *z*-value for the feasible primal solution □ $z \leq w$.

Consequences

- Any feasible solution to the dual can be used to develop a bound on the optimal value of the primal objective function.
- If the primal is unbounded, then the dual problem is infeasible.
- If the dual is unbounded, then the primal is infeasible.
- How to read the optimal dual solution from Row 0 of the optimal tableau if the primal is a max problem:

  „optimal value of dual variable y_i"

  = „coefficient of s_i in optimalrow0" (if const. *i* is a ≤const.)

  = –„coefficient of e_i in optimalrow0" (if const. *i* is a ≥ const.)

  = „coefficient of a_i in optimal row 0" – M (if const. *i* is a = const.)

- How to read the optimal dual solution from Row 0 of the optimal tableau if the primal is a minproblem:

„optimal value of dual variable x_i"

= „coefficient of s_i in optimalrow0" (if const. i is a ≤const.)

=–„coefficient of e_i inoptimalrow0" (if const. i is a ≥const.)

= „coefficient of a_i in optimal row 0" + M (if const. i is a = const.)

Economic Interpretation

When the primal is a normal max problem, the dual variables are related to the value of resources available to the decision maker. For this reason, dual variables are often referred to as ***resource shadow prices***.

Example

PRIMAL

Let x_1, x_2, x_3 be the number of desks, tables and chairs produced. Let the weekly profit be $\$z$. Then, we must

$$\max z = 60x_1 + 30x_2 + 20x_3$$

$8x_1 + 6x_2 + x_3 \leq 48$ (Lumber constraint)

$4x_1 + 2x_2 + 1.5x_3 \leq 20$ (Finishing hour constraint) $2x_1 + 1.5x_2 + 0.5x_3 \leq 8$ (Carpentry hour constraint)

$$x_1, x_2, x_3 \geq 0$$

DUAL

Suppose an entrepreneur wants to purchase all of Dakota"s resources.

In the dual problem y_1, y_2, y_3 are the resource prices (price paid for one board ft of lumber, one finishing hour, and one carpentry hour).

$w is the cost of purchasing the resources.

Resource prices must be set high enough to induce Dakota to sell. i.e. total purchasing cost equals total profit.

$$\min w = 48y_1 + 20y_2 + 8y_3$$

s.t. $8y_1 + 4y_2 + 2y_3 \geq 60$ (Desk constraint) $6y_1 + 2y_2 + 1.5y_3 \geq 30$ (Table constraint) $y_1 + 1.5y_2 + 0.5y_3 \geq 20$ (Chair constraint)

$$y_1, y_2, y_3 \geq 0$$

SENSITIVITY ANALYSIS

Reduced Cost

For any nonbasic variable, the reduced cost for the variable is the amount by which the nonbasic variable's objective function coefficient must be improved before that variable will become a basic variable in some optimal solution to the LP.

If the objective function coefficient of a nonbasic variable $x_k$is improved by its reduced cost, then the LP will have alternative optimal solutions at least one in which $x_k$is a basic variable, and at least one in which $x_k$is not a basic variable.

If the objective function coefficient of a nonbasic variable $x_k$is improved by more than its reduced cost, then any optimal solution to the LP will have $x_k$as a basic variable and $x_k > 0$.

Reduced cost of a basic variable is zero (see definition)!

Shadow Price

We define the shadow price for the *i*th constraint of an LP to be the amount by which the optimal z value is "improved" (increased in a max problem and decreased in a min problem) if the RHS of the *i*th constraint is increased by 1.

This definition applies only if the change in the RHS of the constraint leaves the current basis optimal!

A ≥ constraint will always have a non positive shadow price; a ≤ constraint will always have a nonnegative shadow price.

Conceptualization

$$\max z = 5x_1 + x_2 + 10x_3$$

$$x_1 + x_3 \leq 100$$

$$x_2 \leq 1$$

All variables ≥ 0

This is a very easy LP model and can be solved manually without utilizing Simplex.

$x_2 = 1$ (This variable does not exist in the first constraint. In this case, as the problem is a maximization problem, the optimum value of the variable equals the RHS value of the second constraint).

$x_1 = 0$, $x_3 = 100$ (These two variables do exist only in the first constraint and as the objective function coefficient of x_3 is greater than that of x_1, the optimum value of x_3 equals the RHS value of the first constraint).

Hence, the optimal solution is as follows:

$z = 1001$, $[x_1, x_2, x_3] = [0, 1, 100]$

Similarly, sensitivity analysis can be executed manually.

Reduced Cost

As x_2 and x_3 are in the basis, their reduced costs are 0.

In order to have x_1 enter in the basis, we should make its objective function coefficient as great as that of x_3. In other words, improve the coefficient as 5 (10-5). New objective function would be (max $z = 10x_1 + x_2 + 10x_3$) and there would be at least two optimal solutions for $[x_1, x_2, x_3]$: [0, 1, 100] and [100, 1,0].

Therefore reduced cost of x_1 equals 5.

If we improve the objective function coefficient of x_1 more than its reduced cost, there would be a unique optimal solution: [100, 1, 0].

Shadow Price

If the RHS of the first constraint is increased by 1, new optimal solution of x_3 would be 101 instead of 100. In this case, new z value would be 1011.

If we use the definition: 1011 - 1001 = 10 is the shadow price of the first constraint.

Similarly the shadow price of the second constraint can be calculated as 1 (please find it).

Some important equations

If the change in the RHS of the constraint leaves the current basis optimal (within the allowable RHS range), the following equations can be used to calculate new objective function value:

for maximization problems

new obj. fn. value = old obj. fn. for minimization problems

- new obj. fn. value = old obj. fn. value – (new RHS – old RHS) × shadow price For Lindo example, as the allowable increases in RHS ranges are infinity for each constraint, we can increase RHS of them as much as we want. But according to allowable decreases, RHS of the first constraint can be decreased by 100 and that of second constraint by1.

Lets assume that new RHS value of the first constraint is 60.

As the change is within allowable range, we can use the first equation (max. problem):

$z_{new} = 1001 + (60 - 100) 10 = 601$.

value + (new RHS – old RHS) × shadow price

Utilizing Simplex for Sensitivity

In Dakota furniture example; x_1, x_2, and x_3 were representing the number of desks, tables, and chairs produced.

The LP formulated for profit maximization:

$$
\begin{array}{llllllllll}
\max z = & 60x_1 & 30x_2 & 20x_3 & & & & & & \\
8x_1 & & +6x_2 & +x_3 & +s_1 & & & & =48 & \text{Lumber} \\
4x_1 & & +2x_2 & +1.5x_3 & & +s_2 & & & =20 & \text{Finishing} \\
2x_1 & & +1.5x_2 & +.5x_3 & & & +s_3 & & =8 & \text{Carpentry} \\
 & & x_2 & & & & & +s_4 & =5 & \text{Demand}
\end{array}
$$

The optimal solution was:

$$
\begin{array}{lllllllll}
z & & +5x_2 & & & +10s_2 & +10s_3 & & =280 \\
 & -2x_2 & & & +s_1 & +2s_2 & -8s_3 & & = 24 \\
 & -2x_2 & & +x_3 & & +2s_2 & -4s_3 & & = 8 \\
 & +x_1 & +1.25x_2 & & & -.5s_2 & +1.5s_3 & & = 2 \\
 & x_2 & & & & & & +s_4 & = 5
\end{array}
$$

Analysis 1

Suppose available finishing time changes from 20 □ 20+□ , then we have the system:

$$
\begin{array}{llllllll}
z' = & 60x_1' & +30x_2' & +20x_3' & & & & \\
 & 8x_1' & +6x_2' & +x_3' & +s_1' & & & =48 \\
 & 4x_1' & +2x_2' & +1.5x_3' & & +s_2' & & =20+\square \\
2x_1' & & +1.5x_2' & +.5x_3' & & & +s_3' & =8 \\
 & & +x_2' & & & & +s_4' & = 5
\end{array}
$$

or equivalently:

$$
\begin{array}{llllllll}
z' = & 60x_1' & +30x_2' & +20x_3' & & & & \\
 & 8x_1' & +6x_2' & +x_3' & +s_1' & & & =48 \\
 & 4x_1' & +2x_2' & +1.5x_3' & & +(s_2'-\square) & & =20 \\
2x_1' & & +1.5x_2' & +.5x_3' & & & +s_3' & =8 \\
 & & +x_2' & & & & +s_4' & = 5
\end{array}
$$

That is z', x_1', x_2', x_3', x_4', s_1', $s_2'-\square$, s_3', s_4' satisfy the original problem, and hence (1)

Substituting in:

$$
\begin{array}{llllll}
z' & +5x_2' & & +10(s_2'-\square) & +10s_3' & = 280 \\
 & -2x_2' & +s_1' & +2(s_2'-\square) & -8s_3' & = 24
\end{array}
$$

$$-2x_2' + x_3' + 2(s_2' - \square) - 4s_3' = 8$$
$$+x_1' + 1.25x_2' - .5(s_2' - \square) + 1.5s_3' = 2$$
$$x_2' + s_{4'} = 5$$

and thus

$$z' + 5x_{2'} + 10s_2' + 10s_3' = 280 + 10\square$$
$$-2x_{2'} + s_1' + 2s_2' - 8s_3' = 24 + 2\square$$
$$-2x_2' + x_3' + 2s_2' - 4s_3' = 8 + 2\square$$
$$+x_1' + 1.25x_2' - .5s_2' + 1.5s_3' = 2 - .5\square$$
$$x_2' + s_{4'} = 5$$

For -4 □ □ □ 4, the new system maximizezs''. In this range RHS values are non- negative. As □ increases, revenue increases by 10□ . Therefore, the ***shadow price*** of finishing labor is \$10 per hr. (This is valid for up to 4 extra hours or 4 fewer hours).

Analysis 2

What happens if revenue from desks changes to \$60+□ ? For small □ □ revenue increases by 2□ (as we are making 2 desks currently). But how large an increase is possible?

The new revenue is:

$$z' = (60+\square)x_1 + 30x_2 + 20x_3 = z + \square x_1$$
$$= (280 - 5x_2 - 10s_2 - 10s_3) + \square\,(2 - 1.25x_2 + .5s_2 - 1.5s_3)$$
$$= 280 + 2\square - (5 + 1.25\square)x_2 - (10 - .5\square)s_2 - (10 + 1.5\square)s_3$$

So the top line in the final system would be: $z' + (5 + 1.25\square)x_2 + (10 - .5\square)s_2 + (10 + 1.5\square)s_3 = 280 + 2\square$

Provided all terms in this row are □ □□ we are still optimal.

For -4 □ □ □ 20, the current production schedule is still optimal.

Analysis 3

If revenue from a non-basic variable changes, the revenue is

$$z'' = 60x_1 + (30 + \square)x_2 + 20x_3 = z + \square x_2$$
$$= 280 - 5x_2 - 10s_2 - 10s_3 + \square x_2$$
$$= 280 - (5 - \square)x_2 - 10s_2 - 10s_3$$

The current solution is optimal for □ □ 5. But when □ □ 5 or the revenue per table is increased past \$35, it becomes better to produce tables. We say the **reduced cost** of tables is \$5.00.

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UNIT – II: TRANSPORTATION PROBLEM

FORMULATING TRANSPORTATION PROBLEMS

In general, a transportation problem is specified by the following information:

- A set of m ***supply points*** from which a good/service is shipped. Supply point i can supply at most s_i units.
- A set of n ***demand points*** to which the good/service is shipped. Demand point j must receive at least d_j units.
- Each unit produced at supply point i and shipped to demand point j incurs a variable cost of c_{ij}.

The relevant data can be formulated in a ***transportation tableau***:

	Demand point 1	Demand point 2	.....	demand Point n	SUPPLY
Supply point 1	c_{11}	c_{12}		c_{1n}	s_1
Supply point 2	c_{21}	c_{22}		c_{2n}	s_2
.....					
Supply point m	c_{m1}	c_{m2}		c_{mn}	s_m
DEMAND	d_1	d_2		d_n	

If total supply equals total demand then the problem is said to be a ***balanced transportation problem***.

Let x_{ij} = number of units shipped from supply point i to demand point j

□ Decision variable x_{ij}: number of units shipped from supply point i to demand point j

then the general LP representation of a transportation problem is

$$\min \sum_i \sum_j c_{ij} x_{ij}$$

s.t. $\sum_j x_{ij} \le s_i \quad (i=1,2,...,m)$ Supply constraints

$\sum_i x_{ij} \geq d_j (j=1,2,...,n)$ Demand constraints

$x_{ij} \geq 0$

If a problem has the constraints given above and is a *maximization* problem, it is still a transportation problem.

Formulating Balanced Transportation Problem Example 1.

Powerco

Powerco has three electric power plants that supply the needs of four cities. Each power plant can supply the following numbers of kwh of electricity: plant 1, 35 million; plant 2, 50 million; and plant 3, 40 million. The peak power demands in these cities as follows (in kwh): city 1, 45 million; city 2, 20 million; city 3, 30 million; city 4, 30 million. The costs of sending 1 million kwh of electricity from plant to city is given in the table below. To minimize the cost of meeting each city"s peak power demand, formulate a balanced transportation problem in a transportation tableau and represent the problem as a LPmodel.

	To			
From	City 1	City 2	City 3	City 4
Plant 1	\$8	\$6	\$10	\$9
Plant 2	\$9	\$12	\$13	\$7
Plant 3	\$14	\$9	\$16	\$5

Answer

<u>Representation of the problem as a LP model</u>

x_{ij}: number of (million) kwh produced at plant *i* and sent to city *j*.

min z = 8 x11 + 6 x12 + 10 x13 + 9 x14 + 9 x21 + 12 x22 + 13 x23 + 7 x24 + 14 x31 + 9 x32 + 16 x33 + 5 x34

s.t. x11 + x12 + x13 + x14 ≤ 35 (supply constraints)

x21 + x22 + x23 + x24 ≤ 50

x31 + x32 + x33 + x34 ≤ 40

x11 + x21 + x31 ≥ 45 (demand constraints)

x12 + x22 + x32 ≥ 20

x13 + x23 + x33 ≥ 30

x14 + x24 + x34 ≥ 30

$x_{ij} \geq 0$ (i = 1, 2, 3; j = 1, 2, 3,4)

Formulation of the transportation problem

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8	6	10	9	35
Plant 2	9	12	13	7	50
Plant 3	14	9	16	5	40
DEMAND	45	20	30	30	125

Total supply & total demand both equal 125: “balanced transportion problem”.

Balancing an Unbalanced Transportation Problem

Excess Supply

If total supply exceeds total demand, we can balance a transportation problem by creating a ***dummy demand point*** that has a demand equal to the amount of excess supply. Since shipments to the dummy demand point are not real shipments, they are assigned a cost of zero. These shipments indicate unused supply capacity.

Unmet Demand

If total supply is less than total demand, actually the problem has no feasible solution. To solve the problem it is sometimes desirable to allow the possibility of leaving some demand unmet. In such a situation, *a penalty is often associated with unmet demand.* This means that a ***dummy supply point*** should be introduced.

Example 2. Modified Powerco for Excess Supply

Suppose that demand for city 1 is 40 million kwh. Formulate a balanced transportation problem.

Answer

Total demand is 120, total supply is 125.

To balance the problem, we would add a dummy demand point with a demand of 125 – 120 = 5 million kwh.

From each plant, the cost of shipping 1 million kwh to the dummy is 0. For details see Table 4.

Table 4. Transportation Tableau for Excess Supply

	City 1	City 2	City 3	City 4	Dummy	SUPPLY
Plant 1	8	6	10	9	0	35
Plant 2	9	12	13	7	0	50
Plant 3	14	9	16	5	0	40
DEMAND	40	20	30	30	5	125

Example 3. Modified Powerco for Unmet Demand

Suppose that demand for city 1 is 50 million kwh. For each million kwh of unmet demand, there is a penalty of 80$. Formulate a balanced transportation problem. **Answer**

We would add a dummy supply point having a supply of 5 million kwh representing shortage.

	City1	City2	City3	City4	SUPPLY
Plant 1	8	6	10		35
Plant 2	9	12	13		50
Plant 3	14	9	16		40
Dummy (Shortage)	80	80	80		5
DEMAND	50	20	30	30	130

FINDING BFS FOR TRANSPORT'N PROBLEMS

For a balanced transportation problem, general LP representation may be written as: min $\sum_i$

$\sum_j c_{ij} x_{ij}$

s.t. $\sum_j x_{ij} = s_i (i=1,2,...,m)$ Supply constraints

$\sum_i x_{ij} = d_j (j=1,2,...,n)$ Demand constraints

$x_{ij} \geq 0$

To find a bfs to a balanced transportation problem, we need to make the following important observation:

If a set of values for the xij‟s satisfies all but one of the constraints of a balanced transportation problem, the values for the xij‟s will automatically satisfy the other constraint. This observation shows that when we solve a balanced transportation, we may omit from considerationanyoneoftheproblem‟sconstraintsandsolveanLPhaving$m$+n-1constraints. We arbitrarily assume that the first supply constraint is omitted from consideration. In trying to find a bfs to the remaining m+n-1 constraints, you might think that any collection of m+n-1 variables would yield a basic solution. But this is not the case: If the m+n-1 variables yield a basic solution, the cells corresponding to this set contain ***no loop***.

An ordered sequence of at least four different cells is called a loop if

- Any two consecutives cells lie in either the same row or same column
- No three consecutive cells lie in the same row or column
- The last cell in the sequence has a row or column in common with the first cell in the sequence

There are three methods that can be used to find a bfs for a balanced transportation problem:

1. Northwest Corner method
2. Minimum cost method
3. Vogel‟s method

Northwest Corner Method

We begin in the upper left corner of the transportation tableau and set x_{11} as large as possible (clearly, x_{11} can be no larger than the smaller of s_1 and d_1).

- If $x_{11}=s_1$, cross out the first row of the tableau. Also change d_1 tod_1-s_1.
- If $x_{11}=d_1$, cross out the first column of the tableau. Change s_1 tos_1-d_1.
- If $x_{11}=s_1=d_1$, cross out either row 1 or column 1 (but not both!).
    - If you cross out row, change d_1 to0.
    - If you cross out column, change s_1 to0.

Continue applying this procedure to the most northwest cell in the tableau that does not lie in a crossed out row or column. Eventually, you will come to a point where there is only one cell that can be assigned a value. Assign this cell a value equal to its row or column demand, and cross out both the cells row or column.

Minimum Cost Method

Northwest Corner method does not utilize shipping costs, so it can yield an initial bfs that has a very high shipping cost. Then determining an optimal solution may require several pivots. To begin the minimum cost method, find the variable with the smallest shipping cost (call it

x_{ij}). Then assign x_{ij}its largest possible value, min $\{s_i, d_j\}$.

As in the NWC method, cross out row i or column j and reduce the supply or demand of the non crossedout of row or column by the value of x_{ij}.

Continue like NWC method (instead of assigning upper left corner, the cell with the minimum cost is assigned). See Northwest Corner Method for the details!

2	3	5	6	
2 2	1 8	3	5	X
3	8	4	6	15
10	X	4	6	

2 5	3	5	6	X
2 2	1 8	3	5	X
3	8	4	6	15
5	X	4	6	

2 5	3	5	6	X
2 2	1 8	3	5	X
3 5	8	4 4	6 6	10
5	X	4	6	

Vogel's Method

Begin by computing for each row and column a penalty equal to the difference between the two smallest costs in the row and column. Next find the row or column with the largest penalty. Choose as the first basic variable the variable in this row or column that has the smallest cost. As described in the NWC method, make this variable as large as possible, cross out row or column, and change the supply or demand associated with the basic variable (See Northwest Corner Method for the details!). Now recomputed new penalties (using only cells that do not lie in a crossed out row or column), and repeat the procedure until only one uncrossed cell remains. Set this variable equal to the supply or demand associated with the variable, and cross out the variable's row and column.

Example 3.

				Supply	Row penalty
	6	7	8	10	7-6=1
	15	80	78	15	78-15=63
Demand	15	5	5		
Column penalty	15-6=9	80-7=**73**	78-8=70		

				Supply	Row penalty
	6	**7** 5	8	5	8-6=2
	15	80	78	15	78-15=63
Demand	15	X	5		
Column penalty	15-6=9	---------------------	78-8=**70**		

				Supply	Row penalty
	6	7 5	**8** 5	X	-
	15	80	78	15	-
Demand	15	X	0		
Column penalty	15-6=9	-	-		

	6	7 5	8 5	X
	15 15	80	78 0	15
Demand	15	X	0	

THE TRANSPORTATION SIMPLEX METHOD

Steps of the Method

1. If the problem is unbalanced, balance it
2. Use one of the methods to find a bfs for the problem
3. Use the fact that $u_1 = 0$ and $u_i + v_j = c_{ij}$ for all basic variables to find the u‟s and v‟s for the current bfs.
4. If $u_i + v_j - c_{ij} \leq 0$ for all non-basic variables, then the current bfs is optimal. If this is not the case, we enter the variable with the most positive $u_i + v_j - c_{ij}$ into the basis using the *pivoting procedure*. This yields a new bfs. Return to Step3.

For a maximization problem, proceed as stated, but replace Step 4 by the following step:

If $u_i + v_j - c_{ij} \geq 0$ for all non-basic variables, then the current bfs is optimal. Otherwise, enter the variable with the most negative $u_i + v_j - c_{ij}$ into the basis using the *pivoting procedure*. This yields a new bfs. Return to Step 3.

Pivoting procedure

1. Find the loop (there is only one possible loop!) involving the entering variable (determined at step 4 of the transport‟n simplex method) and some or all of the basic variables.
2. Counting *only cells in the loop*, label those that are an even number (0, 2, 4, and so on) of cells away from the entering variable as *even cells*. Also label those that are an odd number of cells away from the entering variable as *oddcells*.
3. Find the odd cell whose variable assumes the smallest value. Call this value . The variable corresponding to this odd cell will leave the basis. To perform the pivot, decrease the value of each odd cell by and increase the value of each even cell by . The values of variables not in the loop remain unchanged. The pivot is now complete. If = 0, the entering variable will equal 0, and odd variable that has a current value of 0 will leave the basis.
4.

Example 1. Powerco

The problem is balanced (total supply equals total demand).

When the NWC method is applied to the Powerco example, the bfs in the following table is obtained (check: there exist m+n–1=6 basic variables).

	City1	City2	City3	City4	SUPPLY
Plant1	8 35	6	10	9	35
Plant2	9 10	12 20	13 20	7	50
Plant3 10	14 30	9	16	5	40
DEMAND	45	20	30	30	125

u1 = 0

u1 + v1 = 8 yields v1 = 8

u2 + v1 = 9 yields u2 = 1

u2 + v2 = 12 yields v2 = 11

u2 + v3 = 13 yields v3 = 12

u3 + v3 = 16 yields u3 = 4u3

+ v4=5 yields v4 =1

For each non-basic variable, we now compute $\hat{c}_{ij} = u_i + v_j - c_{ij}\hat{c}_{12} = 0$

+ 11 – 6 = 5

$\hat{c}_{13} = 0 + 12 - 10 = 2$

$\hat{c}_{14} = 0 + 1 - 9 = -8$

$\hat{c}_{24} = 1 + 1 - 7 = -5$

$\hat{c}_{31} = 4 + 8 - 14 = -2$

$\hat{c}_{32} = 4 + 11 - 9 = 6$

Since $\hat{c}_{32}$ is the most positive one, we would next enter x_{32} into the basis: Each unit of x_{32} that is entered into the basis will decrease Powerco‟s cost by \$6. The loop involving x_{32} is (3,2)-(3,3)-(2,3)-(2,2). □ = 10 (see table)

	City 1	City2	City3	City4	SUPPLY
Plant 1	8 35	6	10	35 9	
Plant 2	9 10	12 **20–Φ**	13 **20+Φ**	7	50
Plant 3	14	9 **Φ**	16 **10–Φ**	5 30	40
DEMAND	45	20	30	30	125

x_{33} would leave the basis. New bfs is shown at the following table:

u_i/v_j	8	11	12	7	SUPPLY
0	8 35	6	10	9	35
1	9 10	12 10	13 30	7	50
-2	14	1 5	9	6	40
DEMAND	45	10 20	30	30 30	125

$\hat{c}_{12} = 5$, $\hat{c}_{13} = 2$, $\hat{c}_{14} = -2$, $\hat{c}_{24} = 1$, $\hat{c}_{31} = -8$, $\hat{c}_{33} = -6$

Since $\hat{c}_{12}$ is the most positive one, we would next enter x_{12} into the basis. The loop involving x_{12} is (1,2)-(2,2)-(2,1)-(1,1). □ = 10 (see table)

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8 **35–Φ**	6 **Φ**	10	9	35
Plant 2	9 **10+Φ**	12 **10–Φ**	13 30	7	50
Plant 3	14	9 10	16	5 30	40
DEMAND	45	20	30	30	125

x_{22} would leave the basis. New bfs is shown at the following table:

u_i/v_j	8	6	12	2	SUPPLY
0	8 25	6 10	10	9	35
1	9 20		12 30	13 7	50
3	14		9	5 16	40
DEMAND	45	10 20	30	30 30	125

$\hat{c}_{13} = 2$, $\hat{c}_{14} = -7$, $\hat{c}_{22} = -5$, $\hat{c}_{24} = -4$, $\hat{c}_{31} = -3$, $\hat{c}_{33} = -1$

Since $\hat{c}_{13}$ is the most positive one, we would next enter x_{13} into the basis. The loop involving x_{13} is (1,3)-(2,3)-(2,1)-(1,1). □ = 25 (see table)

	City1	City2	City3	City4	SUPPLY
Plant 1	8 **25–Φ**	6 10	10 13	9	35
Plant 2	**20+Φ**		**30–Φ**	7	50
Plant 3		9 10		5 30	40
DEMAND	45	20	30	30	125

x_{11} would leave the basis. New bfs is shown at the following table:

u_i/v_j	6	6	10	2	SUPPLY
0	8	6 10	10	9 25	35
1 3	9		2 7	13	50
3 14	45		5 9 5	16	40
DEMAND	45	20 10	30	30 30	125

$\hat{c}_{11}$ = -2, $\hat{c}_{14}$ = -7, $\hat{c}_{22}$ = -3, $\hat{c}_{24}$ = -2, $\hat{c}_{31}$ = -5, $\hat{c}_{33}$ = -3

Since all $\hat{c}_{ij}$"s are negative, an optimal solution has been obtained.

Report

45 million kwh of electricity would be sent from plant 2 to city 1.

10 million kwh of electricity would be sent from plant 1 to city 2. Similarly, 10 million kwh of electricity would be sent from plant 3 to city 2.

25 million kwh of electricity would be sent from plant 1 to city 3. 5 million kwh of electricity would be sent from plant 2 to city 3.

30 million kwh of electricity would be sent from plant 3 to city 4 and

Total shipping costis:

$$z = .9 (45) + 6 (10) + 9 (10) + 10 (25) + 13 (5) + 5 (30) = \$ 1020$$

TRANSSHIPMENT PROBLEMS

Sometimes a point in the shipment process can both receive goods from other points and send goods to other points. This point is called as **transshipment point** through which goods can be transshipped on their journey from a supply point to demand point.

Shipping problem with this characteristic is a transshipment problem.

The optimal solution to a transshipment problem can be found by converting this transshipment problem to a transportation problem and then solving this transportation problem.

Remark

As stated in "Formulating Transportation Problems", we define a **supply point** to be a point that can send goods to another point but cannot receive goods from any other point.
Similarly, a **demand point** is a point that can receive goods from other points but cannot send goods to any other point.

Steps

1. If the problem is unbalanced, balance it
   Let s = total available supply (or demand) for balanced problem
2. Construct a transportation tableau as follows
   A row in the tableau will be needed for each supply point and transshipment point A column will be needed for each demand point and transshipment point
   Each supply point will have a supply equal to its original supply Each demand point will have a demand equal to its original demand
   Each transshipment point will have a supply equal to "that point‟s original supply + s"
   Each transshipment point will have a demand equal to "that point‟s original demand + s"
3. Solve the transportation problem

Example 1.Bosphorus

(Based on Winston7.6.)

Bosphorus manufactures LCD TVs at two factories, one in Istanbul and one in Bruges. The Istanbul factory can produce up to 150 TVs per day, and the Bruges factory can produce up to 200 TVs per day. TVs are shipped by air to customers in London and Paris. The customers in each city require 130 TVs per day. Because of the deregulation of air fares, Bosphorus believes that it may be cheaper to first fly some TVs to Amsterdam or Munchen and then fly them to their final destinations.

The costs of flying a TV are shown at the table below. Bosphorus wants to minimize the total cost of shipping the required TVs to its customers.

€	Γo					
From	Istanbul	Bruges	Amsterdam	Munchen	London	Paris
Istanbul	0	-	8	13	25	28
Bruges	-	0	15	12	26	25
Amsterdam	-	-	0	6	16	17
Munchen	-	-	6	0	14	16
London	-	-	-	-	0	-
Paris	-	-	-	-	-	0

Answer:

In this problem Amsterdam and Munchen are *transshipment points*.

Step 1. Balancing the problem Total

supply = 150 + 200 = 350

Total demand = 130 + 130 = 260

Dummy‟s demand = 350 – 260 = 90

s = 350 (total available supply or demand for balanced problem)

Step 2. Constructing a transportation tableau

Transshipment point‟s demand = Its original demand + *s* = 0 + 350 = 350 Transshipment point‟s supply = Its original supply + *s* = 0 + 350 = 350

	Amsterdam	Munchen	London	Paris	Dummy	Supply
Istanbul	8	13	25	28	0	150
Bruges	15	12	26	25	0	200
Amsterdam	0	6	16	17	0	350
Munchen	6	0	14	16	0	350
Demand	350	350	130	130	90	

Step 3. Solving the transportation problem

	Amsterdam	Munchen	London	Paris	Dummy	Supply
Istanbul	8 130	13	25	28	0 20	150
Bruges	15 0		12	26	25	200
Amsterdam	130					350
Munchen	70					
	0		6	16	17	
				130	90	1050

Report:

Bosphorus should produce 130 TVs at Istanbul, ship them to Amsterdam, and transship them from Amsterdam to London.

The 130 TVs produced at Bruges should be shipped directly to Paris. The total shipment is 6370 Euros.

ASSIGNMENT PROBLEMS

There is a special case of transportation problems where each supply point should be assigned to a demand point and each demand should be met. This certain class of problems is called as "assignment problems". For example determining which employee or machine should be assigned to which job is an assignment problem.

LP Representation

An assignment problem is characterized by knowledge of the cost of assigning each supply point to each demand point: c_{ij}

On the other hand, a 0-1 integer variable x_{ij} is defined as follows

$x_{ij}=1$ if supply point i is assigned to meet the demands of demand point j $x_{ij}=0$ if supply point i is not assigned to meet the demands of point j

In this case, the general LP representation of an assignment problem is min $\sum_i \sum_j c_{ij} x_{ij}$

s.t. $\sum_j x_{ij}=1$ (i=1,2,..., m) Supply constraints

$\sum_i x_{ij}=1$ (j=1,2,...,n) Demand constraints

$x_{ij}=0$ or $x_{ij}=1$

Hungarian Method

Since all the supplies and demands for any assignment problem are integers, all variables in optimal solution of the problem must be integers. Since the RHS of each constraint is equal to 1, each x_{ij} must be a nonnegative integer that is no larger than 1, so each x_{ij} must equal 0 or 1. Ignoring the $x_{ij}=0$ or $x_{ij}=1$ restrictions at the LP representation of the assignment problem, we see that we confront with a balanced transportation problem in which each supply point has a supply of 1 and each demand point has a demand of 1.

However, the high degree of degeneracy in an assignment problem may cause the Transportation Simplex to be an inefficient way of solving assignment problems.
For this reason and the fact that the algorithm is even simpler than the Transportation Simplex, the Hungarian method is usually used to solve assignment problems.

Remarks

1. To solve an assignment problem in which the goal is to maximize the objective function, multiply the profits matrix through by –1 and solve the problem as a **minimization** problem.
2. If the number of rows and columns in the cost matrix are unequal, the assignment problem is **unbalanced**. Any assignment problem should be balanced by the addition of one or more dummy points before it is solved by the Hungarian method.

Steps

1. Find the minimum cost each row of the $m*m$ cost matrix.
2. Construct a new matrix by subtracting from each cost the minimum cost in its row
3. For this new matrix, find the minimum cost in each column
4. Construct a new matrix (reduced cost matrix) by subtracting from each cost the minimum cost in its column
5. Draw the minimum number of lines (horizontal and/or vertical) that are needed to cover all the zeros in the reduced cost matrix. If m lines are required, an optimal solution is available among the covered zeros in the matrix. If fewer than m lines are needed, proceed to next step
6. Find the smallest cost (k) in the reduced cost matrix that is uncovered by the lines drawn in Step5
7. Subtract k from each uncovered element of the reduced cost matrix and add k to each element that is covered by two lines. Return to Step5

Example 1. Flight Crew

(Based on Winston 7.5.)

Four captain pilots (CP1, CP2, CP3, CP4) has evaluated four flight officers (FO1, FO2, FO3, FO4) according to perfection, adaptation, morale motivation in a 1-20 scale (1: very good, 20: very bad). Evaluation grades are given in the table. Flight

Company wants to assign each flight officer to a captain pilot according to these evaluations. Determine possible flight crews.

	FO1	FO2	FO3	FO4
CP1	2	4	6	10
CP2	2	12	6	5
CP3	7	8	3	9
CP4	14	5	8	7

Answer:

Step 1. For each row in the table we find the minimum cost: 2, 2, 3, and 5 respectively

Step 2 & 3. We subtract the row minimum from each cost in the row. For this new matrix, we find the minimum cost in each column

	0	2	4	8
	0	10	4	3
	4	5	0	6
	9	0	3	2
Column minimum	0	0	0	2

Step 4. We now subtract the column Minimum from each cost in the column obtaining reduced cost matrix.

0	2	4	6
0	10	4	1
4	5	0	4
9	0	3	0

Step 5. As shown, lines through row 3, row 4, and column 1 cover all the zeros in the reduced cost matrix. The minimum number of lines for this operation is 3. Since fewer than four lines are required to cover all the zeros, solution is not optimal: we proceed to next step.

0	2	4	6
0	10	4	1
4	5	0	4
9	0	3	0

Step 6 & 7. The smallest uncovered cost equals 1. We now subtract 1 from each uncovered cost, add 1 to each twice-covered cost, and obtain

0	1	3	5
0	9	3	0
5	5	0	4
10	0	3	0

Four lines are now required to cover all the zeros: An optimal s9olution is available. Observe that the only covered 0 in column 3 is x_{33}, and in column 2 is x_{42}. As row 5 can not be used again, for column 4 the remaining zero is x_{24}. Finally we choose x_{11}.

Report:

CP1 should fly with FO1; CP2 should fly with FO4; CP3 should fly with FO3; and CP4 should fly withFO4.

Example 2. Maximization problem

	F	G	H	I	J
A	6	3	5	8	10
B	2	7	6	3	2
C	5	8	3	4	6
D	6	9	3	1	7
E	2	2	2	2	8

Report:

Optimal profit = 36

Assigments: A-I, B-H, C-G, D-F, E-J

Alternative optimal solution: A-I, B-H, C-F, D-G, E-J

TRAVELING SALESPERSON PROBLEMS

"Given a number of cities and the costs of traveling from any city to any other city, what is the cheapest round-trip route (tour) that visits each city once and then returns to the starting city?"

This problem is called the traveling salesperson problem (TSP), not surprisingly.

An itinerary that begins and ends at the same city and visits each city once is called a ***tour***.

Suppose there are *N* cities.

Let c_{ij} = Distance from city *i* to city *j* (for i□j) and

Let c_{ii} = M (a very large number relative to actual distances) Also define x_{ij} as a 0-1 variable as follows:

x_{ij} = 1 if s/he goes from city *i* to city *j*;

x_{ij} = 0otherwise

The formulation of the TSP is:

min $\sum_i \sum_j c_{ij} x_{ij}$

s.t. $\sum_i x_{ij} = 1$ for all$j$

$\sum_j x_{ij}=1$ for all$i$

$u_i - u_j + N x_{ij} \leq N-1$ for$i$□$j;i,j$>

1 All x_{ij} = 0 or 1, All $u_i \geq 0$

The first set of constraints ensures that s/he arrives once at each city. The second set of constraints ensures that s/he leaves each city once. The third set of constraints ensure the following:

Any set of x_{ij}"s containing a subtour will be infeasible Any set of x_{ij}"s that forms a tour will be feasible

$u_i - u_j + N x_{ij} \leq N - 1$ for i□j; i, j

>1 Assume N=5

Subtours: 1-5-2-1, 3-4-3 ???

Choose the subtour that does not contain city 1:

$u_3 - u_4 + 5\ x_{34} \leq 4$

$u_4 - u_3 + 5\ x_{43} \leq 4\ 5$

$(x_{34} + x_{43}) \leq 8$

This rules out the possibility that $x_{34} = x_{43} = 1$

The formulation of an IP whose solution will solve a TSP becomes unwieldy and inefficient for large TSPs.

When using branch and bound methods to solve TSPs with many cities, large amounts of computer time may be required. For this reason, heuristics, which quickly lead to a good (but not necessarily optimal) solution to a TSP, are often used.

UNIT – IIIA: SEQUENCING

Introduction

Suppose there are *n* jobs to perform, each of which requires processing on some or all of *m* different machines. The effectiveness *(i.e.* cost, time or mileage, etc.) can be measured for any given sequence of jobs at each machine, and the most suitable sequence is to be selected (which optimizes the effectiveness measure) among all *(n!)m* theoretically possible sequences.Although,theoretically,itisa1wayspossibletoselect the best sequence by testing each one, but it is practically impossible because of large number of computations.

In particular, if *m* = 5 and *n* = 5, the total number of possible sequences will be (5 !)5 = 25,000,000,000. Hence the effectiveness for each of (5 !)5 sequences is to be computed before selecting the most suitable one. But, this approach is practically impossible to adopt. So easier methods of dealing with such problems are needed.

Before proceeding to our actual discussion we should explain what the sequencing problem is. The problem of sequencing may be defined as follows:

Definition: Suppose there are n jobs (1, 2, 3,..., n), each of which has to be processed one at a time at each of m machines A, *B,* C, ... The order of processing each job through machines is given(for example, job is processed through machines A, C, and B in this order). The time that each job must require on each machine is known. The problem is to find a sequence among*(n*
!)m number of all possible sequences (or combinations) (or order) for processing the jobs so that the total elapsed time for all the jobs will be minimum. Mathematically, let

Ai = time for job *i* on machine A,

Bi = time for job *i* on machine *B,* etc.

T = time from start of first job to completion of the last job.

Then, the problem is to determine for each machine a sequence of jobs *i1, i2, i3, ...,in*- where

(i1,i2, i3, ..., in) is the permutation of the integers which will minimize *T.*

Terminology and Notations

The following *terminology* and notations will be used in this chapter.

i. Number of Machines. It means the service facilities through which. a job must pass before it is completed..

ii. For example, a book to be published has to be processed through composing, printing, binding, etc. In this example, the book constitutes the *job* and the different processes constitute the *number of machines.* .

iii. Processing order:. It refers to the order in which various machines are required for completing the job.

iv. Processing Time: It means the time required by each job on each machine. The notation *Tij* will denote the processing time required for Ith job b y*jth* machine *(i* = 1,2,..., *n;j*= 1, 2,...,*m)*.

v. Idle Time on a Machine. This is the time for which a machine remains idle during the total elapsed time. The notation *Xij* shall be (used to denote the idle time of machine *j* between the *end* of the (i - 1)th job and the start of the ith job.

vi. Total Elapsed Time. This is the time between starting the first job and completing the last job. This also includes *idle time,* if exists. It will be denoted by the symbol *T*.

vii. No Passing Rule. This rule means that P1lssing is not allowed, *i.e* .the same order of jobs is maintained over each machine. If each of the n-jobs is to be processed through two machines *A* and *B* in the order *AB*, then this rule means that each job will go to machine *A* first and then to *B*.

Principal Assumptions

i. No machine can process more than one operation at a time.

ii. Each operation ,once started ,must be performed till completion.

iii. A job is an entity, *i.e.* even though the job represents a lot of individual parts, no lot may be processed by more than one machine at a time.

iv. Each operation must be completed before any other operation, which it must precede, can begin.

v. Time intervals for processing are independent of the order in which operations are performed.

vi. There is only one of each type of machine.

vii. A job is processed as soon as possible subject to ordering requirements.

viii. All jobs are known and are ready to start processing before the period under consideration begins.

ix. The time required to transfer jobs between machines is negligible.

- Processingnjobsand2Machines
- Processingnjobsand3Machines
- Processing n jobs and m Machines
- PROCESSING N JOBS THROUGH TWO MACHINES:

The problem can be described as: (i) only two machines *A* and *B* are involved, *(ii)* each job is is processed in the order *AB*, and *(iii)* the exactor expected processing times*A),Az,A3,...,An;B),B2,B3,...,* *Bn* are known

The problem is to sequence (order) the jobs so as to minimize the total elapsed time *T*.

The solution procedure adopted by *Johnson* (1954) is given below.

Processing Times	Job (*i*)				
	1	2	3	...	*n*
A_i	A_1	A_2	A_3	...	A_n
B_i	B_1	B_2	B_3	...	B_n

Solution Procedure

Step 1. Select, the least processing time occurring in the list *A* I, *Az, A3,...,* Ar and *Bt, B2, B3'* ..., *BII'* If there is a tie, either of the smallest processing time should be selected.

Step 2. If the least processing time is A_r select *r th* job *first.* If it is *Bs*, do the s th job *last* (as the given order *is AB).*

Step 3. There are now *n* - I jobs left to be ordered. Again repeat steps I and n for the reduced set of processing times obtained by deleting processing times for both the machines corresponding to the job already assigned. .

Continue till all jobs have been ordered. The resulting ordering will minimize the elapsed time T.

Proof. Since passing is not allowed, all *n* jobs must be processed on machine *A* without any *idle time* for it. On the other hand, machine *B is* subject to its remaining idle time at various stages. Let *Yj* be the time for which m machine B remains idle after completing *(i* - l)th job and before starting processing the ith job *(i*=1,2, ..., *n).* Hence, the total elapsed time T is given by

$$T=\sum_{i=1}^{n} B_i+\sum_{i=1}^{n} Y_i$$

Determine a sequence for five jobs that will minimize the elapsed timeT. Calculate

2				

Now, the reduced list of processing times becomes

Job	*A*	*B*
1	5	$\boxed{2}$
3	9	7
4	3	8
5	10	4

Again, the smallest processing time in the reduced list is 2 for job 1 on the machine *B*. So place job 1 last.

2				1

Continuing in the like manner, the next reduced list is obtained

Job	*A*	*B*
3	9	7
4	$\boxed{3}$	8
5	10	4

thetotal idle time for the machines in this period.

Solution. Apply steps I and II of solution procedure. It is seen that the smallest processing time is one hour for job 2 on the machine A. So list the job 2 at first place as shown above.

leading to sequence

2	4			1

and the list

Job	A	B
3	9	$\boxed{7}$
5	10	4

gives rise to sequence

2	4		5	1

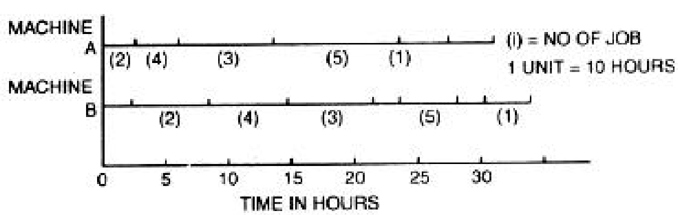
Finally, the optimal sequence is obtained,

2	4	3	5	1

Further, it is also possible to calculate the minimum elapsed time corresponding to the optimal sequencing, using the individual processing time given in the statement of the problem. The details are given in Table

Thus, the minimum time, i.e. the time for starting of job 2 to completion of the last job 1, is 30 hrs only. During this time, the machine A remains idle for 2 hrs (from 28 to 30 hrs) and the machine B remains idle for 3 hrs only (from 0-1,22-23, and 27-28 hrs). The total elapsed time can also be calculated by using Gantt chart as follows:

Now finally we enumerate all these programs one by one using Gantt Chart as shown below:



‘From these charts it is clear that optimum program is 6th and the minimum elapsed time is 18 hours. From the Fig it can be seen that the total elapsed time is 30 hrs, and the idle time of the machine B is 3 hrs. In this problem, it is observed that job may be held in inventory before going to the machine. For example, 4th job will be free on machine A after 4th hour and will start on machine B after 7th hr. Therefore, it will be kept in inventory for 3 hrs. Here it is assumed that the storage space is available and the cost of holding the inventory for each job is either same or negligible. For short duration process problems, it is negligible. Second general assumption is that the order of completion of jobs has no significance, i.e. no job claims the priority.

Processing n Jobs Through Three Machines

The problem can be described as: *(i)* Only three machines *A, B* and C are involved, *(ii)* each job is processed in the prescribed order *ABC, (iii)* transfer of jobs is not permitted, *i.e.* adhere strictly the order over each machine, and *(iv)* exact or expected processing times are given in

Optimal Solution. So far no general procedure is available for obtaining an optimal sequence in this case.

However, the earlier method adopted by *Johnson* (1954) can be extended to cover the special cases where *either one or both* of the following conditions hold:

- The minimum time on machine A the maximum time on machine B.
- The minimum time on machine C the maximum time on machine B.

The procedure explained here (without proof) is to replace the problem with an equivalent problem, involving *n* jobs and two fictitious machines denoted by G

$$G_i = A_i + B_i, \; H_i = B_i + C_i.$$

and *H,* and corresponding time *Gj* and *Hj* are defined by

Here... stands for other machines, if any. Applying the rules to this example, it is observed by taking *A* as X, and D as *Y* (I rule), that delete the programs containing *ad.* Such a program is 16thonly. Again by n rule taking *A* as X and C as *Y,* all those programs are deleted which contain *lie, i.e.,* the 5th program. Other rules are not applicable to our problem.

Here... stands for other machines, if any. Applying the rules to this example, it is observed by taking *A* as X, and D as *Y* (I rule), that delete the programs containing *ad.* Such a program is 16thonly.AgainbynruletakingAasXandCasY, all those programs are deleted which contain *lie, i.e.,* the 5th program. Other rules are not applicable to our problem. Thus we have only following five programs

Further, it is also possible to calculate the minimum elapsed time corresponding to the optimal sequencing, using the individual processing time given in the statement of the problem. The details are given in Table

Thus, the minimum time, i.e. the time for starting of job 2 to completion of the last job 1, is 30 hrs only. During this time, the machine A remains idle for 2 hrs (from 28 to 30 hrs) and the machine B remains idle for 3 hrs only (from 0-1,22-23, and 27-28 hrs).