INSTITUTE OF AERONAUTICAL ENGINEERING

DEPARTMENT OF MASTER OF BUSINESS ADMINISTRATION

QUANTITATIVE ANALISIS FOR BUSINESS DECISION

LECTURE NOTES

MBA III SEMESTER

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1. INTRODUCTION

TERMINOLOGY

The British/Europeans refer to "operationalresearch", the Americans to "operations research" - but both are often shortened to just "OR" (which is the term we willuse).

Another term which is used for this field is "management science" ("MS"). The Americans sometimes combine the terms OR and MS together and say "OR/MS" or "ORMS".

Yet other terms sometimes used are "industrial engineering"("IE"), "decision science" ("DS"), and "problem solving".

In recent years there has beena move towards a standardizationupon a single term for the field, namely the term "OR".

THE METHODOLOGY OF OR

When ORis used to solve a problemof an organization, the following sevenstep procedure should be followed:

Step 1. Formulate the Problem: OR analyst first defines the organization's problem. Defining the problem includes specifying the organization's objectives and the parts of the organization (or system) that must be studied before the problem can be solved.

Step 2. Observe the System: Next, the analyst collects data to estimate the values of parameters that affect the organization's problem. These estimates are used to develop (in Step 3) and evaluate (in Step 4) a mathematical model of the organization's problem.

Step 3. Formulate a Mathematical Model of the Problem: The analyst, then, develops a mathematical model (in other words an idealized representation) of the problem. In this class, we describe many mathematical techniques that can be used to model systems.

Step 4. Verify the Model and Use the Model for Prediction: The analyst now tries to determine if the mathematical model developed in Step 3 is an accurate representation of reality. To determine how well the model fits reality, one determines how valid the model is for the current situation.

Step 5. Select a Suitable Alternative: Given a model and a set of alternatives, the analyst chooses the alternative (if there is one) that best meets the organization's objectives. Sometimes the set of alternatives is subject to certain restrictions and constraints. In many situations, the best alternative may be impossible or too costly to determine.

Step 6. Present the Results and Conclusions of the Study: In this step, the analyst presents the model and the recommendations from Step 5 to the decision making individual or group. In some situations, one might present several alternatives and let the organization choose the decision maker(s) choose the one that best meets her/his/their needs.

After presenting the results of the OR study to the decision maker(s), the analyst may find that s/he does not (or they do not) approve of the recommendations. This may result from incorrect definition of the problem on hand or from failure to involve decision maker(s) from the start of the project. In this case, the analyst should return to Step 1, 2, or 3.

Step 7. Implement and Evaluate Recommendation: If the decision maker(s) has accepted the study, the analyst aids in implementing the recommendations. The system must be constantly monitored (and updated dynamically as the environment changes) to ensure that the recommendations are enabling decision maker(s) to meet her/his/their objectives.

HISTORY OF OR

OR is a relatively new discipline. Whereas 70 years ago it would have been possible to study mathematics, physics or engineering (for example) at university it would not have been possible to study OR, indeed the term OR did not exist then. It was only really in the late 1930's that operational research began in a systematic fashion, and it started in the UK.

Early in 1936 the British Air Ministry established Bawdsey Research Station, on the east coast, near Felixstowe, Suffolk, as the centre where all pre-war radar experiments for both the Air Force and the Army would be carried out. Experimental radar equipment was brought up to a high state of reliability and ranges of over 100 miles on aircraft were obtained.

It was also in 1936 that Royal Air Force (RAF) Fighter Command, charged specifically with the air defense of Britain, was first created. It lacked however any effective fighter aircraft - no Hurricanes or Spitfires had come into service - and no radar data was yet fed into its very elementary warning and control system.

It had become clear that radar would create a whole new series of problems in fighter direction and control so in late 1936 some experiments started at Biggin Hill in Kent into the effective use of such data. This early work, attempting to integrate radar data with ground based observer data for fighter interception, was the start of OR.

The first of three major pre-war air-defense exercises was carried out in the summer of 1937. The experimental radar station at Bawdsey Research Station was brought into operation and the information derived from it was fed into the general air-defense warning and control system. From the early warning point of view this exercise was encouraging, but the tracking information obtained from radar, after filtering and transmission through the control and display network, was not very satisfactory.

In July 1938 a second major air-defense exercise was carried out. Four additional radar stations had been installed along the coast and it was hoped that Britain now had an aircraft location and control system greatly improved both in coverage and effectiveness. Not so! The exercise revealed, rather, that a new and serious problem had arisen. This was the need to coordinate and correlate the additional, and often conflicting, information received from the additional radar stations. With the out-break of war apparently imminent, it was obvious that something new - drastic if necessary - had to be attempted. Some new approach was needed.

Accordingly, on the termination of the exercise, the Superintendent of Bawdsey Research Station, A.P. Rowe, announced that although the exercise had again demonstrated the technical feasibility of the radar system for detecting aircraft, its operational achievements still fell far short of requirements. He therefore proposed that a crash program of research into the operational - as opposed to the technical - aspects of the system should begin immediately. The term "operational research" [RESEARCH into (military) OPERATIONS] was coined as a suitable description of this new branch of applied science. The first team was selected from amongst the scientists of the radar research group the same day.

In the summer of 1939 Britain held what was to be its last pre-war air defense exercise. It involved some 33,000 men, 1,300 aircraft, 110 antiaircraft guns, 700 searchlights, and 100 barrage balloons. This exercise showed a great improvement in the operation of the air defense warning and control system. The contribution made by the OR teams was so apparent that the Air Officer Commander-in-Chief RAF Fighter Command (Air Chief Marshal Sir Hugh Dowding) requested that, on the outbreak of war, they should be attached to his headquarters at Stanmore.

On May 15th 1940, with German forces advancing rapidly in France, Stanmore Research Section was asked to analyze a French request for ten additional fighter squadrons (12 aircraft a squadron) when losses were running at some three squadrons every two days. They prepared graphs for Winston Churchill (the British Prime Minister of the time), based upon a study of current daily losses and replacement rates, indicating how rapidly such a move would deplete fighter strength.

No aircraft were sent and most of those currently in France were recalled. This is held by some to be the most strategic contribution to the course of the war made by OR (as the aircraft and pilots saved were consequently available for the successful air defense of Britain, the Battle of Britain).

In 1941 an Operational Research Section (ORS) was established in Coastal Command which was to carry out some of the most well-known OR work in World War II.

Although scientists had (plainly) been involved in the hardware side of warfare (designing better planes, bombs, tanks, etc) scientific analysis of the operational use of military resources had never taken place in a systematic fashion before the Second World War. Military personnel, often by no means stupid, were simply not trained to undertake such analysis.

These early OR workers came from many different disciplines, one group consisted of a physicist, two physiologists, two mathematical physicists and a surveyor. What such people brought to their work were "scientifically trained" minds, used to querying assumptions, logic, exploring hypotheses, devising experiments, collecting data, analyzing numbers, etc. Many too were of high intellectual caliber (at least four wartime OR personnel were later to win Nobel prizes when they returned to their peacetime disciplines).

By the end of the war OR was well established in the armed services both in the UK and in the USA.

OR started just before World War II in Britain with the establishment of teams of scientists to study the strategic and tactical problems involved in military operations. The objective was to find the most effective utilization of limited military resources by the use of quantitative techniques.

Following the end of the war OR spread, although it spread in different ways in the UK and USA.

You should be clear that the growth of OR since it began (and especially in the last 30 years) is, to a large extent, the result of the increasing power and widespread availability of computers. Most (though not all) OR involves carrying out a large number of numeric calculations. Without computers this would simply not be possible

WHAT IS OPERATIONS RESEARCH?

Operations

- The activities carried out in an organization.

Research

- The process of observation and testing characterized by the scientific method. Situation, problem statement, model construction, validation, experimentation, candidate solutions.

Model

- An abstractrepresentation of reality. Mathematical, physical, narrative, set of rules in computer program.

Systems Approach

- Include broad implications of decisions for the organizationat each stage in analysis. Both quantitative and qualitative factors are considered.

Optimal Solution

- A solution to the model that optimizes (maximizes or minimizes) some measure of merit over all feasible solutions.

Team

- A group of individuals bringing various skills and viewpoints to a problem.

Operations Research Techniques

- A collection of general mathematical models, analytical procedures, and algorithms.

Definition of OR?

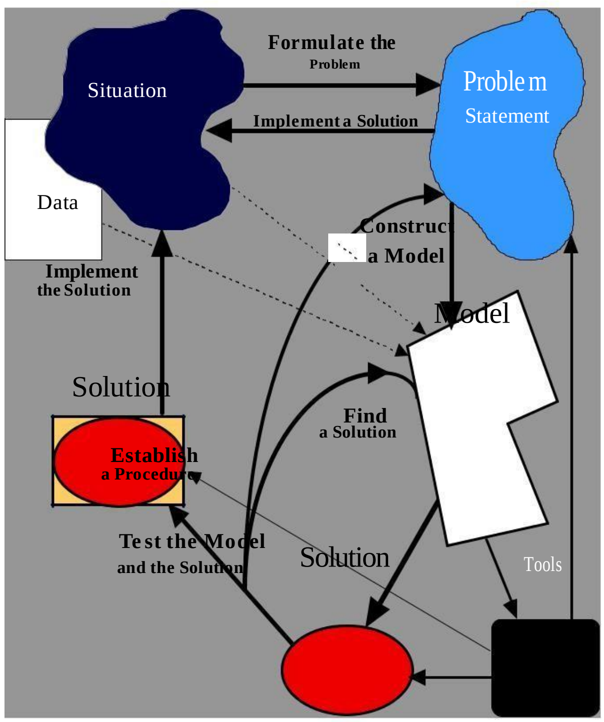
1. OR professionals aim to provide rational bases for decision making by seeking to understand and structure complex situations and to use this understanding to predict system behavior and improve system performance.

2. Much of this work is done using analytical and numerical techniques to develop and manipulate mathematical and computer models of organizational systems composed of people, machines, and procedures.

Problem Solving Process

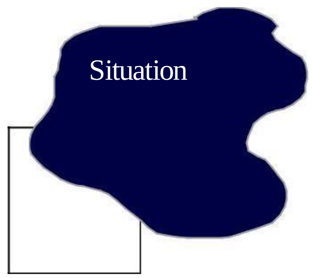
Goal: solve a problem

- Model must be valid
- Model must be tractable
- Solution must be useful



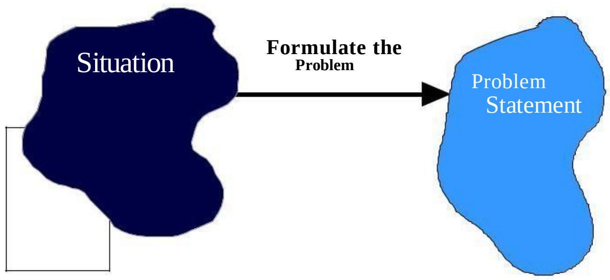
The Situation

- May involve current operations or proposed expansions due to expected market shifts
- May become apparent through consumer complaints or through employee suggestions
- May be a conscious effort to improve efficiency or response to an unexpected crisis.



Example: Internal nursing staff not happy with their schedules; hospital using too many external nurses.

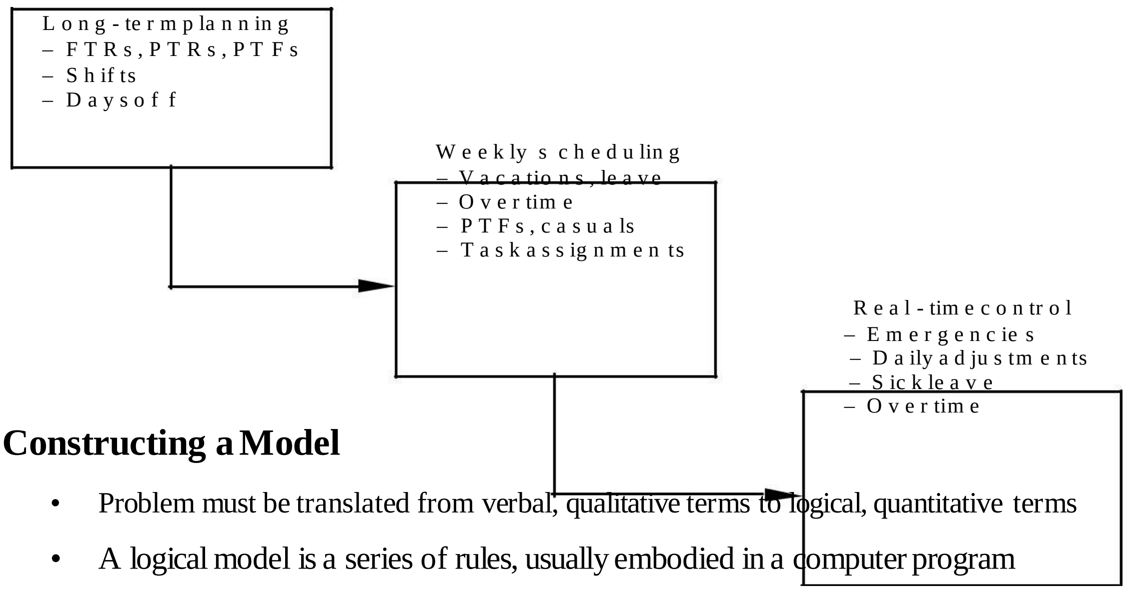
Problem Formulation



- Describe system
- Define boundaries
- State assumptions
- Select performance measures
- Define variables
- Define constraints
- Data requirements

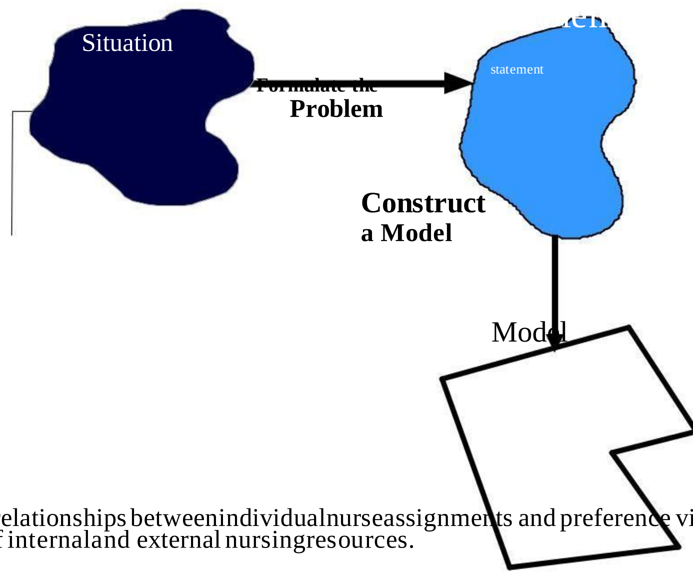
Example: Maximize individual nurse preferences subject to demand requirements.

Personnel Planning and Scheduling: Example of Bounding a Problem



Constructing a Model

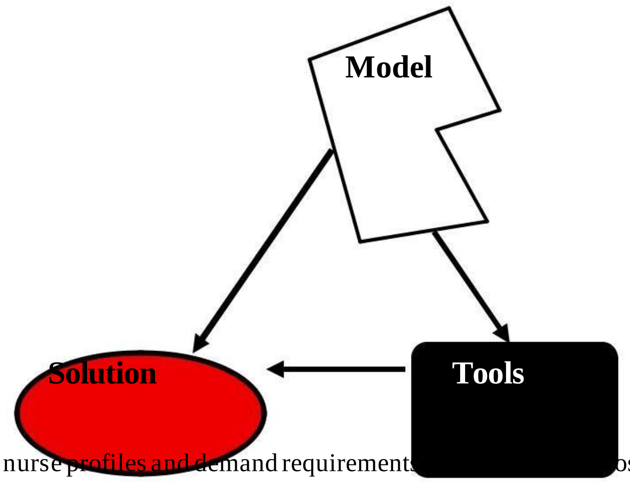
- Problem must be translated from verbal, qualitative terms to logical, quantitative terms
- A logical model is a series of rules, usually embodied in a computer program
- A mathematical model is a collection of functional relationships by which allowable actions are delimited and evaluated.



Example: Define relationships between individual nurse assignments and preference violations; define tradeoffs between the use of internal and external nursing resources.

Solving the Mathematical Model

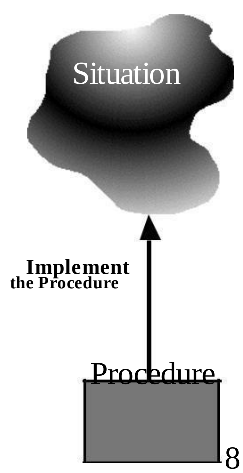
- Many tools are available as will be discussed in this course
- Some lead to “optimal” solutions
- Others only evaluate candidatestrial and error to find “best” course of action



Example: Read nurse profiles and demand requirements [illegible] ost-processes results to get monthly schedules.

Implementation

- A solution to a problem usually implies changes for some individuals in theorganization
- Often there is resistance to change, making the implementation difficult
- User-friendly system needed
- Those affected should go through training



Example: Implement nurse scheduling system in one unit at a time. Integrate with existing HR and T&A systems. Provide training sessions during the workday.

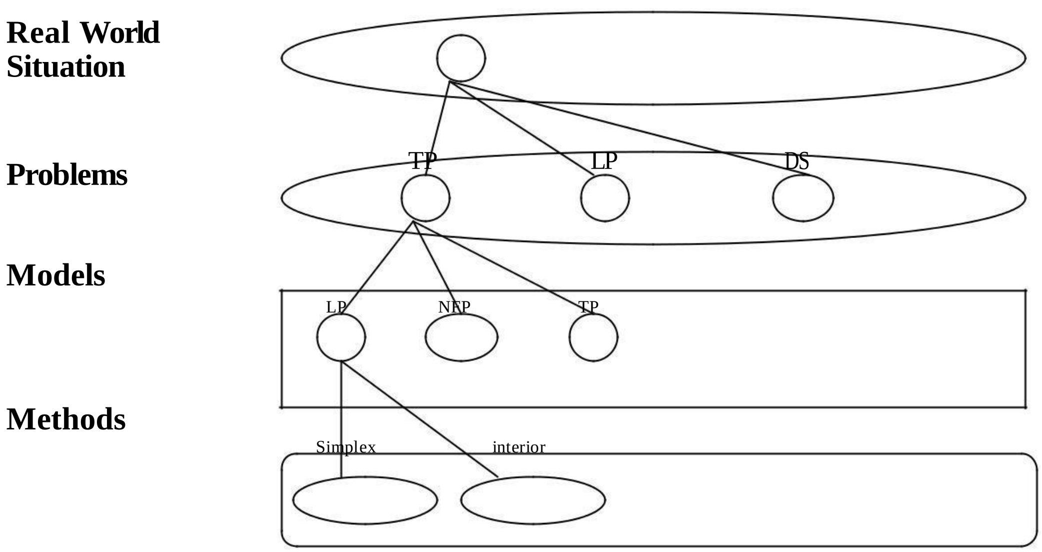
Components of OR-Based Decision Support System:-

Decision support systems based on Operations research models can be of immense value to the decision makers. The Components of a Typical Decision support system are as follows:-

- Data base (nurse profiles, external resources, rules)
- Graphical User Interface (GUI); web enabled using java or VBA
- Algorithms, pre- and post- processor
- What-if analysis
- Report generators

Problems, Models and Methods:-

Operations research consists of the Modeling the real life situations using standard mathematical approaches as problems. These are known as Models. In order to derive solutions to these models specialized techniques such as simplex algorithm, Interior point algorithm etc have been developed. An appreciation of these Problems, Models and Methods forms the Tool kit of Operations Research. The Course on operations research covers the Knowledge on the models and methods using the OR problem solving methodology.



Operations Research Models

Deterministic Models	Stochastic Models
Linear Programming Chains	Discrete-Time Markov
Network Optimization	Continuous-Time Markov Chains
Integer Programming	Queueing
Nonlinear Programming	Decision Analysis

Deterministic vs. Stochastic Models

Deterministic models - assume all data are known with certainty

Stochastic models - explicitly represent uncertain data via Random variables or stochastic processes.

Deterministic models involve optimization
Stochastic models - characterize / estimate system performance.

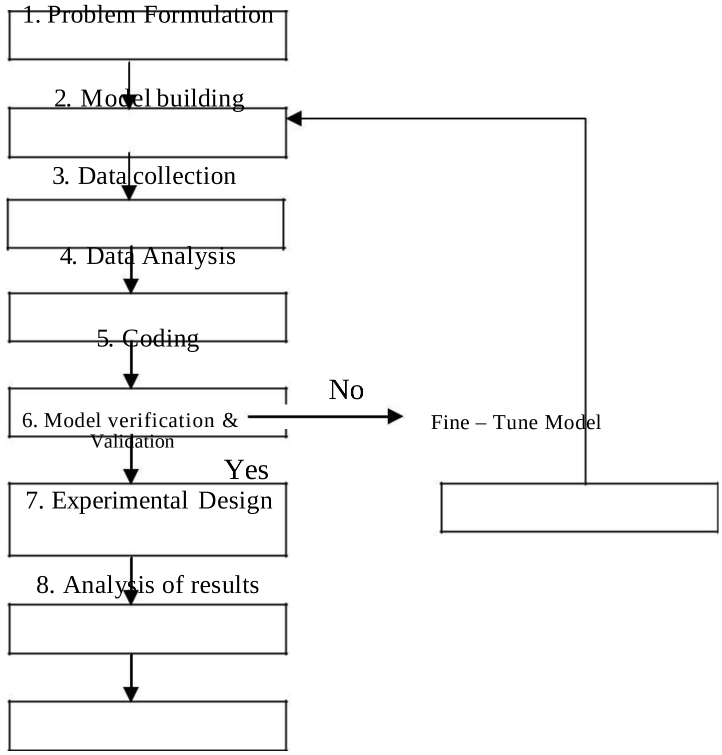
Examples of OR Applications:-

Some typical applications that can be developed using OR methodology includes:-

- Rescheduling aircraft in response to groundings and delays
- Planning production for printed circuit board assembly
- Scheduling equipment operators in mail processing & distribution centers
- Developing routes for propane delivery
- Adjusting nurse schedules in light of daily fluctuations in demand

1.8 Steps in OR Study:-

The typical flow of the OR problem solving methodology is depicted in the flow chart as below.



BASIC OR CONCEPTS

"OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing."
We can also define a mathematical model as consisting of:

Decision variables, which are the unknowns to be determined by the solution to the model.

Constraints to represent the physical limitations of the system
An objective function

An optimal solution to the model is the identification of a set of variable values which are feasible (satisfy all the constraints) and which lead to the optimal value of the objective function.

In generalterms we can regard OR as being the application of scientific methods / thinking to decision making.

Underlying OR is the philosophy that:
decisions have to be made; and

Using a quantitative (explicit, articulated) approachwill lead to better decisions than usingnon-quantitative (implicit, unarticulated) approaches.

Indeed it can be argued that although OR is imperfect it offers the best available approach to making a particular decision in many instances (which is not to say that using OR will produce the right decision).

Two Mines Example

The Two Mines Company own two different mines that produce an ore which, after being crushed, is graded into three classes: high, medium and low -grade. The company has contracted to provide a smelting plant with 12 tons of high-grade, 8 tons of medium-grade and 24 tons of low-grade ore per week. The two mines have different operating characteristics as detailed below.

Mine	**Cost per day (£'000)**	**Production (tons / day)**		
		High	**Medium**	**Low**
X	180	6	3	4
Y	160	1	1	6

How many days per week should each mine be operated to fulfill the smelting plant contract?

Guessing

To explore the Two Mines problem further we might simply guess (i.e. use our judgment) how many days per week to work and see how they turn out.

work one day a week on X, one day a week on Y

This does not seem like a good guess as it results in only 7 tones a day of high-grade, insufficient to meet the contract requirement for 12 tones of high-grade a day. We say that such a solution is infeasible.

work 4 days a week on X, 3 days a week on Y

This seems like a better guess as it results in sufficient ore to meet the contract. We say that such a solution is feasible. However it is quite expensive (costly).

We would like a solution which supplies what is necessary under the contract at minimum cost. Logically such a minimum cost solution to this decision problem must exist. However even if we keep guessing we can never be sure whether we have found this minimum cost solution or not. Fortunately our structured approach will enable us to find the minimum cost solution.

Solution

What we have is a verbal description of the Two Mines problem. What we need to do is to translate that verbal description into an equivalent mathematical description.

In dealing with problems of this kind we often do best to consider them in the order:

.. Variables

.. Constraints

.. Objective

This process is often called formulating the problem (or more strictly formulating a mathematical representation of the problem).

Variables

These represent the "decisions that have to be made" or the "unknowns".

Let

x = number of days per week mine X is operated

y = number of days per week mine Y is operated

Note here that x >= 0 and y >= 0.

Constraints

It is best to first put each constraint into words and then express it in a mathematical form.

ore productionconstraints - balance the amount produced with the quantity required under the smelting plant contract

Ore
High $6x + 1y >= 12$
Medium $3x + 1y >= 8$
Low $4x + 6y >= 24$
days per week constraint - we cannot work more than a certain maximum number of days a week e.g. for a 5 day week we have
$x <= 5$
$y <= 5$

Inequality constraints

Note we have an inequality here rather than an equality. This implies that we may produce more of some grade of ore than we need. In fact we have the general rule: given a choice between an e quality and an inequality choose the inequality

For example - if we choose an equality for the ore production constraints we have the three equations 6x+y=12, 3x+y=8 and 4x+6y=24 and there are no values of x and y which satisfy all three equations (the problem is therefore said to be "over-constrained"). For example the values of x and y which satisfy 6x+y=12 and 3x+y=8 are x=4/3 and y=4, but these values do not satisfy 4x+6y=24.

The reason for this general rule is that choosing an inequality rather than an equality gives us more flexibility in optimizing (maximizing or minimizing) the objective(deciding values for the decision variables that optimize the objective).

Implicit constraints

Constraints suchas days per weekconstraint are often called implicit constraints because theyare implicit in the definition of the variables.

Objective

Again in words our objective is (presumably) to minimize cost which is given by 180x + 160y

Hence we have the complete mathematical representation of the problem:

minimize $180x + 160y$

subject to $6x + y >= 12$

$3x + y >= 8$

$4x + 6y >= 24$

$x <= 5$ y
$<= 5$ x,y
$>= 0$

Some notes

The mathematical problem given above has the form

all variables continuous (i.e. can take fractional values)
a single objective (maximize or minimize)

the objective and constraints are linear i.e. any term is either a constant or a constant multiplied by an unknown (e.g. 24, 4x, 6y are linear terms but xy is a non-linear term)

Any formulation which satisfies these three conditions is called a linear program (LP). We have (implicitly) assumed that it is permissible to work in fractions of days - problems where this is not permissible and variables must take integer values will be dealt with under Integer Programming (IP).

Discussion

This problem was a decision problem.

We have taken a real-world situation and constructed an equivalent mathematical representation - such a representation is often called a mathematical model of the real-world situation (and the process by which the model is obtained is called formulating the model).

Just to confuse things the mathematical model of the problem is sometimes called the formulation of the problem.

Having obtained our mathematical model we (hopefully) have some quantitative method which will enable us to numerically solve the model (i.e. obtain a numerical solution) - such a quantitative method is often called an algorithm for solving the model.

Essentially an algorithm(for a particular model) is a set of instructions which, when followed in a step-by-step fashion, will produce a numerical solution to that model.

Our model has an objective that is something which we are trying to optimize. Having obtained the numerical solution of our model we have to translate that solution bac k into the real-world situation.

"OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing."

Definition:

Linear programming (LP) or Linear Programming Problem (LPP)

The general LPP calls for optimizing (MAX/MIN) a linear function of variables called the OBJECTIVE FUNCTION subject to a set of linear equations and or inequalities called for CONSTRAINTS or RESTRICTIONS.

Linear programming problem arises whenever two or more candidates or activities are competing for limited resources.

Linear programming applies to optimization technique in which the objective and constraints fu nctions are strictly linear.

Application of Linear Programming

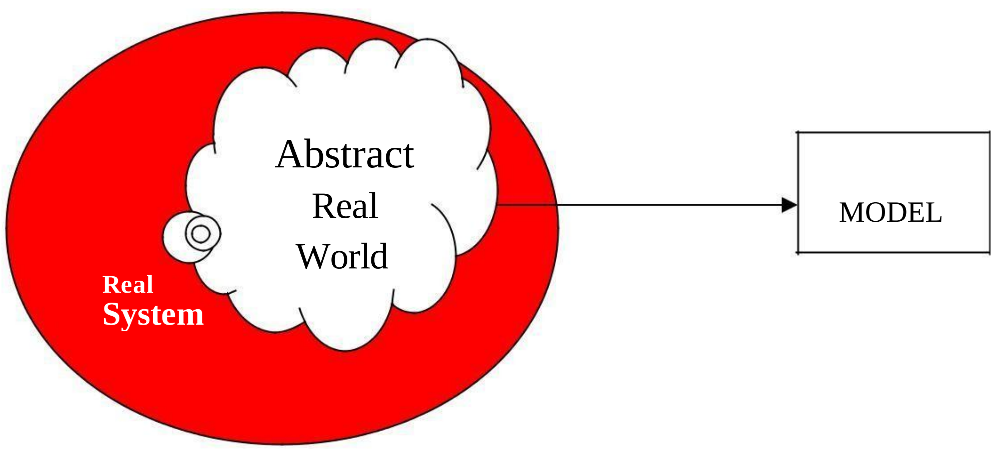
Agriculture, industry, transportation, economics, health Systems, behavioraland socialsciences and the military

It can be computerized for 10000 of constraints and variables.

Art of Modeling:

Models are developedas exact representation of realsituations in the sense that no approximations are used.

The figure below depicts the level of abstraction from the real situation by concentrating on the dominant variables that control the behavior of the real system.



Levels of abstractionin modeldevelopment Example:
A Plastic Manufacture company
Step 1: An order is issued to the Production Department
Step 2: Acquires the required raw materials or procures from outside
Step 3: Once the product ion batchis completed thesales departmenttakes Chargeof distributing the product to customers.
The overall system, a number of variables involved are

1. Production department: Production capacity expressed in terms of available machine and labor hours in-process inventory and quality control standards.
2. Materials Department: Available stockof raw materials, delivery schedules from outside sources and storage limitations.
3. Sales Department: sales forecast capacity of distribution facilities, effectiveness of the advertisement campaign and effect of competition.

A first level defining the boundaries of the assumed real world and the two dominate variables

1. Production rate.
2. Consumption rate.

Mathematical formulation:

A mathematical program is an optimization problem in which the objective and constraints are given as mathematical functions and functional relationships.

The procedure for mathematical formulation of a LPP consists of the following steps

Step1: write down the decision variables (Products) of the problem

Step2: formulate the objective function to be optimized (maximized or minimized) as linear function of the decision variables

Step3: formulate the other conditions of the problem such as resource limitation, market, constraints, and interrelations between variables etc., linear in equations or equations in terms of the decision variables.

Step4: add non-negativity constraints

The objective function set of constraint andthe non-negativeconstraint together forma Linear Programming Problem.

The general formulations of the LPP can be stated as follows:
In order to find the values of n decision variables

x_1, x_2, x_3---------- x_n to MAX or MIN the objective function.

Max $Z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$ (a) $\longrightarrow$ Objective Function

Also satisfy m – constraints or Subject to Constraint

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2 \\ | \quad | \\ | \quad | \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_n \end{array}\right\} \text{(b)} \longrightarrow \text{Constant}$$

$$x_1 >= 0,\ x_2 >= 0,\ \ldots\ldots + x_n >= 0 \quad \text{(c)} \longrightarrow \text{Non Negative Restriction}$$

c_j (j = 1,2----n) is the objective function in equations (a) are called cost coefficient (max profit or min cost)

b_i (i= 1,2,---- m) defining the constraint requirements or available in equation(b) or available in equation (b) is called stipulations and the constants a_{ij} (i=1,2,----m; j= 1,2,---

n)are called structural co-efficient in equation (c) are known as non-negative restriction Matrix form

$$\underset{(m \times n)}{A} = \begin{pmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} \ldots\ldots & a_{2n} \\ & & \\ a_{m1} & a_{m2} \ldots\ldots & a_{mn} \end{pmatrix} \qquad \underset{(m \times n)}{X} = \begin{pmatrix} x_1 \\ x_2 \\ -- \\ -- \\ x_n \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ -- \\ -- \\ b_n \end{pmatrix}$$

and $C = (c_{1n}, c_{2n} \ldots\ldots\ldots c_n)$

A is called the coefficient matrix X is the decisions Vector
b is the requirement Vector and c is the profit (cost) vector of the linear program.

The LPP can be expressed in the matrix as follows

Max or Min Z= CX $\longrightarrow$ **Objective Function**
Subject to Constraint
AX=b $\longrightarrow$ **Structural coefficient**
X>=0 $\longrightarrow$ **Non negativity**

Problem 1

A Manufacture produces two types of models M_1 and M_2 each model of the type M_1 requires 4 hrs of grinding and 2 hours of polishing, where as each model of the type M_2 requires 2 hours of grinding and 5 hours of polishing. The manufactures have 2 grinders and 3 polishers. Each grinder works 40 hours a week and each polishers works for 60 hours a week. Profit on M_1 model is Rs. 3.00 and on Model M_2 is Rs 4.00. Whatever produced in week is sold in the market. How should the manufacturer allocate is production capacity to the two types models, so that he may make max in profit in week?

Solutions:

Decision variables. Let X and X be the numbers of units of M_1 and M_2 Model

Objective function: since, the profit on M_1 and M_2 is Rs. 3.0 and Rs 4. Max $Z = 3x_1 + 4x_2$

Constraint: there are two constraints one for grinding and other is polishing

No of grinders are 2 and the hours available in grinding machine is 40 hrs per week, therefore, totalno of hours available of grinders is 2 X 40 = 80 hours

No of polishers are 3 and the hours available in polishing machine is 60 hrs per week, therefore, totalno of hours available of polishers is 3 X 60 = 180 hours

The grinding constraint is given by: $4x_1 + 2x_2 <= 80$

The Polishing Constraint is given by: $2x_1 + 5x_2 <= 180$

Non negativity restrictions are

$x_1 , x_2 >= 0$ if the company is not manufacturing any products

The LPP of the given problem Max $Z = 3x_1 + 4x_2$

STC

$4x_1 + 2x_2 <= 80$

$2x_1 + 5x_2 <= 180$

$x_1 , x_2 >= 0$

Problem 2:

Egg contains 6 units of vitamin A per gram and 7 units of vitamin B per gram and cost 12 paise per gram. Milk contains 8 units of vitamin A per gram and 12 units of vitamin B per gram and costs 20 paise per gram. The daily requirements of vitamin A and vitamin B are 100 units and 120 units respectively. Find the optimal product mix.

	EGG	MILK	Min Requirements
Vitamin A	6	8	100
Vitamin B	7	12	120
Cost	12	20	

The LPP of the given Problem

Min $Z = 12x_1 + 20x_2$

STC

$6x_1 + 8x_2 \geq 100$

$7x_1 + 12x_2 \geq 120$ $x_1, x_2 \geq 0$

Problem 3: A farmer has 100 acre. He can sell all tomatoes. Lettuce or radishes he raise the price. The price he can obtain is Re 1 per kg of tomatoes, Rs 0.75 a head for lettuce and Rs 2 per kg of radishes. The average yield per acre is 2000kg tomatoes, 3000 heads of lettuce and 1000 kgs of radishes. Fertilizer is available at Rs 0.5 per kg and the amount required per acre 100 kgs each for tomatoes and lettuce, and 50 kgs for radishes. Labor required for sowing, cultivating and harvesting per acre is 5 man-days for tomatoes and radishes, 6 man-days for lettuce. A total of 400 man days of labor available at Rs 20 per man day formulate the problem as linear programming problem model to maximize the farmers' total profit.

Formulation:

Farmer's problemis to decide howmuch area should be allottedto eachtypeof crop. He wants to growto maximize his totalprofit.

Let the farmer decide to allot X_1, X_2 and X_3 acre of his land to growtomatoes, lettuce andradishes respectively.

So the farmer will produce 2000 X_1 kgs of tomatoes, 3000 X_2head of lettuce and 1000 X_3 kgs of radishes.

Profit= sales – cost

= sales – (Labor cost +fertilizer cost)

Sales = $1 \times 2000 X_1 + 0.75 \times 3000 X_2 + 2 \times 1000 X_3$ Labor cost = $5 \times 20 X_1 + 6 \times 20 X_2 + 5 \times 20 X_3$ Fertilizer cost = $100 \times 0.5 X_1 + 0.5 \times 100 X_2 + 0.5 \times 50 X_3$

Max Z= $1850 X_1 + 2080 X_2 + 1875 X_3$
STC
$X_1 + X_2 + X_3 <= 100$ $5X_1 + 6X_2$
$+ 5X_3 <= 400$ $X_1 , X_2 , X_3 >= 0$

Problem 4:

A Manufacturer of biscuits is considering 4 types of gift packs containing 3 types of biscuits, orange cream (oc), chocolate cream (cc) and wafer's(w) market research study conducted recently to assess the preferences of the consumers shows the following types of assortments to be in good demand.

Assortments	Contents	Selling Price per kg in Rs
A	Not less than 40% of OC Not more than 20% of CC Any quantity of W	29
B	Not less than 20% of OC Not more than 40% of CC Any quantity of W	25
C	Not less than 50% of OC Not more than 10% of CC Any quantity of W	22
D	No restrictions	12

For the biscuits the manufacture capacity and costs are for given below.

Biscuits variety	Plant Capacity Kg/ day	Manufacture cost Rs / Kg
OC	200	8
CC	200	9
W	150	7

Formulate a LP model to find the productionschedule which maximizes the profit assuming that there are no market restrictions.

Formulation: the company manufacturer 4 gift packs which oc, cc and w. the quantity of ingredients in each pack is not known.

Let x_{11} denotes the quantities OC of gift pack A
x_{12} denotes the quantities CC of gift pack A
x_{13} denotes the quantities W of gift pack A
x_{21} denotes the quantities OC of gift pack B
x_{22} denotes the quantities CC of gift pack B
x_{23} denotes the quantities W of gift pack B
x_{31} denotes the quantities OC of gift pack C
x_{32} denotes the quantities CC of gift pack C
x_{33} denotes the quantities W of gift pack C
x_{41} denotes the quantities OC of gift pack D
x_{42} denotes the quantities CC of gift pack D
x_{43} denotes the quantities W of gift pack D

The objective Function is to max total profit

Max Z = $20(x_{11}+x_{12}+x_{13}) + 25(x_{21}+x_{22}+x_{23}) + 22(x_{31}+x_{32}+x_{33}) + 12(x_{41}+x_{42}+x_{43}) - 8(x_{11}+x_{21}+x_{31}+x_{41}) - 9(x_{12}+x_{22}+x_{32}+x_{42}) - 7(x_{13}+x_{23}+x_{33}+x_{43})$

STC

Gift pack A

$x_{11} >= 0.4\ (x_{11}+x_{12}+x_{13})$ $x_{12} <= 0.2\ (x_{11}+x_{12}+x_{13})$

Gift pack B

$x_{21} >= 0.2\ (x_{21}+x_{22}+x_{23})$ $x_{12} <= 0.2\ (x_{21}+x_{22}+x_{23})$

Gift pack C

$x_{31} >= 0.2\ (x_{31}+x_{32}+x_{33})$ $x_{32} <= 0.2\ (x_{31}+x_{32}+x_{33})$

$\sum X_{ij} \sum X_{ij} \sum X_{ij} \sum X_{ij} \geq 0$

Graphical Method:

The graphical procedure includes two steps

1. Determination of the solution space that defines all feasible solutions of the model.

2. Determination of the optimum solution from among all the feasible points in the solution space.

There are two methods in the solutions for graphical method

Extreme point method
Objective function line method

Steps involved in graphical method are as follows:

Consider each inequality constraint as equation

Plot each equation on the graph as each will geometrically represent a straight line.

Mark the region. If the constraint is <= type then region below line lying in the first quadrant (due to non negativity variables) is shaded.

If the constraint is >= type then region above line lying in the first quadrant is shaded.

Assign an arbitrary value say zero for the objective function.

Draw the straight line to represent the objective function with the arbitrary value Stretch the objective function line till the extreme points of the feasible region. In the maximization case this line will stop farthest from the origin and passing through at least one corner of the feasible region.

In the minimization case, this line will stop nearest to the origin and passing through at least one corner of the feasible region.

Find the co-ordination of the extreme points selected in step 6 and find the maximum or minimum value of Z.

Problem 1

Max $Z = 3x_1 + 5x_2$

STC

$x_1 \le 4$

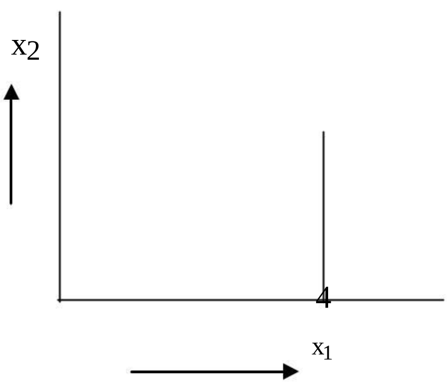
$2x_2 \le 12$

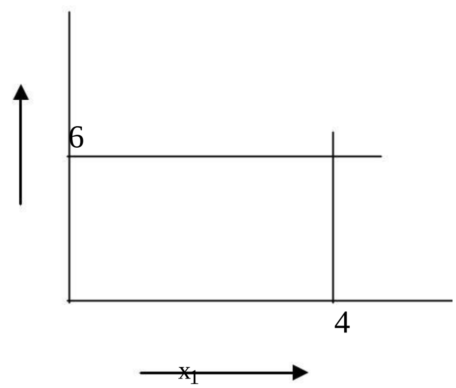
$3x_1 + 2x_2 \le 18$

$x_1, x_2 \ge 0$

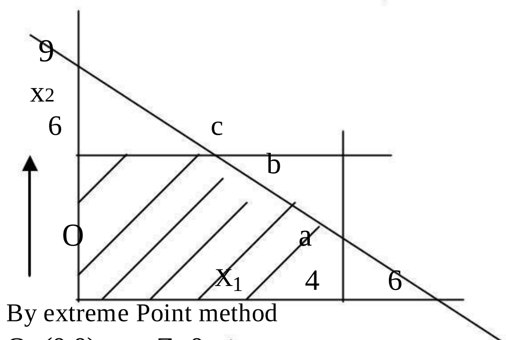
Solution

$x_1 \le 4$ $\quad$ $x_1 = 4$ $\quad$ $2x_2 \le 12$ $\quad$ $2x_2 = 12,\ x_2 = 6$





$3x_1 + 2x_2 \le 18$ $\quad$ put $x_1 = 0$, $x_1 = 6$, put $x_2 = 0$, $x_2 = 9$



By extreme Point method

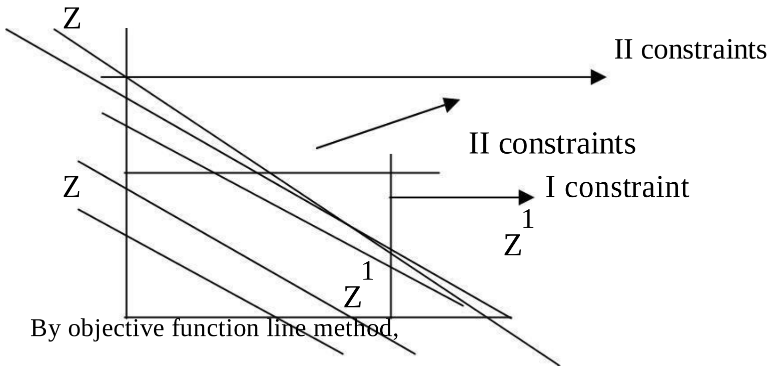
O=(0,0) $\rightarrow$ Z=0

A=(4,0) Z=12

B=(4,3) Z=12+15=27

C=(2,6) Z= 6+30 =36

D= (0,6) Z= 30



By objective function line method,

To find the point of Maxvalue of Z in the feasible region we use objective function line as ZZ^1 , same type of lines are used for different assumed Z value to find the MaxZ in the solutionspace as shown in the above figure.

Let us start = 10

Max Z=10 =$3x_1 + 5x_2$ this will show the value as (3.33,2) by plotting this points in the solution space it explains that Z must be large as 10 we can see many points above this line and within the region.

When Z=20

Max Z=20 =$3x_1 + 5x_2$ this will show the value as (6.66, 4) by plotting this points in the solution space z must be at least 20. The trial and error procedure involves nothing more than drawing a family of parallel lines containing at least one point in the permissible region and selecting the distance from the origin. This lines passes through the points (2,6) or Z=36

Max Z = 36= $3x_1 + 5x_2$ the points are (12, 7.2) this points lies at the intersection of the 2 lines $2x_2$ =12 and $3x_1 + 2x_2 = 18$. so, this point can be calculated algebraically as the simultaneous solutions of these 2 equation.

Conclusions:

The solution indicates thatthe company should produceproducts 1 & 2 at the rate of 2 per minute an d6 / minute respectively with resulting profitable of 36 / minute.

No other mix of 2 products would be profitable according to the model.

Problem 1: find the max Value of the given LPP

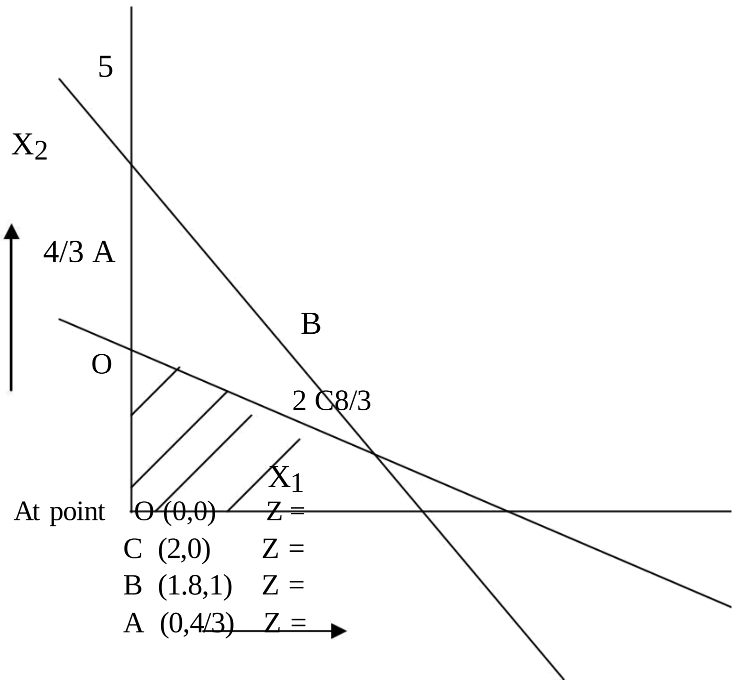
Max Z =$x_1 + 3x_2$

STC

$3x_1 + 6x_2 <= 100$ ⟹ (8/3, 4/3)

$5x_1 + 2x_2 <= 120$ ⟹ (2,5)

$x_1 , x_2 >= 0$



At point O (0,0) Z =

C (2,0) Z =

B (1.8,1) Z =

A (0,4/3) Z =

Problem 2: find the max Value of the given LPP Max Z =5x1 + 2x2

STC

x 1 + x2 <= 4

3x1 + 8x2 <= 24

10x1 +78x2 <= 35

x1 , x2 >= 0

Problem 3: find the max Value of the given LPP Max Z = -x1 + 2x2

STC

-x1 +3x2 <= 10

x1 +x2 <= 6

x1 - x2 <= 2

x1 , x2 >= 0

Problem 4: find the max Value of the given LPP Max Z =7x1 + 3x2

STC

x1 + 2x2 <= 3

x1 + x2 <= 4

0<= x1<=5/2

0 <= x2 <= 3/2

x1 , x2 >= 0

Problem 5: find the max Value of the given LPP

Max Z =20x1 + 10x2

STC

x1 +2x2 <= 40

3x1 + x2 <= 30

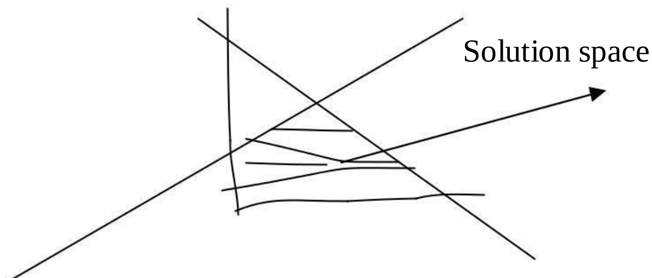
4x1 + 3x2 <= 30

x1 , x2 >= 0

Solution space

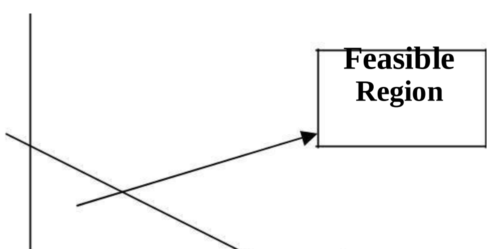
Solutions mean the final answer to a problem

Solutions space to a LPP is the space containing such points. The co-ordinates of which satisfy all the constraints simultaneously. The region of feasibility of all the constraints including non -negativity requirements.



Feasible:

The feasible region for an LP is the set of all points thatsatisfies all the LPs constraints andsign restrictions.



Basic feasible

A b[redacted] a basic solution which alsosatisfies that is allbasic variables are non-negative.

Example:

$4x_1 + 2x_2 <= 80$ we add x3 as slackvariable $2x_1 + 5x_2 <=$ 180 we add x4 as slackvariable

$$\begin{bmatrix} 4 & 2 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ is unit matrix is called basic feasible solution

Optimal

Any feasible solution which optimizes (Min or Max) the objective function of the LPP is called its optimum solution.

Infeasible / Inconsistency in LPP

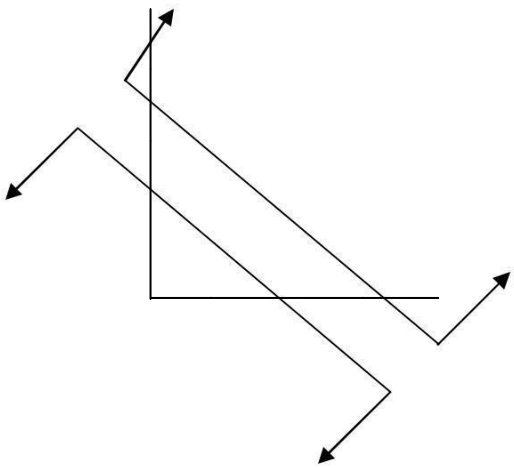
Inconsistency also known as infeasibility

The constraint systemis such that one constraint opposes one or more. It is not possible to find one common solutions to satisfy all the constraints in the system.

Ex:-

$2x_1 + x_2$ <= 20 → (10,20)

$2x_1 + x_2$ <= 40 → (20,40)



If both the constraint cannot be satisfied simultaneously. Such constraint system is said to be raise to inconsistency or infeasibility

Redundancy;

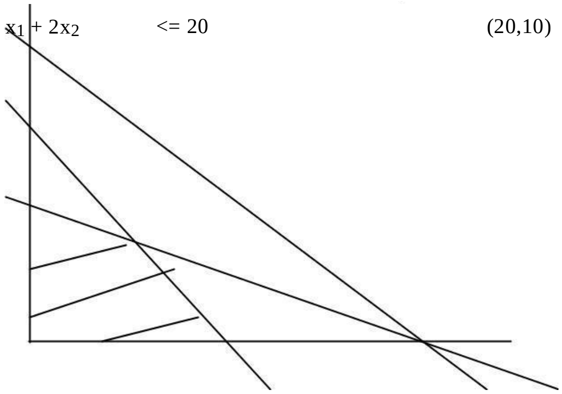
A set of constraintis said to be redundant if one or more of them are automatically satisfied on the basis of the requirement of the others.

Ex:

$2x_1 + x_2$ <= 20 → (10,20)

$2x_1 + x_2$ <= 35 → (17.5, 35)

$x_1 + 2x_2$ <= 20 (20,10)



A redundant constraint system is one in which deletion of at least one of the constraint will not alter the solution space.

Standard form

The standard form of a linear programming problem with m constraints and n variables can be represented as follows:

$$\text{Max } Z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$$

Also satisfy m – constraints or Subject to

Constraint $a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2$

| |

$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_n$

$x_1 >= 0$, $x_2 >= 0$, ……. + $x_n >= 0$

The main features of the standard form

1. the objective function is of the maximization or minimization type
2. all constraints are expressed as equations
3. all variables are restricted to be nonnegative
4. The right-hand side constant of each constraint is nonnegative.

Now, considering how a LPP can be formulated in the standard form will be as follows:

Case (a): if a problem aims at minimizing the objective function. Then it can be converted into a maximization problem simply by multiplying the objective by (-1)

Case (b): if a constraint is of <= type , we add a non negative variable called slack variables is added to the LHS of the constraint on the other hand if the constraint is of >= type, we subtract a non -negative variable called the surplus variable from the LHS.

Case (c) when the variables are unrestricted in sign it can be represented as $X_j = X^1{}_j - X^{11}{}_j$ or $X_1 = X^1{}_1 - X^{11}{}_1$

It may become necessary to introduce a variable that can assume both +ve and –ve values. Generally, unrestricted variable is generally replaced by the difference of 2 non - ve variables.

Problem 1:

Rewrite in standard form the following linear programming problem Min $Z = 12x_1 + 5x_2$

STC

$5x_1 + 3x_2 \quad >= 15$

$7x_1 + 2x_2 \quad <= 14$

$x_1 , x_2 >= 0$

Solution:

Since, the given problem is minimization then it should be converted to maximization by just multiply by (-1) and the first constraint is >= type it is standard by adding by surplus variable as $x_3>=0$ and 2^{nd} constraint is <= type it is standard by adding slack variable and then the given problem is reformulated as follows:

Max $Z = -12x_1 - 5x_2 - 0x_3 + 0x_4$

STC

$5x_1 + 3x_2 - x_3 = 15$ $7x_1 + 2x_2 + x_4 = 14$ $x_1 , x_2 , x_3 , x_4 >= 0$

The matrix form

Max $Z = (-12, -5, 0, 0)\ (-x_1, -x_2, -x_3, x_4)$

STC

$$\begin{Bmatrix} 5 & 3 & -1 & 0 \\ 7 & 2 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 15 \\ 14 \end{Bmatrix}$$

$x_1 , x_2 , x_3 , x_4 >= 0$

Problem 2:

Rewrite in standard form the following linear programming problem Max $Z = 2x_1 + 5x_2 + 4x_3$

STC

$-2x_1 + 4x_2 <= 4$ $x_1 +$
$2x_2 + x_3 >= 5$ $2x_1 + 3x_3$
$<= 2$
$x_1, x_2 >= 0$ x_3 is unrestricted in sign

Solution:

In the given problem it is maximization problem and the constraint are of in equations. The first constraint is <= type we introduce slack variable as $x_4 >= 0$, 2nd constraint is of >= type, we introduce surplus variable as $x_5 >= 0$ and third constraint is <= type we introduce slack variable as $x_6 >= 0$. the x_3 variable is un restricted in sign. So, this can be written as $X_3 = X^1_3 - X^{11}_3$

Then, the given LPP is rewritten as

Max $Z = 2x_1 + 5x_2 + 4X^1_3 - 4X^{11}_3 + 0x_4 - 0x_5 + 0x_6$

STC

$-2x_1 + 4x_2 + x_4 = 4$
$x_1 + 2x_2 + X^1_3 - X^{11}_3 - x_5 = 5$
$2x_1 + 3X^1_3 - 3X^{11}_3 + 0x_6 = 2$
$x_1, x_2, x^1_3, x^{11}_3, x_4, x_5, x_6 >= 0$

The matrix form

Max $Z = (2,5,4,-4,0,0,0)\ (x_1, x_2, x^1_3, x^{11}_3, x_4, x_5, x_6)$

STC

$$\begin{Bmatrix} -2 & 4 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & -1 & 0 & -1 & 0 \\ 2 & 0 & 3 & -3 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 5 \\ 2 \end{Bmatrix}$$

$x_1, x_2, x^1_3, x^{11}_3, x_4, x_5, x_6 >= 0$

Reference Books:

1 Taha H A, Operation Research - An Introduction, Prentice Hall of India, 7th edition, 2003

2 Ravindran, Phillips and Solberg, Operations Research : Principles and Practice, John Wiely & Sons, 2nd Edition

3 D.S.Hira, Operation Research, S.Chand & Company Ltd., New Delhi, 2004

4.

In order to develop a general solutions for LPP having more than 2 variables the LPP must be put in the standard form.

The ideas conveyed by the graphical LP solution lay the foundation for the development of the algebraic simplex method.

Graphical Method	Algebraic simplex method
Solution space consists of infinity of feasible points	The system has infinity of feasible solution
Candidate for the optimum solution are given by a finite number of corner points	Candidate for the optimum solution are given by a finite number of basic feasible solution
Use the objective function to determine the optimum corner point	Use the objective function to determine the optimum basic feasible solution

Principles of the simplex method.

The simplex method is developed by G.B Dantzig is an iterative procedure for solving linear programming problem expressed in standard form. In addition to this simplex method requires constraint equations to be expressed as a canonical system form which a basic feasible solution can be readily obtained

The solutions of any LPP by simplex algorithm the existence of an IBFS is always assumed, the following steps help to reach an optimum solution of an LPP.

Procedure for Maximization

Step 1 write the given LPP in standard form

Step 2 check whether the objective function of the given LPP is to be Maximized or minimized if its to be minimized then we convert it into a problem of maximizing using the result.

Min z = - Max Z or (-Z)

Step 3 check whether all b_i (i=123…..m0 are non-negative. if anyone b_i is –VE, then multiply the corresponding in equation of the constraint by-1.

Step 4 Convert all the in equations of the constraint into equations by introducing slack or surplus variable in the constraints. Put these costs equal to zero in objective cost.

Step 5 Obtain an IBFS to problem in the form identity matrix form in canonical form 1 0 0 and put it in the 1st column of simples table.

0 1 0
0 0 1

Step 6 compute the net evaluation row ($Z_j - C_j$) (j=1,2,3….n) $Z_j - C_j$ = P/U (Profit / unit) x X_j -C_j (j=1,2,3…n)

examine the sign of $Z_j - C_j$

i) if all ($Z_j - C_j$) >= 0 then the IBFS solutions column is an optimum basic feasible solutions.

ii) if at least one ($Z_j - C_j$) <=0 proceed to the next step

Step 7 if there are more than one –ve ($Z_j - C_j$) then choose the most –ve of them then it will become key column

i) if al**l** the no's in the key column is –ve then there is an unbounded solutions to the given problem

ii) if at least one X_m >0 (m =123…n) then the corresponding vector X_m >=0 (m=1,2,3…n) then the corresponding vector X_m entry the basis of solution column

Step 8 compute the ratios = solutions column no / key column no.

And choose the minimum of them. Let the minimum of these ratios be the key row these variable in the basic variable column of the key row will become the leaving element or variable.

Step 9 using the below relation to find new no of other than key row and new no for key row also

New no for pivot row = current pivot row / pivot element

Other than key row

New element = old element – (PCE * NPRE)

PCE= Pivot column element, NPRE=new pivot row element

New no= old no – (corresponding Key column / Key element)

x (corresponding key row)

Step 10 go to step5 and repeat the computational procedure until either an optimum solutions is obtained or there is an indication of an unbounded solution.

Note : case 1 in case of a tie for entering basis vector. i.e., there are 2 or more $Z_j – C_j$ which are equal and at the same time the highest –ve values then arbitrary selection of any one of them will not alter optimality.

Case 2 in case of a tie for the leaving variable i.e., there are 2 or more min ratio column i.e., (solution no / key column no) which are equal and greater than zero then arbitrary select any one of them will not alter optimality. But, if the tied ratios are zeros then charnes method of penalty should be followed.

Problem 1:

Use simplex method to solve the given LPP

Max Z $=5x_1 + 3x_2$

STC

$x_1 + x_2 <= 25x_1$
$+ 2x_2 <= 10\ 3x_1$
$+ 8x_2 <= 12\ x_1,$
$x_2 >= 0$

Solution:

Step 1: Since the problem is maximization problem all the constraint are <= type and the requirements are +ve. This satisfies the simplex method procedure.

Step 2: since all the constraints are <= type we introduce the slack variables for all the constraints as $x_3 >=0$, $x_4 >=0$, $x_5 >=0$ for the I II and III constraint

Step 3: the given LPP can be put in standard form

Max Z $=5x_1 + 3x_2 + (0)\ x_3 + (0)\ x_4 + (0)x_5$

STC

$x_1 + x_2 + x_3 <= 2$

$5x_1 + 2x_2 + x_4 <= 10$

$3x_1 + 8x_2 + x_5 <= 12$

$x_1, x_2, x_3, x_4, x_5 >= 0$

Step 4: matrix form

$(5,3,0,0,0)\ (x_1, x_2, x_3, x_4, x_5$

Max Z =)

STC

$$\begin{Bmatrix} 1 & 1 & 1 & 0 & 0 \\ 5 & 2 & 0 & 1 & 0 \\ 3 & 8 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 10 \\ 12 \end{Bmatrix}$$

$x_1, x_2, x_3, x_4, x_5 >= 0$

Step 5 : since, considering sub-matrix from the matrix are which form basic variables for the starting table of simplex

(1 0 0) (0 1 0) (0 0 1) are linearly independent column vectors of A.

Therefore, the sub Matrix is

$$B= \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

The corresponding variables of the sub matrix is (x_3, x_4 ,x_5) and these variables are the basic variables for the starting iteration of the simplex problem and there obvious initial basic feasible solutions are (x_3, x_4 ,x_5) = (2, 10, 12)

			5	3	0	0	0	
Basic Variable (B V)	Profit / Unit (P/U)	solution	x_1	x_2	x_3	x_4	x_5	Min Ratio = soln. no /key column No.
x_3	0	2	1	1	1	0	0	2/1 = 2
x_4	0	10	5	2	0	1	0	10/5 = 2
x_5	0	12	3	8	0	0	1	12/3 = 4
	Max Z= 0x2+0x10 +0x12		=0x1 +0x5 +0x3 -5	=0x1 +0x2 +0x8 -3	=0x1 +0x0 +0x0 -0	=0x0 +0x1 +0x0 -0	=0x0 +0x0 +0x1 -0	
			-5	-3	0	0	0	

Leaving variable

Key Row

Key Column

Entering Variable

5 3 0 0 0

Basic Variable (B V)	Profit / Unit (P/U)	solution	x_1	x_2	x_3	x_4	x_5	Min Ratio = soln. no /key column No.
x_1	5	2	1	1	1	0	0	
x_4	0	0	0	-3	-5	1	0	
x_5	0	6	0	5	-3	0	1	
	Max Z= 5x2+0x0+0x6 =10		=5x1+ 0x0 +0x0- 5	=5x1+0 x-3 +0x5 -3	=5x1 +0x-5 + 0x-3 - 0	=5x0+ 0x1+0 x0-0	=5x0+ 0x0+0 x1-0	
			0	2	5	0	0	

New Numbers for Key row

Soln.= old no / Key element

= 2/ =2

x_1 = 1/1 = 1

x_2 = 1/1 = 1

x_3 = 1/1 = 1

x_4 = 0/1 = 0

x_5 = 0/1 = 0

other than key rows new no is found by using the following formulae

New No=old element – PCE*NPRE

for x_4 row newno are

Soln. = 10-5*2= 0

x_1 = 5-5*1 = 0

x_2 = 2-5*1 = -3

x_3 = 0-5*1 = -5

x_4 = 1-5*0 =1

x_5 = 0 -5*0 =0

for x_5 row the newno are

Soln. = 12-3*2 =6

x_1 =3-3*1 = 0

x_2 = 8-3*1 = 5

x_3 = 0-3*1 = -3

x_4 = 0-3*0 = 0

x_5 = 1 -3*0 = 1

Since, the given problems net evaluation row is +ve, then given problem as attained the optimum

Therefore, x_1 =2, x_2 =0, x_3 =0, x_4 =0, x_5 =6,
Substitute in the objective function

Max Z =$5x_1 + 3x_2 + (0)\, x_3 + (0)\, x_4 + (0)x_5$
Max Z =5x2+3x0+0x0+0x0+0x6
Max Z = 10

Problem no 2

Solve the given problem by simplex method
Max Z =$107x_1 + x_2 + 2\, x_3$
STC

$14x_1 + x_2 - 6x_3 + 3x_4 = 7$

$16x_1 + 1/2x_2 - 6x_3 <= 5$

$16x_1 - 8x_2 - x_3 <= 0$

$x_1, x_2, x_3, x_4 >= 0$

Solutions:

In the given problem the objective function is MaxZ and it has only three variables.
The I constraint is of standard form already slack variable is introduced as $x_4 >= 0$ and the value should be one but, it is having 3 due this it should be divided by three for enter equation on both sides and II & III constraint are of <= type so we introduce $x_5 >= 0$ x_6 >= 0 as slack variable.

Then the given problem can be rewritten as
Max Z =$107x_1 + x_2 + 2\, x_3 + 0x_4 + 0\, x_5 + 0\, x_6$
STC

$14/3x_1 + 1/3\, x_2 – 6/3x_3 + 3/3x_4 = 7/3$

$16x_1 + 1/2x_2 - 6x_3 + x_5 = 5$

$16x_1 - 8x_2 - x_3 + x_6 = 0$

$x_1, x_2, x_3, x_4\ x_5\ x_6 >= 0$

The matrix form

Max Z= (107, 1, 2, 0, 0, 0) $(x_1, x_2, x_3, x_4, x_5, x_6)$

STC

$$\begin{Bmatrix} 14/3 & 1/3 & -2 & 1 & 0 & 0 \\ 16 & 1/2 & -6 & 0 & 1 & 0 \\ 16 & -1 & -1 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} \begin{Bmatrix} 7/3 \\ =5 \\ 0 \end{Bmatrix}$$

$x_1, x_2, x_3, x_4 \; x_5 \; x_6 \;\; >= 0$

$$\begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

Now, the canonical form from the above matrix is the corresponding variables are $(x_4 \; x_5 \; x_6)$ and their obvious solution is (7/3 5 0)

Starting Table: 107 1 2 0 0 0

Basic variable (B.V)	Profit /unit P/U	Solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratio =Soln. no/ Pivot column no
x_4	0	7/3	14/3	1	-6	1	0	0	=7/3 / 14/3=0.5
x_5	0	5	16	½	-6	0	1	0	=16/5= 3.2
x_6	0	0	16	-1	-1	0	0	1	=16/0=∞
	Max Z = 0 x 7/3 + 0 x 5 + 0 x 0 = 0		=(0x14/3 +0x16+0x 16) -107	=(0x1+0 x1/2+0x -1) -1	= (0 x-6 +0x- 6+0x-1) -2	=(0x1 +0x0 +0x0) - 0	=(0x0+ 0x1+0x 0)- 0	=(0x0+0 x0+0x1) - 0	
			-107	-1	-2	0	0	0	

First Table 1:

			107	1	2	0	0	0	
Basic variable (B.V)	Profit /unit P/U	Solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratio =Soln. no/ Pivot column no
x_1	107	0.5	1	3/14	1	3/14	0	0	
x_5	0	-3	0	-41/14	-22	-48/14	1	0	
x_6	0	-8	0	-62/14	- 17	-48/14	0	1	
	Max Z = 107 x 0.5 + 0 x -3 +0 x -8 = 0		=(1x107 +0x0 +0x0) -107	=(107x3 /14 + 0 x -41/14 + 0 x -62/14) - 1	= (107 x 1 + 0 x -22 + 0 x-17) -2	=(107x3 /14 + 0 x -48/14 + 0 x -48/14) - 0	=(107 x 0 +0 x 1+ 0 x 0) - 0	=(107x0 + 0x0 +0x1) - 0	
	Max Z=53.5		0	+ve	+ve	+ve	+ve	0	

New Numbers for Key row

Soln.= old no / Key element

= 7/3/ 14/3 =0.5

x_1 = 14/3 / 14/3 = 1

x_2 = 1/14/3 = 3/14

x_3 = -6 / 14/3 = 1

x_4 = 1/14/3 = 3/14

x_5 = 0/14/3 = 0

x_6 = 0/14/3 = 0

for x_6 row the new no are

Soln. = 0-16*0.5 = -8

x_1 =16-16*1 = 0

x_2 = -1-16*3/14 = -62/14

x_3 = -1-16*1 = -17

x_4 = 0-16*3/14 = -48/14

x_5 = 0 -16*0 = 0

x_6 = 1 -16*0 = 1

other than key rows new no is found by using the following formulae

New No=old element – PCE*NPRE

for x_5 row new no are

Soln. = 5-16*0.5= -3

x_1 = 16-16*1 = 0

x_2 = 1/2-16*3/14 = -41/14

x_3 = -6-16*1 =-22

x_4 = 0-16*3/14= - 48/14

x_5 = 1 -16*0 =1

x_6 = 0 -16*0 =0

Since, all the NER is positive then given problem is optimal
Therefore, $x_1 = 0.5$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, $x_5 = -3$, $x_6 = -8$
Max Z = 107*0.5 + 1*0 +2*0+ 0*0 +0*-3 +0* -8
= 53.5

Problem 3:

Solve the given LPP by simplex method
Max $Z = 4x_1 + 5x_2 + 9x_3 + 11x_4$
STC

$x_1 + x_2 - 6x_3 + 3x_4 \quad <= 7$

$7x_1 + 5x_2 + 3x_3 + 2x_4 \quad <= 120$

$3x_1 + 5x_2 + 10x_3 + 15x_4 \quad <= 100$

$x_1, x_2, x_3, x_4 >= 0$

Solution:

The given problem is maximization problem and all the constraints are <= type, so, we introduce the slack variable as $x_5 >= 0$,
$x_6 >= 0$, $x_7 >= 0$ for I , II and III constraints respectively.

The given LPP can be written as
Max $Z = 4x_1 + 5x_2 + 9x_3 + 11x_4$
STC

$x_1 + x_2 + x_3 + x_4 + x_5 \quad = 7$

$7x_1 + 5x_2 + 3x_3 + 2x_4 + x_6 \quad = 120$

$3x_1 + 5x_2 + 10x_3 + 15x_4 + x_7 \quad = 100$

$x_1, x_2, x_3, x_4 >= 0$

Matrix form

Max Z = (4, 5, 9, 11, 0, 0, 0) (x_1, x_2, x_3, x_4 x_5, x_6, x_7)

STC

$$\begin{Bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 7 & 5 & 3 & 2 & 0 & 1 & 0 \\ 3 & 5 & 10 & 15 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 7 \\ 120 \\ 100 \end{Bmatrix}$$

x_1, x_2, x_3, x_4 x_5, x_6, x_7 >= 0

Starting table 4 5 9 11 0 0 0

Basic variable	Profit / unit	solution	x_1	x_2	x_3,	x_4	x_5	x_6	x_7	Min ratio
x_5	0	7	1	1	1	1	1	0	0	7/1
x_6	0	120	7	5	3	2	0	1	0	120/2
x_7	0	100	3	5	10	15	0	0	1	100/15
	MaxZ = 0x7+ 0x120 +0x100		-4	-5	-9	-11	0	0	0	

BIG M Method or Methods of Penalties

Whenever the objective function is MinZ and when all or some of the constraints are of >= type or = type. We introduce surplus variable and a artificial variable to LHS of the constraint when it is necessary to complete the identity matrix I.

The general practice is to assign the letter M as the cost in a minimization problem, and –M as the profit in the maximization problem with assumption that M is a very large positive number to the artificial variables in the objective function.

The method of solving a LPP in which a high penalty cost has been assigned to the artificial variables is known as the method of penalties or BIG m Method.

Procedures

Step1: At any iteration of the usual simplex method can arise any one of the following three cases:

Case a) if there is no vector corresponding to some artificial variable in the solution column in such case, we proceed to step 2.

Case b) if at least one vector corresponding to some artificial variable, in the basis is basic variable column at the zero level i.e., corresponding entry in solution column is zero and the co-efficient of m in each net evaluation Z_j - C_j is non negative.

In such case, the current basic feasible solution is a degenerate one.

If this is a case when an optimum solution. The given LPP includes an artificial basic variable and an optimum basic feasible solution does not exist.

Case c) if at least one artificial vector is in the basis Y_b but, not at zero level i.e., the corresponding entry in X_b is non zero. Also co-efficient of M in each net evaluation Z_j - C_j is non negative.

In the case, the given LPP does not possess an optimum basic feasible solution. Since, M is involved in the objective function. In such case, the given problem has a pseudo optimum basic feasible solution.

Step 2: application of simplex method is continued until either an optimum basic feasible solution is obtained or there is an indication of the existence of an unbounded solution to the given LPP.

Note: while applying simplex method, whenever a vector corresponding to some artificial variable happens to leave the basis, we drop that vector and omit all the entries corresponding to that vector from the simplex table.

Problem 1:

Solve the give LPP by BIG M Method

Max Z $=3x_1 - x_2$

STC

$2x_1 + x_2 \quad >= 2$

$x_1 + 3x_2 \quad <= 3$

$8x_2 \quad <= 4$

$x_1, \; x_2 >= 0$

Since, the given problem is max z and the I constraint is >= type we introduce surplus variable as $x_3 >= 0$, and a artificial variable as $x_4 >= 0$, the II & III constraint are of <= type, so we introduce $x_5 >= 0$, $x_6 >= 0$.

The standard of the given LPP as follow as.

Max $Z = 3x_1 - x_2 + 0x_3 - M x_4 + 0 x_5 + 0x_6$

STC

$2x_1 + x_2 - x_3 + x_4 = 2$

$x_1 + 3x_2 + x_5 = 3$

$x_2 + x_6 = 4$

$x_1, x_2, x_3, x_4\ x_5, x_6 >= 0$

The Matrix form

Max $Z = (3, -1, 0, -M, 0, 0)\ (x_1, x_2, x_3, x_4\ x_5, x_6)$

STC

$$\begin{Bmatrix} 2 & 1 & -1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 3 \\ 4 \end{Bmatrix}$$

$x_1, x_2, x_3, x_4\ x_5, x_6 >= 0$

$$\begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

$(x_4\ x_5, x_6)$ canonical system.

The basic variable and their obvious solution

Starting table:

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
			3	-1	0	-M	0	0	
x_4	-M	2	2	1	-1	1	0	0	2/2=1
x_5	0	3	1	3	0	0	1	0	3/1=3
x_6	0	4	0	8	0	0	0	1	4/0=∞
	MaxZ= -mx2+0x3 +0x4= -2m		=-mx2+ 0x1+0x4 -3=- 2m-3	=- mx1+0x 3+0x0- (-1)= - m+1	=-mx- 1+0x0 +0x0- 0= m	=- mx1+0x 0+0x0-)- m)=0	=- mx0+0 x1+0x 0-0=0	=- mx0+0x 0+0x1- 0=0	

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
			3	-1	0	-M	0	0	
x_1	3	1	1	½	-1/2	1/2	0	0	1/-1/2=-2
x_5	0	2	0	5/2	1/2	-1/2	1	0	2/1/2=4
x_6	0	4	0	1	0	0	0	1	4/0=∞
	MaxZ=3x1+0x2+0 x4=3		=3x1+0 x0+0x0 – 3	3x1/2+0 x5/2+0x 1-(-1)	=3x- 1/2+0x 1/2+0x 0-0	=3x1/2+ ox- 1/2+0x0 – (-M)	=3x0+ 0x1+0 x0-0	=3x0+0 x0+0x1- 0	
			0	5/2	-3/2	3/2+m	0	0	

New Number for Key row is

New no for key row =old element/pivot element

Soln= 2/2=1, x_1 =2/2= 1, x_2 =1/2, x_3 = -1/2, x_4 =1/2, x_5 =0/2=0, x_6 =0/2=0

Other than key row new no. = old element - PCE*NPRE

Soln= 3-1*1= 2, x_1 =1-1*1=0, x_2 =3-1*1/2= 5/2, x_3 =0-1*-1/2= 1/2, x_4 =0-1*1/2= -1/2, x_5 =1-1*0=1, x_6 =0-1*0=0

For x_6 new no are

Soln= 4-0*1=4, x_1 =0-0*1=0, x_2 =8-0*1/2= 8, x_3 =0-0*-1/2=0, x_4 =0-0*1/2= 0, x_5 =0-0*0=0, x_6 =1-0*0=1

			3	-1	0	-M	0	0	
Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
x_1	3	3	1	3	0	0	1	0	
x_3	0	4	0	5	1	-1	2	0	
x_6	0	4	0	1	0	0	0	1	
	Max Z =3x3+0x4+0x4=9		=3x1+0 x0+0x0-3	=3x3+0 x5+0x1-(-1)	=3x0+ 0x1+0 x0-0	3x0+0x-1+0x0-(-M)	=3x1+ 0x2+0 x0-0	=0x3+0 x0+0x1-0	
			0	10	0	M	3	0	

New Number for Key row is

New no for key row =old element/pivot element (x_3)

Soln= 2/1/2=4, x_1 =0/1/2=0, x_2 =5/2/1/2=5, x_3 = 1/2/1/2=1, x_4 =-1/2/1/2=-1, x_5 =1/1/2=2, x_6 =0/1/2=0

Other than key row new no. x_1 row = old element – PCE *NPRE

Soln = 1-(-1/2)*4=3, x_1 =1+1/2*0=1, x_2 =1/2+1/2*5=3, x_3 =-1/2-(-1/2)*1=0, x_4 =1/2+1/2*-1=0, x_5 =0+1/2*2=1, x_6 =0-(-1/2)*0=0

For x_6 new no are

Soln= 4-0*4=4, x_1 =0-0*0=0, x_2 =1-0*5=1, x_3 =0-0*1=0, x_4 =0-0*-1=0, x_5 =0-0*0=0, x_6 =1-0*0=1

Since all the NER is +ve and at zero level which is optimum.

This problem is of case A, which means no artificial vector appears at the optimal table and therefore, the given problem as attained the optimality.

$x_1=3, x_2=0, x_3=4, x_4=0, x_5=0 x_6=4$ substitute this values in the objection function. Max Z =3x3 - 0 +0x4 –Mx0+0x0+ 0x4 = 9

Problem:2

Solve the given problem by charnes penalty method

Max $Z = 3x_1 + 2x_2$

STC $2x_1$
$+ x_2$
$3x_1 + 4x_2$
$x_1,$

Solution:

Since, the given problem is max z

The I constraint is <= type we introduce slack variable as $x_3 >= 0$ and the II constraint is of >= type we introduce surplus variable as $x_4 >= 0$ and a artificial variable as $x_5 >= 0$. In the constraint the value of artificial variable will be 1 and in the objective function is –M. Therefore, the standard form of the problem

Max $Z = 3x_1 + 2x_2 + 0x_3 + 0x_4 - Mx_5$

STC

$2x_1 + x_2 + x_3 = 2$

$3x_1 + 4x_2 - x_4 + x_5 = 12$

$x_1, x_2, x_3, x_4\ x_5 >= 0$

The Matrix form

Max Z= (3,2, 0,0,-M) $(x_1, x_2, x_3, x_4\ x_5)$

STC

$$\begin{Bmatrix} 2 & 1 & 1 & 0 & 0 \\ 3 & 4 & 0 & -1 & 1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix}$$

$x_1, x_2, x_3, x_4\ x_5 >= 0$

$$\left\{ \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\}$$

Canonical system. The basic variable (x_3,x_5) and their obvious solution (2,12)

Starting table: 3 2 0 0 -M

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	Min ratios
x_3	0	2	2	1	1	0	0	2/1=2
x_5	-M	12	3	4	0	-1	1	12/4=3
			2+- mx3-3					

Starting table: 3 2 0 0 -M

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	Min ratios
x_2	2	2	2	1	1	0	0	
x_5	-m	4	-5	0	-4	-1	1	
	Maxz= 2x2+-mx4		5m+1	0	4m+2	m	0	

Since, the co-efficient of m in each $Z_j – C_j$ is non-negative and an artificial vector appeared in the basis, not at zero level.

Then, given LPP does not possess an optimum basic feasible solution. So, that is existence of pseudo optimum BFS to the given LPP.

Problem:3

Solve the given problem by charnes penalty method

Max $Z = 3x_1 + 2x_2 + 3x_3$

STC

$2x_1 + x_2 + x_3 \quad <= 2$

$3x_1 + 4x_2 + 2x_3 \quad >= 8$

$x_1, \; x_2, \; x_3 \quad >= 0$

Solution:

Since, the given problem is max z

The I constraint is <= type we introduce slack variable as $x_4 >= 0$ and the II constraint is of >= type we introduce surplus variable as $x_5 >= 0$ and a artificial variable as $x_6 >= 0$.

In the constraint the value of artificial variable will be 1 and in the objective function is – M. Therefore, the standard form of the problem.

Max $Z = 3x_1 + 2x_2 + 3x_3 + 0\,x_4 + 0\,x_5 - M\,x_6$

STC

$2x_1 + x_2 + x_3 + x_4 = 2$

$3x_1 + 4x_2 - x_5 + x_6 = 8$

$x_1, x_2, x_3, x_4\; x_5\; x_6 >= 0$

The Matrix form

Max Z= (3,2,3, 0,0,-M) $(x_1, x_2, x_3, x_4\; x_5, x_6$

)

STC

$$\begin{Bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & -1 & 1 \end{Bmatrix} \quad \begin{Bmatrix} 2 \\ = \; 8 \end{Bmatrix} \qquad \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}$$

Canonical system. The basic variable

$x_1, x_2, x_3, x_4\; x_5 x_6 >= 0$

$(x_4\,,\, x_6)$ and their obvious solution

(2, 8)

Starting table: 3 2 3 0 0 -M

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
X_4	0	2	2	1	1	1	0	0	2/1=2
x_6	-M	8	3	4	2	0	-1	1	8/4=2
			=0x2+-mx3-3	=0x1+-mx4-2	=0x1+-mx2-3	=0 x 1 +- mx0-0	=0x0+-m x-1- 0	=0x0+-mx1-(-m)	
			-3m-3	-4m-2	-2m-3	0	m	0	

Starting table: 3 2 0 0 0 -M

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
x_2	2	2	2	1	1	1	0	0	
x_6	-m	0	-5	0	-2	-4	-1	1	
	maxZ=2x2+-mx0=4		5m+1	1	2m+1	4m+2	m	0	

Since, the co-efficient of m in each $Z_j - C_j$ is non-negative and an artificial vector appears in the basis, at zero level.

Then indicates the existence of an optimum basic feasible solution to the given LPP.
Thus, an optimum basic feasible solution to the given LPP.

$x_1 = 0$	$x_2 = 2$	$x_3 = 0$	$x_4 = 0$	$x_5 = 0$	$x_6 = 0$

Max $Z = 3x_1 + 2x_2 + 3x_3 + 0\, x_4 + 0\, x_5 - M\, x_6$

=3*0+2*2+3*0+0*0+0*0-M*0

=4

Problem No : 4

Solve the given problem by charnes penalty method

Max Z =$3x_1 + 2x_2 + 4x_3$

STC

$2x_1 + 5x_2 + x_3 = 12$

$3x_1 + 4x_2 = 11$

$x_1, x_2, x_3 >= 0$

Problem No 5

Solve the given LPP

Min Z = $4x_1 + x_2$

STC

$3x_1 + 2x_2 = 3$

$4x_1 + 3x_2 > = 6$

$x_1 + 2x_2 < = 4$

$x_1, x_2 > = 0$

The I constraint is already in standard form, only artificial variable is added as $x_3 >= 0$ and II constraint is of >= type we introduce

$x_4 > = 0$ as surplus variable and an artificial variable as $x_5 > = 0$ and third constraint is of <= type we introduce $x_6 > = 0$ as slack variable.

Then, the given LPP is in standard form

Min Z = $4x_1 + x_2 + Mx_3 + 0x_4 + Mx_5 + 0x_6$ converting MinZ= -Z

or

Max Z = $-4x_1 - x_2 - Mx_3 - 0x_4 - Mx_5 - 0x_6$

STC

$3x_1 + 2x_2 + x_3 = 3$

$4x_1 + 3x_2 - x_4 + x_5 = 6$

$x_1 + 2x_2 + x_6 = 4$

$x_1, x_2, x_3, x_4, x_5, x_6 > = 0$

The matrix form

Max Z = (-4, -1, -M, 0, -M, 0) $(x_1, x_2, x_3, x_4, x_5, x_6)$

STC

$$\begin{Bmatrix} 3 & 2 & 1 & 0 & 0 & 0 \\ 4 & 3 & 0 & -1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 6 \\ 4 \end{Bmatrix}$$

$x_1, x_2, x_3, x_4, x_5, x_6 >= 0$

Starting table: - 4 -1 -m 0 -m 0

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
X_3	-m	3	3	2	1	0	0	0	3/3=1
x_5	-m	6	4	3	0	-1	1	0	6/4=1.5
x_6	0	4	1	2	0	0	0	1	4/1
	Max z= -9m		-7m+4	-5m+1	0	m	0	0	

Starting table: - 4 -1 -m 0 -m 0

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
X_1	-4	1	1	2/3	1/3	0	0	0	3/2
x_5	-m	2	0	1/3	-4/3	-1	1	0	6
x_6	0	3	0	4/3	-1/3	0	0	1	9/4
	Max z= -4-2m		0	.-5-m /3	4m-4 /3	m	0	0	

Starting table: - 4 -1 -m 0 -m 0

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
X_2	-1	3/2	3/2	1	1/2	0	0	0	
x_5	-m	3/2	-1/2	0	-2	-1	1	0	
x_6	0	1	-2	0	-3	0	0	1	
	Max z= -3/2-3/2m		5+m/2	0	-1/2+2m	m	0	0	

Problem 6

Solve the given problem by BIG M Method

Min Z = $-3x_1 + x_2 + x_3$

STC

$x_1 + -2x_2 + x_3$

$-4x_1 + x_2 + 2x_3$

$24x_1 - x_3$

x_1, x_2, x_3

Solution:

The I constraint is of <= type so add $x_4 >= 0$ as slack variable, II constraint is of >= type so we introduce $x_5 >= 0$ as surplus variable and $x_6 >= 0$ as artificial variable and III constraint is already a standard form and right side is –ve. So it should be multiplied by - 1 both sides of the equation and $x_7 >= 0$ as artificial variable and the standard form of the given LPP is written below.

Min Z = $-3x_1 + x_2 + x_3 + 0x_4 + m\, x_5 + x_6 + m\, x_7$

STC

$x_1 + -2x_2 + x_3 + x_4 = 11$

$-4x_1 + x_2 + 2x_3 + x_5 + x_6 = 3$

$-2x_1 + x_3 + x_7 = 1$

$x_1, x_2, x_3, x_4\ x_5\ x_6\ x_7 >= 0$

The matrix form

Min z = (-3,1,1,0,M ,1,M) ($x_1, x_2, x_3, x_4, x_5, x_6, x_7$)

STC

$$\begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 & 0 \\ -4 & 1 & 2 & 0 & -1 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 11 \\ 3 \\ 1 \end{bmatrix}$$

$x_1, x_2, x_3, x_4, x_5, x_6, x_7 >= 0$

Starting table:

			-3	1	1	0	m	1	m
Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_4	0	11	1	-2	1	1	0	0	0
x_6	M	3	-4	1	2	0	-1	1	0
x_7	M	1	-2	0	1	0	0	0	1
MaxZ=0x11+3xM+1xm=4m			-3+6m	1-m	1-3m	0	2m	1-m	0

NER= C_j –(P/U* x_j) (j=12….n)

x_1= -3-(0x1+mx-4+-2m) = -3+6m

x_2= 1- (0x-2+mx1+0xm) = 1-m

x_3= 1- (0x1+mx2+1xm) = 1-3m

x_4= 0- (0x1+mx0+0xm) = 0

x_5= m- (0x0+mx-1+0xm) = 2m

x_6= 1- (0x0+mx1+0xm) = 1-m

x_7= m- (0x0+mx0+1xm) = 0

First table: -3 1 1 0 m 1 m

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Min ratios
X_4	0	10	3	-2	0	1	0	0	-1	10/-2= -ve
X_6	M	1	0	1	0	0	-1	1	-2	1/1=1
X_3	1	1	-2	0	1	0	0	0	1	1/0=0
	Min Z= 0x10 + mx1 +1x1 =1+m		-3-(0x3+mx0+1x-2) = -1	1-(0x-2+mx1+1x0)=1-m	1-(0x0+mx0+1x1) = 0	0-(0x1 +mx0+1x0)= 0	m-(0x0+mx-1 + 1x0)= 2m	=1-(0x0+mx1+1x0) = 1-m	m -(0x-1+mx-2+1x1) = 3m-1	

The values of key row remain the same because 1 is the key element

New nos for other than key rows i.e X_4

Soln= 11-1*1=10

x_1= 1-1*-2 =3

x_2= -2-1*0=-2

x_3=1-1*1=0

x_4=1-1*0=1

x_5=0-1*0=0

x_6=0-1*0=0

x_7=0-1*1=-1

X_5,Soln= 3-2*1=1

x_1= -4-2*-2= 0

x_2= 1-2*0=1

x_3= 2-2*1=0

x_4= 0-2*0=0

x_5= -1-2*0=-1

x_6= 1-2*0=1

x_7=0-2*1=-2

Third table: -3 1 1 0 m 1 m

Basic Variable	Profit / unit	Solution	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Min ratios
x_4	0	12	3	0	0	1	2	-2	-5	12/3
x_2	1	1	0	1	0	0	1	-1	-2	1/0
x_3	1	1	-2	0	1	0	0	0	1	1/-2
			-1	0	0	0	m-1	1	m+1	

Fourth table: -3 1 1 0 m 1 m

Basic Variable	Profit / unit	Solution	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Min ratios
x_1	-3	4	1	0	0	1/3	2/3	-2/3	-5/3	
x_2	1	1	0	1	0	0	1	-1	-2	
x_3	1	9	0	0	1	2/3	4/3	-4/5	-7/3	
			0	0	0	1/3	m-1/3	1/3	m-2/3	

Table 4 is optimal and the value of the decision variable is x_1 =4, x_2 =1, x_3 =9, x_4 =0, x_5 =0 ,x_6 =0 x_7 = 0

Min Z = -3*4 +1*1 + 1*9 + 0*0 +0*0+ 0*0 +0*) =-12+1+9 = -2

The linear programming model we develop for a situation is referred to as the **primal** problem. The dual problem can be derived directly from the primal problem.

The standard form has three properties

1. All the constraints are equations (with nonnegative right-hand side)
2. All the variables are nonnegative
3. The sense of optimization may be maximization of minimization.

Comparing the primal and the dual problems, we observe the following relationships.

1. The objective function coefficients of the primal problem have become the right-hand side constants of the dual. Similarly, the right-hand side constants of the primal have become the cost coefficients of the dual
2. The inequalities have been reversed in the constraints
3. The objective is changed from maximization in primal to minimization in dual
4. Each column in the primal correspondence to a constraint (row) in the dual. Thus, the number of dual constraints is equal to the number of primal variables.
5. Each constraint (row) in the primal corresponds to a column in the dual. Hence, there is one dual variable for every primal constraints
6. The dual of the dual is the primal problem.
7. if the primal constraints >= the dual constraints will be <= & vice versa

Duality is an extremely important and interesting feature of linear programming. The various useful aspects of this property are

i) If the primal problem contains a large no of rows constraints and smaller no of columns variables computational procedure can be considerably reduced by converting it into dual and then solving it.

ii) It gives additional information as to how the optimal solution changes as a result of the changes in the coefficients and the formulation of its problem.

iii) Calculations of the dual checks the accuracy of the primal solution

iv) This indicates that fairly close relationships exist between LP and theory of games

Primal Problem	Dual Problem
Max Z = CX STC Ax<=b x>=0	Min Z = bY STC Ay>=c y>=0
Max Z = CX STC ex: Ax=b x1+x2< = 2 x>=0 -x1-x2< =-2	Min Z = bY STC Ay>=c Y is unrestricted in sign
Max Z = CX STC Ax=b X is unrestricted in sign	Min Z = bY STC Ay=c y is unrestricted in sign
Max Z = CX STC A1x<=b1 A2x=b2 x>=0	Min Z = b1Y1+b2y2 STC A1y1+A2y2>=c Y1>=0 y2 is unrestricted in sign

Note: it is not necessary that only the Max problem be taken as the primal problem we can as well consider the minimization LPP as the primal

Formulation of dual to the primal problem

Problem no 1

Write the dual of the primal problem

Max Z = $3x_1 + 5x_2$

S
T
C

2
x
1

+
6
x
2

$3x_1 + 2x_2$

$5x_1 - 3x_2$

x_2

x_1, x_2, x_3

To write dual for the above primal, since the primal as 4 constraints the dual will have 4 variables as $y_1\ y_2\ y_3\ y_4$ then dual for the primal will be as follows.

Min Z = $50y_1 + 35y_2 + 10y_3 + 20y_4$

STC

$2\ y_1 + 3y_2 + 5y_3 + 0y_4 >= 3$

$6y_1 + 2y_2 - 3y_3 + y_4 \quad > = 5$

$>= 0$

it can be observed from the dual problem has less no constraint as compared to the primal problem (in case of primal they are 4 and in case of dual they are 2) which requires less work and effort to solve it

Problem 2

Construct the dual of the given problem

Min Z = $3x_1 - 2x_2 + 4x_3$

STC

$3x_1 + 5x_2 + 4\ x_3 \quad >= 7$

$6x_1 + x_2 + 3x_3 \quad > = 4$

$7x_1 - 2\ x_3 - x_3 \quad < = 10$

$x_1 - 2x_2 + 5\ x_3 \quad >= 3$

$4x_1 + 7x_2 - 2\ x_3 \quad >= 2$

$x_1,\ x_2,\ x_3 \quad >= 0$

Solution:

The given problem is minimization type and all the constraint should be >= type. In the given problem third constraint is <= type so we must convert constraint to >= type by multiply both side of the constraint by -1, we get

$-7x_1 +2 x_3 +x_3 \quad >= -10$

Then given the problem can be written restated

$\text{Min } Z = 3x_1 -2x_2 + 4x_3$

STC

$3x_1 +5x_2+4 x_3 \quad >=7$

$6x_1 +x_2+ 3x_3 \quad >= 4$

$-7x_1 +2 x_2 +x_3 \quad >= -10$

$x_1 -2x_2+5 x_3 \quad >= 3$

$4x_1 +7x_2-2 x_3 \quad >= 2$

$x_1, x_2, x_3 \quad >= 0$

Then dual of the given problem is as follows and the dual variables are $y_1 ,y_2 ,y_3,y_4\ y_4$

$\text{Max } Z = 7y_1 +4y_2 -10y_3+3y_4 +2y_5$

STC

$3 y_1 +6y_2 -7y_3+y_4+4y_4 >= 3$

$5y_1 +y_2 +2y_3-2y_4 +7y_4 >= -2$

$4y_1 +3y_2 +y_3+5y_4 -2y_4 >= 4$

$y_1 ,y_2 ,y_3,y_4 ,y_5 >= 0$

Problem 3

Obtain the dual problem of the following LPP

Max $Z = 2x_1 + 5x_2 + 6x_3$

STC

$5x_1 + 6x_2 - 4x_3 \quad <= 3$

$-2x_1 + x_2 + 4x_3 \quad < = 4$

$x_1 - 5x_2 + 3x_3 \quad < = 1$

$-3x_1 - 3x_2 + 7x_3 \quad <= 6$

$x_1, x_2, x_3 \quad >= 0$

Also, verify that the dual of the dual problem is the primal problem
The given primal can be restated in the matrix form

Max $Z = (2,5,6)\,(x_1, x_2, x_3)$

STC

$$\begin{Bmatrix} 5 & 6 & -4 \\ -2 & 1 & 4 \\ 1 & -5 & 3 \\ -3 & -3 & 7 \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} <= \begin{Bmatrix} 3 \\ 4 \\ 1 \\ 6 \end{Bmatrix}$$

$x_1, x_2, x_3 \quad >= 0$

The dual of the above primal can be written as and the dual variables are (y_1, y_2, y_3, y_4)

Min $w = (y_1, y_2, y_3, y_4)\,(3.4.1.6)$

STC

$$\begin{Bmatrix} 5 & -2 & 1 & -3 \\ 6 & 1 & -5 & 3 \\ -4 & 4 & 3 & 7 \end{Bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} \geq \begin{Bmatrix} 2 \\ 5 \\ 6 \end{Bmatrix}$$

$y_1, y_2, y_3, y_4 >= 0$

Now, we can restate this dual problem as follows

Max w = (-3.-4.-1.-6) (y_1, y_2, y_3, y_4)

$$(-1)\begin{pmatrix} 5 & -2 & 1 & -3 \\ 6 & 1 & -5 & -3 \\ -4 & 4 & 3 & -7 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \\ -6 \end{pmatrix}$$

$y_1, y_2, y_3, y_4 >= 0$

MaxZ = (-1)

Minz

The dual problem in this form looks like the primal problem and this we may write down the dual of this dual problem .

MinZ= (-2,-5, -6) (x_1, x_2, x_3)

STC

$$(-1)\begin{pmatrix} 5 & 6 & -4 \\ -2 & 1 & 4 \\ 1 & -5 & 3 \\ -3 & -3 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} <= \begin{pmatrix} -3 \\ -4 \\ -1 \\ -6 \end{pmatrix}$$

$x_1, x_2, x_3 >= 0$

or Max z = (2 5 6) (x_1, x_2, x_3)

STC

$$\begin{pmatrix} 5 & 6 & -4 \\ -2 & 1 & 4 \\ 1 & -5 & 3 \\ -3 & -3 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} <= \begin{pmatrix} -3 \\ -4 \\ -1 \\ -6 \end{pmatrix}$$

$x_1, x_2, x_3 >= 0$

Problem 4

Construct the dual of the primal problem

Max $Z = 3x_1 + 17x_2 + 4x_3$

STC

$x_1 - x_2 + x_3$

$-3x_1 + 2x_3$

x_1, x_2, x_3 $\quad$ >=3 <

= 1 >=

0

Problem 5

Construct the dual of the following primal

Max $Z = x_1 - 2x_2 + 3x_3$

STC

$-2x_1 + x_2 + 3x_3 = 2$

$2x_1 + 3x_2 + 4x_3 = 1$

$x_1, x_2, x_3 >= 0$

The given problem is 2nd variety in the duality and it is in the form of Max Z, then the constraint in the standard form can be written as

$-2x_1 + x_2 + 3x_3 <= 2$

$2x_1 - x_2 - 3x_3 <= -2$

and

$2x_1 + 3x_2 + 4x_3 <= 1$

$-2x_1 - 3x_2 - 4x_3 <= -1$

Then, given primal can restated as

Max $Z = x_1 - 2x_2 + 3x_3$

STC

$-2x_1 + x_2 + 3x_3 <= 2$

$2x_1 - x_2 - 3x_3 <= -2$

$2x_1 + 3x_2 + 4x_3 <= 1$

$-2x_1 - 3x_2 - 4x_3 <= -1$

$x_1, x_2, x_3 >= 0$

The dual of the above primal is written as and its dual variables are y_1, y_2, y_3

$\text{Min } W = 2y_1 - 2y_2 + y_3 - y_4$
STC
$-2y_1 + 2y_2 + 2y_3 - 2y_4 >= 1$
$y_1 - y_2 + 3y_3 - 3y_4 >= -2$
$3y_1 - 3y_2 + 4y_3 - 4y_4 >= 3$
$y_1, y_2, y_3\ y_4 >= 0$

This can be written
$\text{Min } W = 2y_1 + y_2$

STC

$-2y_1 + 2y_2 >= 1$

$y_1 + 3y_2 >= -2$

$3y_1 + 4y_2 >= 3$
y_1, y_2 are unrestricted in sign

Problem 6

Write the dual of the given LPP
$\text{Max } Z = 3x_1 + x_2 + 2x_3 - 2x_4$
STC
$2x_1 - x_2 + 3x_3 + x_4 = 1$

$x_1 + x_2 - x_3 + x_4 = -3$
$x_1, x_2 >= 0$, x_3 & x_4 are unrestricted in sign

Problem 7

Write the dual of the following LPP
$\text{Min } Z = 2x_1 + 3x_2 + 4x_3$
STC

$2x_1 + 3x_2 + 5x_3 >= 2$

$3x_1 + x_2 + 7x_3 = 2$

$x_1 + 4x_2 + x_3 <= 5$
$x_1, x_2, >= 0$ x_3 are is unrestricted in sign

Use duality to solve the following LPP

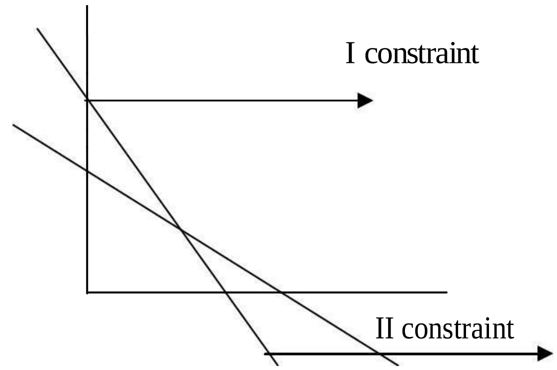
Max $Z = 3x_1 + 2x_2$

STC

$2x_1 + x_2 <= 5$ the points are (5/2,5)

$x_1 + x_2 <= 3$ (3,3)

$x_1, x_2 >= 0$



Use duality concept the dual of the above primal as follows

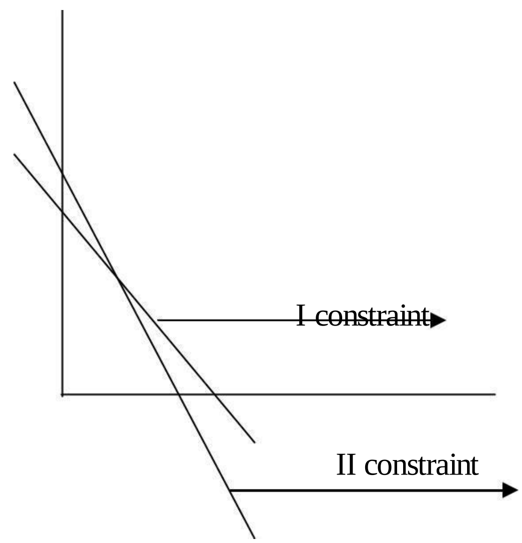
Min $Z = 5y_1 + 3y_2$

STC

$2y_1 + y_2 >= 3$ (3/2,3)

$y_1 + y_2 >= 2$ (2,2)

$y_1, y_2 >= 0$



Using duality theory solve the following LPP by simplex method

Min $Z = 4x_1 + 3x_2 + 6x_3$

STC

$x_1 + x_3 >= 2$

$x_2 + x_3 >= 5$

$x_1, x_2, x_3 >= 0$

the dual of the above primal

Max Z = $2y_1 + 5y_2$
STC

$y_1 <= 4$
$y_2 <= 3$
$y_1 + y_2 <= 6$
$y_1, y_2 >= 0$

Using duality theory solve the following LPP by simplex method

Max Z = $x_1 + 6x_2$
STC

$x_1 + x_2 >= 2$

$x_1 + x_2 <= 3$

$x_1, x_2 >= 0$

The given problem is max Z so all the constraint should be <= type. In this problem I constraint is of >= type so we converting by multiply b -1 both sides of the constraint the resulting will be as

$-x_1 - x_2 <= -2$

Then, the given LPP is restated as

Max Z = $x_1 + 6x_2$
STC

$-x_1 - x_2 <= -2$

$x_1 + x_2 <= 3$

$x_1, x_2 >= 0$

The dual of the above LPP will be as

Min Z = $2y_1 + 3y_2$
STC

$-y_1 + y_2 >= 1$

$-y_1 + y_2 >= 6$

$x_1, x_2 >= 0$

Solving the dual using Big M method

Dual simplex method

Dual simplex method applies to problems which start with dual feasible solns. The objective function may be either in the maximization form or in the minimization form. After introducing the slack variables, if any right-hand side element is –ve and if the optimality condition is satisfied.

The problem can be solved by the dual simplex method Procedure for dual simplex method

Step1) obtain an initial basic solution to the LPP and put the solution in the starting dual simplex table

Step2) test the nature of Z_j-C_j in the starting simplex table

a)if all Z_j-C_j and solution column are non-negative for all I and j, then an optimum basic feasible solution has been obtained.

b) if all Z_j-C_j are non negative and at least on basic variable in the solution column is negative go step 3

c) if at least one Z_j-C_j is –ve the method is not applicable to the given problem

Step 3) selects the most –ve in solution column
step4) test the nature of

a) if all x_{ij} are non negative the given problem does not exist any feasible solution
b) b) if at least one x_{ij} is –ve, compute the ratios Z_j-C_j / x_{ij} , x_{ij} >=0 chose the maximum of theses ratios

Step 5) test the new iterated dual simplex table for optimality

Repeat the procedure until either an optimum feasible solution has been obtained or there is an indication of the non existence of a feasible solution.

Use dual simplex method to solve the LPP

Min $Z = x_1 + x_2$

STC

$2x_1 + x_2 \quad >= 2$

$-x_1 - x_2 >= 1$

$x_1, x_2 \quad >= 0$

Since, the given problem in min z form it is converted to max z and all the constraint is of >= type it should be converted to <= type by multiplying -1 on the both sides then given problem will be as follows

Max $Z = -x_1 - x_2$

STC

$-2x_1 - x_2 <= -2$

$x_1 + x_2 <= -1$

$x_1, x_2 >= 0$

Now, introducing slack variable for the I and II constraint as $x_3 >= 0, x_4 >= 0$

The standard form as follows

Max $Z = -x_1 - x_2 + 0x_3 + 0x_4$

STC

$-2x_1 - x_2 + x_3 \quad = -2$

$x_1 + x_2 + x_4 \quad = -1$

$x_1, x_2, x_3\ x_4 \quad >= 0$

The matrix form

Max $Z = (-1, -1, 0, 0)\,(x_1, x_2, x_3\ x_4)$

STC

$$\begin{pmatrix} -2 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$x_1, x_2, x_3\ x_4 \quad >= 0$

the identity matrix form are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the correspond variables $(x_3\ x_4)$ and their solutions are -2 -1 starting table -1 -1 0 0

Basic Variable	Profit / Unit	solution	x_1	x_2	x_3	x_4
x_3	0	-2	-2	-1	1	0
x_4	0	-1	1	1	0	1
	Max Z= 0*-2+0*-1		=0*-2+0*1+1	=0*-1+0*1+1	=0*1+0*0 -0	=0*0+0*1-0
	0		1	1	0	0

Ratios = NER / X_{1j}

= [1/-2, 1/-1, 0/1, 0/0]

The max negative in the ratios are 1/-2

First table -1 -1 0 0

Basic Variable	Profit / Unit	solution	x_1	x_2	x_3	x_4
x_1	-1	1	1	0.5	-1/2	0
x_4	0	-2	0	0.5	1/2	1
	Max Z= -1*1=0*-2		=-1*1+0*0+1	=0.5*-1+0.5*0+1	=-1*-1/2+0*1/2 -0	=-1 * 0 + 0*1-0
	-1		0	0.5	0	0

New no for key row are

Soln = -2/-2 = 1, -2/-2 =1, -1/-2=0.5, - 1/2,0/-2
New no for other than key rows is x_4
Soln=-1-1*1=-2, 1-1*1=0, 1-1*0.5=0.5, 0-1*-1/2=1/2, 1-1*0=1

Since, all the values in NER is positive and in solution column one variable is –ve and that is selected as key row and to select key column or pivot column at least on variable in the row should be –ve but, no vector corresponding to that row is –ve and we cannot find the ratios and So we cannot select the key column and the given problem does not given any feasible solution to the LPP.

Use dual simplex table to solve the LPP

Min Z = $x_1 + 2x_2 + 3x_3$

STC

$x_1 - x_2 + x_3 \quad >= 4$

$x_1 + x_2 + 2x_3 \quad <= 8$

$x_2 - x_3 \quad >= 2$

$x_1, x_2, x_3 \quad >= 0$

the given problem is set for the requirement of dual simplex method

Max Z = $-x_1 - 2x_2 - 3x_3$

STC

$-x_1 + x_2 - x_3 \quad <= -4$

$x_1 + x_2 + 2x_3 \quad <= 8$

$-x_2 + x_3 \quad <= -2$

$x_1, x_2, x_3 \quad >= 0$

since, all the constraint are of <= type we introduce three slack variable for I II and III constraint as $x_4 >= 0$, $x_5 >= 0$, $x_6 >= 0$

the standard form

Max Z = $-x_1 - 2x_2 - 3x_3 + 0x_4 + 0x_5 + 0x_6$

STC

$-x_1 + x_2 - x_3 + x_4 \quad = -4$

$x_1 + x_2 + 2x_3 + x_5 \quad = 8$

$-x_2 + x_3 + x_6 \quad = -2$

$x_1, x_2, x_3\ x_4\ x_5\ x_6 \quad >= 0$

The matrix form

Max Z (-1,-2,-3,0,0,0) (x_1, x_2 , x_3 x_4 x_5 x_6)

STC

$$\begin{Bmatrix} -1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} -4 \\ 8 \\ -2 \end{Bmatrix} \quad \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

The IBFS are x_4 x_5 x_6

x_1, x_2 , x_3 x_4 x_5 x_6 >= 0 and their obvious soln (-4,8,-2)

Starting table -1 -2 -3 0 0 0

Basic Variable	Profit / unit	soln	x_1	x_2	x_3	x_4	x_5	x_6
x_4	0	-4	-1	1	-1	1	0	0
x_5	0	8	1	1	2	0	1	0
x_6	0	-2	0	-1	1	0	0	1
	MaxZ 4+0*8+0*-8	=0*-	1	2	3	0	0	0

The ratios =(1/-1, 2/-1, 3/-1, 0/0, 0/0, 0/1)

Other than key row new nos
are x_5

soln = 8-1*4=4, x_1 =1-1*1=0, x_2 =1-1*-1=0, x_3 =2-1*1=1, x_4 =0-1*-1=1

x_5 = 1-1*0=1, x_6 =0-1*0=0

x_6

soln= -2-0*4=-2, all other value in the row remains the same because of corresponding value of pivot column element is zero

Starting table -1 -2 -3 0 0 0

Basic Variable	Profit / unit	soln	x_1	x_2	x_3	x_4	x_5	x_6
x_1	-1	4	1	-1	1	-1	0	0
x_5	0	4	0	0	1	1	1	0
x_6	0	-2	0	-1	1	0	0	1
Max Z =			0	3	2	1	0	0

Ratios = (0/0, 3/-1,2/1,1/0,0/0,0/1)

For x_1 row

soln=4-(-1)*2=6, x_1 =1-(-1)*0=1, x_2 =-1-(-1)1=0, x_3 =1-(-1)-1=0, x_4 =-1-(-1)*0=-1, x_5 =0-(-1)*0=0, x_6 =0-(-1)*-1=-1

For x_5 row

soln =4 - 0*2=4, any thing multiplied by 0 is 0 so the value in this row remains same.

Starting table -1 -2 -3 0 0 0

Basic Variable	Profit / unit	soln	x_1	x_2	x_3	x_4	x_5	x_6
x_1	-1	6	1	0	0	-1	0	-1
x_5	0	4	0	0	1	1	1	0
x_2	-2	2	0	1	-1	0	0	-1
	Max Z =		0	0	5	1	0	3

Since all the NER and Solution column are non negative and the given problem as attained the optimum

Therefore, $x_1 = 6$, $x_2 = 2$, $x_3 = 0$, $x_4 = 0$, $x_5 = 4$, $x_6 = 0$

These values are substituted in the objective function

Max Z = -6*1 -2*2 -3*0 +0*0+ 0*4+ 0*0

Dual Readoff Technique

Solve the following the LPP and read the solution for the real problem

Min $Z = 2x_1 + x_2$

STC

$x_1 + x_2 \quad >= 15$

$x_1 - x_2 \quad <= 3$

$2x_1 + 5x_2 \quad >= 20$

$-x_1 + 3x_2 \quad >= 10$

$x_1, x_2 \quad >= 0$

To write the dual of the above primal all the constraint should be of >= type, since, II constraint is <= type we must convert it into >= type by -1, then, the given primal can be written as follows

Min $Z = 2x_1 + x_2$

STC

$x_1 + x_2 \quad >= 15$

$-x_1 + x_2 \quad >= -3$

$2x_1 + 5x_2 \quad >= 20$

$-x_1 + 3x_2 \quad >= 10$

$x_1, x_2 \quad >= 0$

The dual for the above primal problem as follows

The dual variables ($y_1\ y_2\ y_3\ y_4$)

Max $Z = 15y_1 - 3y_2 + 20y_3 + 10y_4$

STC

$y_1 - y_2 + 2y_3 - y_4 \quad <= 2 \quad y_5$ >= 0 as slack variable

$y_1 + y_2 + 5y_3 + 3y_4 <= 1 \quad y_6$ >= 0 as slack variable

$y_1\ y_2\ y_3\ y_4 >= 0$

Matrix form

Max Z = (y_1 y_2 y_3 y_4) (15,-3,20,10)

STC

$$\begin{Bmatrix} 1 & -1 & 2 & -1 & 1 & 0 \\ 1 & 1 & 5 & 3 & 0 & 1 \end{Bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$$

$y_1\ y_2\ y_3\ y_4\ y_5\ y_6 >= 0$

$\begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}$ is identity matrix which form the basic variable (y_5 y_6) for the I table and it's solution is (2,1)

Starting table 15 -3 20 10 0 0

Basic Variable	Profit / unit	soln	y_1	y_2	y_3	y_4	y_5	y_6	Min ratio
y_5	0	2	1	-1	2	-1	1	0	2/2=1
y_6	0	1	1	1	5	3	0	1	1/5=
	Max Z = 0		-15	3	-20	-10	0	0	

Starting table 15 -3 20 10 0 0

Basic Variable	Profit / unit	soln	y_1	y_2	y_3	y_4	y_5	y_6	Min ratio
y_5	0	8/5	3/5	0	-10/5	1	-1	-2/5	8/3
y_3	20	1/5	1/5	1/5	1	3/5	0	1/5	1
	Max Z = 0		-11	7	0	2	0	20	

Starting table 15 -3 20 10 0 0

Basic Variable	Profit / unit	soln	y_1	y_2	y_3	y_4	y_5	y_6	Min ratio
y_5	0	∞	0	-2	-3/5	-4	1	∞	
y_1	15	1	1	1	5	3	0	1	
	Max Z = 0		0	18	55	35	0	15	

Surplus variable Real Primal

Since all the NER non negative the given problem as attained optimum

Sl. No.	Primal	Dual
1	$X_1 = 0$	$Y_1 = 1$
2	$X_2 = 15$	$Y_2 = 0$
3	$X_3 = 0$	$Y_3 = 0$
4	$X_4 = 18$	$Y_4 = 0$
5	$X_5 = 55$	$y_5 = \infty$
6	$x_6 = 35$	$Y_6 = 0$

Reference Books:

1 Taha H A, Operation Research - An Introduction, Prentice Hall of India, 7th edition, 2003

2 Ravindran, Phillips and Solberg, Operations Research : Principles and Practice, John Wiely & Sons, 2nd Edition

3 D.S.Hira, Operation Research, S.Chand & Company Ltd., New Delhi, 2004

4.

The assignment problem is a special case of transportation problem in which the objective is to assign ‘m’ jobs or workers to ‘n’ machines such that the cost incurred is minimized.

J
O
B
S
----

1 2 n

1

2 C11 C12 ---------------- C1n

-- C21 C22---------------- C2n

- - -

WORKERS -- - - -

--

n - - -

Cn1 Cn2 ------------------ Cmn

The element Cij represents the cost of assigning worker I to job (I,j= 1,2,---n). There is no loss in generality in assuming that the number of workers always equals the number of jobs because we can always add fictitious (untrue or fabricated) workers or fictitious jobs to effect this result.

The assignment model is actually a special case of the transportation model in which the workers represent the sources and the jobs represent the destinations.

The supply amount at each source and the demand amount at each destination exactly equal 1.

The cost of transporting workers I to job j is C_{ij} .

The assignment model can be solved directly as a regular transportation model.

The fact that all the supply and demand amounts equal 1 has led to the development of a simple solution algorithm called the Hungarian method.

Difference between transportation and Assignment problems		
Sl. No.	Transportation	Assignment
1	This problem contains specific demand and requirement in columns and rows	The demand and availability in each column or row is one
2	Total demand must be equal to the total availability	It is a square matrix. The no of rows must be equal to the no of columns.
3	The optimal solution involves the following conditions M+N-1 M $\longrightarrow$ rows N $\longrightarrow$ columns	The optimal solutions involves one assignment in each row and each column
4	There is no restriction in the number of allotments in any row or column	There should be only one allotment in each row and each column
5	It is a problem of allocating multiple resources to multiple markets	It is a problem of allocation resources to job j

Assignment Algorithm (Hungarian Method)

Step I :- Create Zero elements in the cost matrix by subtract the smallest element in each row column for the corresponding row and column.

Step II:- Drop the least number of horizontal and vertical lines so as to cover all zeros if the no of there lines are 'N'

i) If N = n (n=order of the square matrix) then an optimum assignment has been obtained

ii) If N<n proceeds to step III

Step III :- determine the smallest cost cell from among the uncrossed cells subtract. This cost from all the uncrossed cells and add the same to all those cells laying in the intersection of horizontal and vertical lines.

Step IV:- repeat steps II and III until N=n.

Step V:- examine the rows (column) successively until a row (column) with are zero is found enclose the zero in a square (0) and cancel out (~~0~~) any other zeros laying in the column (row) of the Matrix. Continue in this way until all the rim requirements are satisfied i.e N=n.

Step VI:- repeat step 5 successively one of the following arises.

i) No unmarked zero is left

ii) If more then one unmarked zeros in one column or row.

In case i) the algorithm stops

ii)Encircle one of the unmarked zeros arbitrary and mark a cross in the cells of remaining zeroes in it's row and column. Repeat the process until no unmarked zero is left in the cost matrix.

Step VII) we now have exactly one encircled zero in each row and each column of the cost matrix. The assignment schedule corresponding to there zeros is the optimum (maximal) assignment.

Note: the above procedure for assignment is Hungarian assignment method

Problem 1.

Three jobs A B C are to be assigned to three machines x Y Z. The processing costs are as given in the matrix shown below. Find the allocation which will minimize the overall processing cost.

Machines				
Jobs		X	Y	Z
	A	19	28	31
	B	11	17	16
	C	12	15	13

Solution:

Step 1: create zero in each row or column by subtracting by selecting least number in each row and column

Row Minimization

0	9	12
0	6	5
0	3	1

Column Minimization

0	6	11
0	3	4
0	0	0

Now draw Horizontal and vertical lines

0	6	11
0	3	4
0	0	0

Here, no of horizontal lines is one and vertical line is one

The order of matrix is 3 x 3, therefore, N≠ n

Now, in the uncrossed cell the least cost is selected and subtracted for the remaining uncrossed cell by the least value and for the intersection of the horizontal line and vertical line the least value should be added and the resulting matrix.

0	3	8
0	0	1
3	0	0

The above matrix has two horizontal line and one vertical line which satisfies our condition N= n

{0}	3	8
~~0~~	{0}	~~1~~
3	~~0~~	{0}

The assignment are A ⟶ X = 19

B ⟶ Y = 17

C ⟶ Z = 13

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Problem 2

Solve of the assignment problem

	I	II	III	IV	V
1	11	17	8	16	20
2	9	7	12	6	15
3	13	16	15	12	16
4	21	24	17	28	26
5	14	10	12	11	15

Solns:

Row Minimization

3	9	0	8	12
3	1	6	0	9
1	4	3	0	4
4	7	0	11	9
4	0	2	1	5

Column Minimization

2	9	0	8	8
2	1	6	0	1
0	4	3	0	0
3	7	0	11	5
3	0	2	1	1

N≠n

5≠4

2	9	0	8	8
2	1	6	0	1
0	4	3	0	0
3	7	0	11	5
3	0	2	1	1

The least value in the uncrossed cell is 1, it is subtracted for the uncrossed cell and added for intersection of the vertical line and horizontal.

1	9	0	8	7
1	1	6	0	0
0	4	3	0	0
2	7	0	11	4
2	0	2	1	0

Again N ≠ n

least value is one again

0	8	0	7	6
1	1	6	0	0
0	4	3	0	0
1	6	0	10	3
2	0	2	1	0

Here, it satisfies our condition N=n

Now, the assignment for the optimum table

[0]	8	~~0~~	7	6
1	1	6	[0]	~~0~~
~~0~~	4	3	~~0~~	[0]
1	6	[0]	10	3
2	[0]	2	1	~~0~~

Assignment

1 ⟶ I = 11

2 ⟶ IV = 6

3 ⟶ V = 16

4 ⟶ III = 17

5 ⟶ II = 10

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Problem 3

Using the following cost matrix, determine a) optimal job assignment b) the cost of assignments

JOB						
MECHANIC		1	2	3	4	5
	A	10	3	3	2	8
	B	9	7	8	2	7
	C	7	5	6	2	4
	D	3	5	8	2	4
	E	9	10	9	6	10

Row Minimization

8	1	1	0	6
7	5	6	0	5
5	3	4	0	2
1	3	6	0	2
3	4	3	0	4

Column minimization

7	0	0	0	4
6	4	5	0	3
4	2	3	0	0
0	2	5	0	0
2	3	2	0	2

Draw the horizontal and vertical lines

7	0	0	0	4
6	4	5	0	3
4	2	3	0	0
0	2	5	0	0
2	3	2	0	2

Here, N≠n, 4≠ 5

Then, we have to select the least value in the uncrossed cell i.e 2 the result table.

9	0	0	2	6
6	2	2	0	3
4	0	1	0	0
0	0	3	0	0
2	1	0	0	2

N=n satisfies our condition, so optimal assignment can be done

9	[0]	~~0~~	2	6
6	2	2	[0]	3
4	~~0~~	1	~~0~~	[0]
[0]	~~0~~	3	~~0~~	~~0~~
2	1	[0]	~~0~~	2

A $\longrightarrow$ 2 = 3

B $\longrightarrow$ 4 = 2

C $\longrightarrow$ 5 = 4

D $\longrightarrow$ 1 = 3

E $\longrightarrow$ 3 = 9

21 the minimum cost is Rs. 21

Problem 4

Job shop needs to assign 4 jobs to 4 workers. The cost of performing a job is a function of the skills of the workers. Table summarizes the cost of the assignments. Worker1 cannot do job3, and worker 3 cannot do job 4. Determine the optimal assignment using the Hungarian method.

Job					
Worker		1	2	3	4
	1	Rs. 50	Rs.50	--	Rs.20
	2	Rs.70	Rs.40	Rs.20	Rs.30
	3	Rs.90	Rs.30	Rs.50	--
	4	Rs.70	Rs.20	Rs60	Rs70

Row Minimization

30	30	--	0
50	20	0	10
60	0	20	--
50	0	40	50

Column minimizati on

0	30	--	0
20	20	0	10
30	0	20	--
20	0	40	50

N≠n

3≠4

The least value in the uncrossed cell is 10 and the resulting table will be as follows

~~0~~	40	~~--~~	~~0~~
~~10~~	20	~~0~~	~~0~~
20	0	20	--
10	0	40	40

N≠n

3=4

The least value in the uncrossed cell is 10

~~0~~	~~50~~	~~--~~	~~0~~
~~10~~	~~20~~	~~0~~	~~0~~
~~10~~	~~0~~	~~10~~	~~--~~
~~0~~	~~0~~	~~30~~	~~30~~

N=n

The given problem satisfies the condition, the assignment can be made for the optimal table.

~~0~~	50	--	[0]
10	20	[0]	~~0~~
10	[0]	10	--
[0]	~~0~~	30	30

1--------- 4 = 20

2--------- 3 = 20

3--------- 2 = 30

4--------- 1 = 70

140

4----------1-------- 4, 2-------- 3------- 2

There are two sequences in the given problem

Problem 5

A typical assignment problem, presented in the classic manner, is shown in Fig. Here there are five machines to be assigned to five jobs. The numbers in the matrix indicate the cost of doing each job with each machine. Job s with costs of M are disallowed assignments. The problem is to find the minimum cost matching of machines to jobs.

		J1	J2	J3	J4
M1	M1	M	8	6	12
M2	M2	15	12	7	M
M3	M3	10	M	5	14
M4	M4	12	M	12	16
M5	M5	18	17	14	M

Fig 1 M atrix model of the assignment problem.

The network model is in shown in Fig.2. It is very similar to the transportation m odel except the external flows are all +1 or -1. T he only relevant parameter for the assignment model is arc cost (not shown in the figure for clarity) ; all other parameters should be set to default values. The assignment network also hasth e bipartite structure.

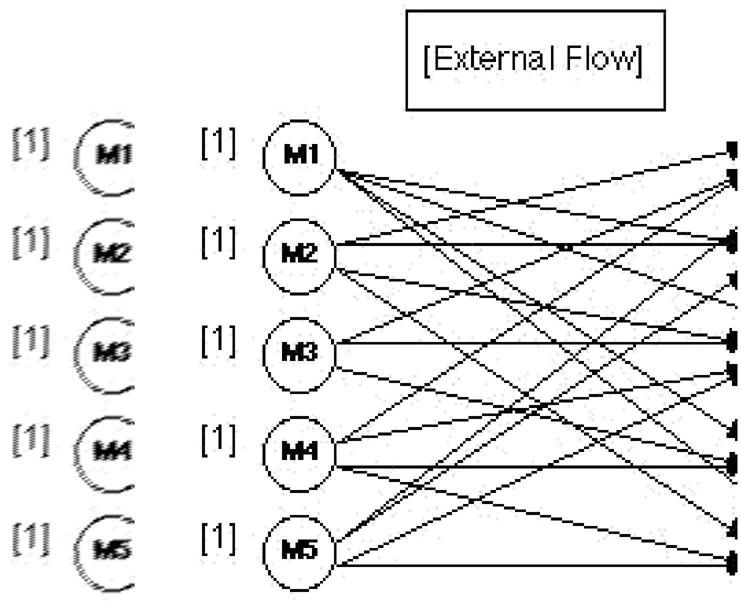

Figure 2. Network model of the assignment problem.

M	8	6	12	1
15	12	7	M	10
10	M	5	14	M
12	M	12	16	15
18	17	14	M	13

This is the given problem, using Hungarian method we solve the problem

Row minimization

M	7	5	11	0
8	5	0	M	3
5	M	0	9	M
0	M	0	4	3
5	4	1	M	0

Column minimization

M	3	5	7	0
8	1	0	M	3
5	M	0	5	M
0	M	0	0	3
5	0	1	M	0

N≠n, 4≠5 so select least value in

The uncrossed cell and subtract

M	3	5	7	0
8	1	0	M	3
5	M	0	5	M
0	M	0	0	3
5	0	1	M	0

M	2	5	6	0
7	0	0	M	3
4	M	0	4	M
0	M	1	0	3
2	0	1	M	0

N≠n, 4≠5 so select least value in The uncrossed cell and subtract

M	2	5	2	0
3	0	0	M	3
0	M	0	0	M
0	M	1	0	3
2	0	1	M	0

M	2	5	2	[0]
3	~~0~~	[0]	M	3
~~0~~	M	~~0~~	[0]	M
[0]	M	1	~~0~~	3
2	[0]	1	M	~~0~~

N=n and the assignment can be done

The solution to the assignment problem as shown in Fig. 3 has a total flow of 1 in every column and row, and is the assignment t hat minimizes total cost.

		J1	J2	J3	J4
M1	M1	0	0	0	0
M2	M2	0	0	1	0
M3	M3	0	0	0	1
M4	M4	1	0	0	0
M5	M5	0	1	0	0

Figure 3. Solution to the assignment Problem

Problem 6.

Four different jobs can be done on four different machines and take down time costs are prohibitively high for change ov ers. The matrix below gives the cost in rupees of producing job on machine j;

Jobs	Machine			
	M1	M2	M3	M4
J1	5	7	11	6
J2	8	5	9	6
J3	4	7	10	7
J4	10	4	8	3

How the jobs should be assigned to the various machines so that the total cost is minimized.

Row minimization

0	2	6	1
3	0	4	1
0	3	6	3
7	1	5	0

Column minimization

0	2	2	1
3	0	0	1
0	3	2	3
7	1	1	0

Draw the horizontal and vertical lines which covers max no of zeros

0	2	2	1
3	0	0	1
0	3	2	3
7	1	1	0

N≠n 3≠4

The least value is 1, the resulting table

0	1	1	1
4	0	0	2
0	2	1	3
7	0	0	0

N≠n

3≠4

The least value is 1, the resulting table

0	0	0	0
5	0	0	2
0	1	0	2
8	0	0	0

N=n

4=4 the assignment can be made for the above optimal table

~~0~~	~~0~~	~~0~~	[0]
5	[0]	0	2
[0]	1	~~0~~	2
8	~~0~~	[0]	~~0~~

J1 ----------M4 = 6

J2----------M2 = 5

J3----------M1 = 4

J4----------M3 = 8

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Alternate solution

[0]	~~0~~	~~0~~	~~0~~
5	[0]	~~0~~	2
~~0~~	1	[0]	2
8	~~0~~	~~0~~	[0]

J1 --------- M1 = 5

J2-------- M2 = 5

J3-------- M3 = 10

J4-------- M4 = 3

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Problem 7

A company has 5 jobs tobe done the following matrix shows the return in Rs. of assigning ith machine (i=1,2,3,---5) to the jth job (j=1,2,3,----n). Assign the 5 jobs to the 5 machines so as to maximize the expected profit.

JOB						
Machine		1	2	3	4	5
	1	5	11	10	12	4
	2	2	4	6	3	5
	3	3	12	5	14	6
	4	6	14	4	11	7
	5	7	9	8	12	8

Since, the given problem is maximum

Step 1: to convert the problem to a minimum by multiply all elements C_{ij} Of the assignment matrix by -1

Then the given problem will in the form as shown below

JOB						
Machine		1	2	3	4	5
	1	-5	-11	-10	-12	-4
	2	-2	-4	-6	-3	-5
	3	-3	-12	-5	-14	-6
	4	-6	-14	-4	-11	-7
	5	-7	-9	-8	-12	-8

Step2: select the most –ve and subtract with other elements of the matrix minz= - (-maxZ)

In the matrix the most –ve value is -14. Using this value the matrix is subtracted and the resulting is the minimization matrix. This can be used for finding the optimal assignment table using usual procedure to solve the problem.

JOB						
Machine		1	2	3	4	5
	1	9	3	4	2	10
	2	12	10	8	11	9
	3	11	2	9	0	8
	4	8	0	10	3	7
	5	7	5	6	2	6

Example

C11= -5 – (-14) = 9 and continued for all other element

Step 3:- Using the above table i.e. minZ matrix table and all cost elements non-ve. The Hungarian method can be applied to find the optimal assignment problem.

Row minimization

7	1	2	0	8
4	2	0	3	1
11	2	9	0	8
8	0	10	3	7
5	3	4	0	4

Column minimization

3	1	2	0	7
0	2	0	3	0
7	2	9	0	7
4	0	10	3	6
1	3	4	0	3

N=3, n=5x5

N≠n, select the minimum value from the uncrossed cell and subtract for all the elements of uncrossed cell and add for the intersection of horizontal and vertical.

	2	1	1	0	6
	0	3	0	4	0
	6	2	8	0	6
	3	0	9	3	5
	0	3	3	0	2

N=4 n=5x5

N≠n

The least value in the uncrossed cell is 1 again and subtracts using this value for all other elements and add for intersection of horizontal and vertical

	2	1	0	0	5
	1	4	0	5	0
	6	2	7	0	5
	3	0	8	3	4
	0	3	2	0	1

N=n, for assignment the optimal table is obtained

2	1	[0]	~~0~~	5
1	4	~~0~~	5	[0]
6	2	7	[0]	5
3	[0]	8	3	4
[0]	3	2	~~0~~	1

Now the assignment is

1----------- 3 = 10

2----------- 5 = 5

3-----------4 = 14

4-----------2 = 14

5----------- 1 = 7

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Problem 7

A marketing manager has 5 elements and there are 5 sales districts. Consideri ng the capabilities of the salesmen and the nature of districts, the estimates made by the marketing manager for the sales per month (in 1000 rupees) for each salesman in each district would be as follows.

	A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

Find the assignment of salesmen to the districts that will result in the maximum sale.

Solution:

The given problem is profit matrix. To maximize the profit, first we must convert it minimization. To convert to minimization we must select the maximum value of the matrix i.e., 41. This value is subtracted for all other elements in the matrix and the resulting matrix is minimization.

	A	B	C	D	E
1	9	3	1	13	1
2	1	17	13	20	5
3	0	14	8	11	4
4	19	3	0	5	5
5	12	8	1	6	2

Using above table we can solve the given problem by the Hungarian method.

Row minimization

8	2	0	12	0
0	16	12	19	4
0	14	8	11	4
19	3	0	5	5
11	7	0	5	1

Column minimization

8	0	0	7	0
0	14	12	14	4
0	12	8	6	4
19	1	0	0	5
11	5	0	0	1

Drawing the horizontal and vertical
lines N=4 n5x5

N≠n, now select the least value in the uncrossed cell and subtract to all the uncrossed cell and add to the intersection of horizontal and vertical line. The least value is 4

8	0	0	7	0
0	10	8	10	0
0	8	4	2	0
19	1	0	0	5
11	5	0	0	1

N=n, it satisfies our condition and now we can assign the workers to jobs using above table

8	[0]	~~0~~	7	~~0~~
[0]	10	8	10	~~0~~
~~0~~	8	4	2	[0]
19	1	[0]	~~0~~	5
11	5	~~0~~	[0]	1

1--------B = 38

2--------A = 40

3--------E = 37

4--------C = 41

5--------D = 35

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Problem 8

A company has four territories open and four salesmen available for assignment. The territories are not equally rich in their sales potential; it is estimated that a typical salesman operating in each territory would bring in the following annual sales:

Territory :	I	II	III	IV
Annual Sales (Rs)	60,000	50,000	40,000	30,000

Four salesmen are also considered to differ in chair ability: it is estimated that, working under the same conditions, their yearly sales would be proportionately as follow:

Salesmen:	A	B	C	D
Proportion:	7	5	5	4

If the criterion is maximum expected total sales, then including answer is to assign the best salesman to the richest territory, the next best salesman to the second richest, and so on, verify this answer by the assignment technique.

Solution:

Step 1 to construct the effectiveness of the matrix

By taking Rs. 10000/- as one unit and the sales proportion and the maximum sales matrix is obtained as follows:

		Sales in 10 thousand of rupees			
Sales Proportion		6	5	4	3
		I	II	III	IV
7	A	42	35	28	21
5	B	30	25	20	15
5	C	30	25	20	15
4	D	24	20	16	12

To find the value of c11= sales proportion X sales of territory = 7X6=42

In the same it is continued for the remaining cells

Step 2: to convert the maximum sales matrix to minimum sales matrix

By simply multiplying each element of given matrix by -1. Thus resulting matrix becomes:

	I	II	III	IV
A	-42	-35	-28	-21
B	-30	-25	-20	-15
C	-30	-25	-20	-15
D	-24	-20	-16	-12

Step 3: select the most negative in the matrix i.e. is -42. With this element subtract all the elements in the matrix. The resulting is minimization table

	I	II	III	IV
A	0	7	14	21
B	12	17	22	27
C	12	17	22	27
D	18	22	26	30

Now, using the above table we can apply the Hungarian method to find the assignment for the given problem and the value should be taken from the original table since, it is a maximization problem

Row minimization

0	7	14	21
0	5	10	15
0	5	10	15
0	4	8	12

Column minimization

0	3	6	9
0	1	2	3
0	1	2	3
0	0	0	0

Draw horizontal and vertical lines

0	3	6	9
0	1	2	3
0	1	2	3
0	0	0	0

N≠n, 2≠4 so select the least value of the uncrossed cell and subtract

The least value is 1

0	~~2~~	5	8
0	~~0~~	1	2
0	~~0~~	1	2
0	~~0~~	~~0~~	~~0~~

N≠n, 3≠4 so select the least value of the uncrossed cell and subtract The least value is 1

0	2	~~4~~	7
0	0	~~0~~	1
0	0	~~0~~	1
0	0	~~0~~	~~0~~

N=n, 4=4, the assignment of the given problem

[0]	2	4	7
~~0~~	[0]	~~0~~	1
~~0~~	~~0~~	[0]	1
~~0~~	~~0~~	~~0~~	[0]

A-------- I

B ------------II

C ------------ III

D --------IV

Problem 9:

Alpha Corporation has four plants each of which can manufacture any one of four products production costs differ from one plant to another as do sales revenue. Given the revenue and cost data below, obtain which product each plant should produce to maximize profit.

	Sales revenue (Rs. 000s Product)			
Plant	1	2	3	4
A	50	68	49	62
B	60	70	51	74
C	55	67	53	70
D	58	65	54	69

	Production costs (Rs. 000s Product)			
Plant	1	2	3	4
A	49	60	45	61
B	55	63	45	69
C	52	62	49	68
D	55	64	48	66

Solution:

Now, we have found the profit matrix by using sales revenue and production cost. Profit = sales - cost

Profit matrix

	1	2	3	4
A	1	8	4	1
B	5	7	6	5
C	3	5	4	2
D	3	1	6	3

Now we find the minimization matrix, by selecting the highest profit in the profit matrix i.e. 8 is subtract all the elements in the matrix and resulting will be the minimization matrix of the given problem.

7	0	4	7
3	1	2	3
5	3	4	6
5	7	2	5

Using Hungarian method

Row minimization

7	0	4	7
2	0	1	2
2	0	1	3
3	5	0	3

Column minimization

5	0	4	5
0	0	1	0
0	0	1	1
1	5	0	1

Draw the horizontal and vertical lines

5	0	4	5
0	0	1	0
0	0	1	1
1	5	0	1

N=n, the assignment can be done for the above table

5	[0]	4	5
~~0~~	~~0~~	1	[0]
[0]	~~0~~	1	1
1	5	[0]	1

A --------- 2

B --------- 4

C------ 1

D -------- 3

Problem 11

An air-line operates seven days a week has time-table shown below. Crews must have a minimum layover (rest) time of 5 hrs, between flights. Obtain the pair of flights that minimimizes layover time away from home. For any given pair the crews will e based at the city that result in the smaller layover.

Delhi - Jaipur			Jaipur-Delhi		
Flight No.	Depart	Arrive	Flight No.	Depart	Arrive
1	7.00 AM	8.00 AM	101	8.00 AM	9.15 AM
2	8.00 AM	9.00 AM	102	8.30 AM	9.45 AM
3	1.30 PM	2.30 P.M	103	12.00 NOON	1.15 PM
4	6.30AM	7.30 PM	104	5.30 PM	6.45 PM

for each pair, mention the town where the crews should be based.

SOLUTION:

Step1 construct the table for layour times between flights when crew is based at Delhi, for simplicity consider 15 minutes = 1unit.

Table 1: layover times when crew based at Delhi

Flights	101	102	103	104
1	96	98	112	38
2	92	94	108	34
3	70	72	86	108
4	50	52	66	88

Since, the crew must have a minimum layover of 5 hrs between flights

The layover time between flights 1 and 101 will be 24 hrs (96 units)from 8.00 AM to 8.00 AM next day i.e flight 1 arrives jaipur at 8.00 am and leaves the jaipur 8.00 am next day because of minimum layover is 5 hrs between flights and other flights is there in between so flight will be there next day only.

Flight 1 to 102 will be (98units) 8.00 am arrives jaipur leaves jaipur 8.30 am next day= 24 hrs+30 minutes

Flight 1 to 103 will be (112 units) 8.00 am arrives jaipur leaves jaipur 12.00 noon next day= 24 hrs +4 hrs =112 units

Flight 1 to 104 will be (38 units)8.00 am arrives jaipur leaves jaipur 5.30 pm on the same day = 9 hrs +30 min = 38 mins

The layover time between Flight 2 to 101 will be (9.00 am arrival and depart from jaipur 8.00 am next day) =23 hrs = 92 units

Flight 2 to 102 will be (9.00 am arrives jaipur and depart from jaipur 8.30 am next day) = 23 hrs +30 minutes = 94 units

Flight 2 to 103 will be (9.00 am arrives jaipur and depart from jaipur 12.00 noon next day) = 24 hrs +3 hrs = 108 units

Flight 2 to 104 will be (9.00 am arrives jaipur and depart from jaipur 5.30 pm same day) = 8 hrs +30 minutes= 34 units

The layover time between Flight 3 to 101 will be (2.30 pm arrival and depart from jaipur 8.00 am next day) =17 hrs + 30 minutes = 70 units

Flight 3 to 102 will be (2.30 pm arrives jaipur and depart from jaipur 8.30 am next day) = 18hrs = 72 units

Flight 3 to 103 will be (2.30 pm arrives jaipur and depart from jaipur 12.00 noon next day) = 21hrs + 30 minutes = 86 units

Flight 3 to 104 will be (2.30 pm arrives jaipur and depart from jaipur 5.30 pm next day) = 24hrs+3hrs= 108 units

The layover time between Flight 4 to 101 will be (7.30 pm arrival and depart from jaipur 8.00 am next day) =12 hrs +30 minutes = 50 units

Flight 4 to 102 will be (7.30 pm arrives jaipur and depart from jaipur 8.30 am next day) = 13hrs = 52 units

Flight 4 to 103 will be (7.30 pm arrives jaipur and depart from jaipur 12.00 noon next day) = 16hrs + 30 minutes = 66 units

Flight 4 to 104 will be (7.30 pm arrives jaipur and depart from jaipur 5.30 pm next day) = 22hrs= 88 units

Table 2: layover times when crew based at jaipur

Flights	101	102	103	104
1	87	85	71	49
2	91	89	75	53
3	113	111	97	75
4	37	35	21	95

Arrival and depart when crew is based in jaipur

Flight No.	Arrive(delhi)	Flight No.	Depart (Delhi)
101	9.15 AM	1	7.00 AM
102	9.45 AM	2	8.00 AM
103	1.15 PM	3	1.30 PM
104	6.45 PM	4	6.30AM

Since, the crew must have a minimum layover of 5 hrs between flights

The layover time between flights 101 and 1 will be 21 hrs+ 45 minutes (87 units) from 9.15 AM to 7.00 AM next day by flight no 1 i.e flight 101 arrives Delhi at 9.15 am and leaves the Delhi 7.00 am next day by flight no 1 because of minimum layover is 5 hrs between flights and no other flights is there in between so flight will there next day only.

Flight 101 to 2 will be (91 units) 9.15 am arrives Delhi leaves Delhi 8.00 am next day= 22 hrs+45 minutes

Flight 101 to 3 will be (113 units) 9.15 am arrives Delhi leaves Delhi 1.30 pm next day= 28 hrs +15 minutes = 113 units

Flight 101 to 4 will be (38 units) 9.15 am arrives Delhi leaves Delhi 6.30 pm on the same day = 9 hrs +15 min = 37 mins

The layover time between Flight 102 to 1 will be (9.45 am arrival and depart from Delhi 7.00 am next day) =21 hrs+15 minutes = 85 units

Flight 102 to 2 will be (9.45 am arrives Delhi and depart from Delhi 8.00 am next day) = 22hrs + 15 minutes = 89 units

Flight 102 to 3 will be (9.45 am arrives Delhi and depart from Delhi 1.30 pm next day) = 27 hrs +45 minutes = 111 units

Flight 102 to 4 will be (9.45 am arrives Delhi and depart from Delhi 6.30 pm same day) = 8 hrs +45 minutes= 35 units

The layover time between Flight 103 to 1 will be (1.15 pm arrival and depart from Delhi 7.00 am next day) =17hrs + 45 minutes = 71 units

Flight 103 to 2 will be (1.15 pm arrives Delhi and depart from Delhi 8.00 am next day) = 18hrs + 45 minutes = 75 units

Flight 103 to 3 will be (1.15 pm arrives Delhi and depart from Delhi 1.30 pm next day) = 24 hrs +15 minutes = 97 units

Flight 103 to 4 will be (1.15 pm arrives Delhi and depart from Delhi 6.30 pm same day) = 5 hrs + 15 minutes= 21 units

The layover time between Flight 104 to 1 will be (6.45 pm arrival and depart from Delhi 7.00 am next day) =12hrs + 15 minutes = 49 units

Flight 104 to 2 will be (6.45 pm arrives Delhi and depart from Delhi 8.00 am next day) = 13hrs + 15 minutes = 53 units

Flight 104 to 3 will be (6.45 pm arrives Delhi and depart from Delhi 1.30 pm next day) =18 hrs +45 minutes = 75 units

Flight 104 to 4 will be (6.45 pm arrives Delhi and depart from Delhi 6.30 pm same day) = 23 hrs + 45 minutes= 95 units

Step 3:construct the table for minimum layover times between flights with the help of Table 1 and Table 2 layover times marked * denote that the crew is based at jaipur.

Table 3:

Flights	101	102	103	104
1	87*	85*	71*	38
2	91*	89*	75	34
3	70	72	86	75
4	37*	35*	21*	88

Using Hungarian method we solve the above table and the assignment are as shown in the table

~~0~~	~~0~~	[0]	~~0~~
12	8	8	[0]
[0]	~~0~~	28	50
4	[0]	~~0~~	100

The optimal assignments are

Flight 1-103

Flight 2-104

Flight 3-101

Flight 4-102

Unbalanced Assignment Problem

If the cost matrix of an assignment problem is not a square matrix (number of sources is not equal to the number of destinations),

The assignment problem is called an unbalanced assignment problem.

In such cases, fictitious rows and columns are added in the matrix so as to form a square matrix. Then the usual assignment algorithm can be applied to this resulting balanced problem.

Problem 1:

A company is faced with the problem of assigning six different machines to five different jobs. The costs are estimated as follows (in hundreds of rupees):

		Jobs				
		1	2	3	4	5
Machines	1	2.5	5.0	1.0	6	1.0
	2	2.0	5.0	1.5	7	3.0
	3	3.0	6.5	2.0	9	4.5
	4	3.5	7.0	2.0	9	4.5
	5	4.0	7.0	3.0	9	6.0
	6	6.0	9.0	5.0	10	6.0

Solve the problem assuming that the objective is to minimize the total cost.

Solution:

The matrix is 6x5, and then given problem is unbalanced assignment problem. So we introduce fictitious job i.e Job 6 in the cost matrix in order to get the balanced assignment problem. The costs corresponding to such column are always taken as zero.

	Jobs						
		1	2	3	4	5	6
Machines	1	2.5	5.0	1.0	6	1.0	0
	2	2.0	5.0	1.5	7	3.0	0
	3	3.0	6.5	2.0	9	4.5	0
	4	3.5	7.0	2.0	9	4.5	0
	5	4.0	7.0	3.0	9	6.0	0
	6	6.0	9.0	5.0	10	6.0	0

Then, the problem can be solved by usual manner by Hungarian method.

Row Minimization

2.5	5.0	1.0	6	1.0	0
2.0	5.0	1.5	7	3.0	0
3.0	6.5	2.0	9	4.5	0
3.5	7.0	2.0	9	4.5	0
4.0	7.0	3.0	9	6.0	0
6.0	9.0	5.0	10	6.0	0

Row minimization remains as same as original problem

Column minimization

~~0.5~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~
~~0~~	~~0~~	~~0.5~~	~~1~~	~~2.0~~	~~0~~
1.0	1.5	1.0	3	3.5	~~0~~
1.5	2.0	1.0	3	3.5	~~0~~
2.0	2.0	2.0	3	5.0	~~0~~
4.0	4.0	4.0	4	5.0	~~0~~

N=3, n=5x5, N≠n, so the least value in the uncrossed value is 1.0 and with this value all the uncrossed cell is subtracted and the resulting matrix

~~0.5~~	~~0~~	~~0~~	~~0~~	~~0~~	~~1~~
~~0~~	~~0~~	~~0.5~~	~~1~~	~~2.0~~	~~1~~
~~0~~	~~0.5~~	~~0~~	~~2~~	~~3.5~~	~~1~~
~~0.5~~	~~1.0~~	~~0~~	~~2~~	~~3.5~~	~~1~~
1.0	1.0	1.0	2	5.0	~~0~~
3.0	3.0	3.0	3	5.0	~~0~~

N=5, n=6x6, N≠n

The least value in the uncrossed cell is 1 and using this value it is subtracted to the entire uncrossed cell and for the intersection of horizontal and vertical lines it is added and the resulting matrix is given below.

~~0.5~~	~~0~~	~~0~~	~~0~~	~~0~~	~~2~~
0	0	0.5	1	2.0	2
0	0.5	0	2	3.5	2
0.5	1.0	0	2	3.5	2
0	0	0	1	5.0	0
2.0	2.0	2.0	2	5.0	0

N=5, n=6, N≠n, the least value is 1 and using this it is subtracted to the uncrossed cell of the above matrix and the resulting is shown below.

~~0.5~~	~~0~~	~~0~~	~~0~~	~~0~~	~~2~~
~~0~~	~~0~~	~~0.5~~	~~0~~	~~1.0~~	~~2~~
0	0.5	0	1	2.5	~~2~~
0.5	1.0	0	1	2.5	~~2~~
~~0~~	~~0~~	~~0~~	~~0~~	~~4.0~~	~~0~~
~~2.0~~	~~2.0~~	~~2.0~~	~~1~~	~~4.0~~	~~0~~

The table satisfies N=n and the optimal assignment table is obtained and using the matrix the assignment can be done.

~~0.5~~	~~0~~	~~0~~	~~0~~	[0]	2
~~0~~	[0]	~~0.5~~	~~0~~	1.0	2
[0]	0.5	~~0~~	1	2.5	2
0.5	1.0	[0]	1	2.5	2
~~0~~	~~0~~	~~0~~	[0]	4.0	~~0~~
2.0	2.0	2.0	1	4.0	[0]

Machine 1---------------Job5 = 1

Machine 2---------------Job2 = 5

Machine 3---------------Job1 = 3

Machine 4---------------Job3 = 2

Machine 5---------------Job4 = 9

Machine 6---------------Job6 = 6

26

Problem 2.

Manager of Transportation Company must order 5 trucks out of a fleet to be present at 5 specific locations for loading goods that are awaiting shipment eight trucks are at different location. The costs are given in the table below. Assign 5 trucks so as to minimize the cost.

		Loading Locations				
		A	B	C	D	E
Trucks	1	300	290	280	290	210
	2	250	310	290	300	200
	3	180	190	300	190	180
	4	320	180	190	240	170
	5	270	210	190	250	160
	6	190	200	220	190	140
	7	220	300	230	180	160
	8	200	190	260	210	180

Solution;

The given problem is 8x5, so it is a unbalanced assignment problem, here we must had 3 fictitious loading locations as shown below.

		Loading Locations							
		A	B	C	D	E	F	G	H
Trucks	1	300	290	280	290	210	0	0	0
	2	250	310	290	300	200	0	0	0
	3	180	190	300	190	180	0	0	0
	4	320	180	190	240	170	0	0	0
	5	270	210	190	250	160	0	0	0
	6	190	200	220	190	140	0	0	0
	7	220	300	230	180	160	0	0	0
	8	200	190	260	210	180	0	0	0

Once the matrix is balanced and then usual procedure is used to solve the problem for assignment.

Row minimization

300	290	280	290	210	0	0	0
250	310	290	300	200	0	0	0
180	190	300	190	180	0	0	0
320	180	190	240	170	0	0	0
270	210	190	250	160	0	0	0
190	200	220	190	140	0	0	0
220	300	230	180	160	0	0	0
200	190	260	210	180	0	0	0

Since, each row has zero (0) the row minimization remains same as the original problem.

Column minimization

120	110	90	110	70	0	0	0
70	130	100	120	60	0	0	0
0	10	110	10	40	0	0	0
140	0	0	60	30	0	0	0
90	30	0	70	20	0	0	0
10	20	30	10	0	0	0	0
40	120	40	0	20	0	0	0
20	10	70	30	40	0	0	0

120	110	90	110	70	~~0~~	[0]	~~0~~
70	130	100	120	60	[0]	~~0~~	~~0~~
[0]	10	110	10	40	~~0~~	~~0~~	~~0~~
140	[0]	~~0~~	~~60~~	~~30~~	~~0~~	~~0~~	~~0~~
90	30	[0]	70	20	~~0~~	~~0~~	~~0~~
10	20	30	10	[0]	~~0~~	~~0~~	~~0~~
40	120	40	[0]	20	~~0~~	~~0~~	~~0~~
20	10	70	30	40	~~0~~	~~0~~	[0]

T1 ----------------------------------G =0

T2 ----------------------------------F = 0

T3 ----------------------------------A = 180

T4 ----------------------------------B = 180

T5 ----------------------------------C = 190

T6 ----------------------------------E = 140

T7 ----------------------------------D = 180

T8 ----------------------------------H =0

870

The Travelling salesman (routing) problem

The travelling salesman problem is one of the problems considered as puzzles by the mathematicians.

Suppose a salesman wants to visit a certain number of cities allotted to him.

He knows the distances (or cost or time) of journey between every pair of cities, usually denoted by c_{ij} i.e. city I to city j.

His problem is to select such a route that stars from his home city. Passes through each city once and only once and returns to his home icty in the shortest possible distance (or at the least cost or in least time).

FORMULATION OF A TRAVELLING – SALESMAN PROBLEM AS ASSIGNMENT PROBLEM

Suppose c_{ij} is the distance 9or cost or time) from city I to city j and x_{ij} =1, if the salesman goes directly from city I to city j, and zero otherwise. Then minimize $\sum_i \sum_j x_{ij} c_{ij}$ with the additional restriction that the x_{ij} must be so chosen that no city is visited twice before the tour of all cities is completed. In particular, he cannot go directly from city I to I itself. This possibility may be avoided in the minimization process by adopting the convention $c_{ij} = \infty$ which ensures that x_{ij} can never be unity.

Alternatively, omit the variable x_{ij} from the problem specification. It is also important to note that onl single x_{ij} =1 for each value of I and j. the distance (or cost or time) matrix for this problem is given in table 1.

	To			
From	∞	C12	---	C1n
	C21	∞	---	C2n
	---	--	∞	--
	Cn1	Cn2	---	∞

Problem 1:

Given the matrix of set-up costs, show how to sequence the production so as to minimize the set-up cost per cycle.

	To					
From		A1	A2	A3	A4	A5
	A1	∞	2	5	7	1
	A2	6	∞	3	8	2
	A3	8	7	∞	4	7
	A4	12	4	6	∞	5
	A5	1	3	2	8	∞

Solution:

Row minimization

	A1	A2	A3	A4	A5
A1	∞	1	4	6	0
A2	4	∞	1	6	0
A3	4	3	∞	0	3
A4	8	0	2	∞	1
A5	0	2	1	7	∞

Column minimization

	A1	A2	A3	A4	[redacted]
A1	∞	1	3	6	~~0~~
~~A2~~	~~4~~	~~∞~~	~~0~~	~~6~~	~~0~~
~~A3~~	~~4~~	~~3~~	~~∞~~	~~0~~	~~3~~
~~A4~~	~~8~~	~~0~~	~~1~~	~~∞~~	~~1~~
~~A5~~	~~0~~	~~2~~	~~0~~	~~7~~	~~∞~~

N=n, 5=5x5

∞	1	3	6	[0]
4	∞	[0]	6	$\cancel{0}$
4	3	∞	[0]	3
8	[0]	1	∞	1
[0]	2	$\cancel{0}$	7	∞

A1-------A5 -----A1, A2------- A3------ A4------ A2

Cost = 1+3+4+4+1=13

As per the sequence from the above assignment indicates to produce the products A1, then A5 and then again A1, without producing the products A2,A3 and A4 thereby violates the additional restriction of producing each product once and only once before returning to the first product.

Step 2: next to examine the matrix for the best solutions to the assignment problem and first we try with value one (1) the cells having 1 are c12, c43, c45, using this cells we try for one sequence

Let us try with assigning with c15 to c12 and c42 to c45

∞	[1]	3	6	$\cancel{0}$
4	∞	[0]	6	$\cancel{0}$
4	3	∞	[0]	3
8	$\cancel{0}$	1	∞	[1]
[0]	2	$\cancel{0}$	7	∞

The sequence will be A1---A2---A3---- A4---A5--- A1

The cost will be 2+3+4+5+1 =15

Here the cost is increased by Rs.2. if you see in the matrix to get one sequence we have changed the assignment from c15 to c12 contains 1 and c42 to c45 contains due this penalty cost. The cost is increased by Rs 2.

Problem 2

A machine operator processes 5 types of items on his machine each week, and must choose a sequence for them. The set-up coast per change depends on the item presently on the machine and the set-up to be made according to the following table:

	To Item				
	A	B	C	D	E
A	∞	4	7	3	4
B	4	∞	6	3	4
C	7	6	∞	7	5
D	3	3	7	∞	7
E	4	4	5	7	∞

if the processes each type of item once and only once each week how should he sequence the items on his machine in order to minimize the total set-up cost?

Solution:

Using usual assignment problem Row Minimization

∞	1	4	1	1
1	∞	3	0	1
2	1	∞	2	0
0	0	4	∞	4
0	0	1	3	∞

Column minimization

∞	1	3	1	1
1	∞	2	0	1
2	1	∞	2	0
0	0	3	∞	4
0	0	0	3	∞

Draw horizontal and vertical lines

∞	1	3	~~1~~	~~1~~
1	∞	2	~~0~~	~~1~~
2	1	∞	~~2~~	~~0~~
~~0~~	~~0~~	~~3~~	~~∞~~	~~4~~
~~0~~	~~0~~	~~0~~	~~3~~	~~∞~~

N≠n, 4≠5, least value is 1

∞	0	2	1	1
0	∞	1	0	1
1	0	∞	2	0
0	0	3	∞	5
0	0	0	4	∞

N=n

∞	[0]	2	1	1
~~0~~	∞	1	[0]	1
1	~~0~~	∞	2	[0]
[0]	~~0~~	3	∞	5
~~0~~	~~0~~	[0]	4	∞

A---B----D ----A, C E------- C, THE COST =4+3+5+3+5=20

In the given problem we have 2 sequence and doesn't satisfies the travelling salesmen procedure

So we try with shifting of cells from c12 to c15 and c35 to c32

∞	~~0~~	2	1	[1]
~~0~~	∞	1	[0]	1
1	~~[0]~~	∞	2	~~0~~
[0]	~~0~~	3	∞	5
~~0~~	~~0~~	[0]	4	∞

A---E---B---D—A, C--- B, This also not satisfied the travelling salesmen

COST =4+3+6+3+5=21

Now we try with shifting c32 to c31 and c41 to c42

∞	~~0~~	2	1	[1]
~~0~~	∞	1	[0]	1
[1]	~~0~~	∞	2	~~0~~
~~0~~	[0]	3	∞	5
~~0~~	~~0~~	[0]	4	∞

A---E---C----A, B---D---B this is also not suitable to travelling salesmen

COST= 4+3+7+3+5=22

NOW WE TRY WITH SHIFITNG OF C15 TO C14 AND C24 TO C25

∞	~~0~~	2	[1]	~~1~~
~~0~~	∞	1	~~0~~	[1]
[1]	~~0~~	∞	2	~~0~~
~~0~~	[0]	3	∞	5
~~0~~	~~0~~	[0]	4	∞

A---D—B---E---C—A COST=4+4+7+3+5=23

Problem3

Solve the given travelling salesman Problem

	1	2	3	4	5
1	∞	20	4	10	∞
2	20	∞	5	∞	10
3	4	5	∞	6	5
4	10	∞	6	∞	2
5	∞	10	5	2	∞

Problem 4.

Solve the travelling salesman problem given by the following data

C_{12} =20, C_{13} =4, C_{14} =10, C_{23} =5, C_{34} =6, C_{25} =10, C_{35} =6, C_{45} =20 When = C_{ij}= C_{ji} there is no route between cities I and j if the value is not shown.

Solution :

Step1 first express the given problem in the form of an assignment problem by taking

$C_{ij} = \infty$, for i=j

Apply the usual assignment
problem Row minimization:

∞	16	0	6	∞
15	∞	0	∞	5
0	1	∞	2	1
8	∞	4	∞	0
∞	8	3	0	∞

Column minimization

∞	15	0	6	∞
15	∞	0	∞	5
0	0	∞	2	1
8	∞	4	∞	0
∞	7	3	0	∞

Draw the lines horizontal and vertical lines

∞	15	~~0~~	6	∞
15	∞	~~0~~	∞	5
~~0~~	~~0~~	~~∞~~	~~2~~	~~1~~
~~8~~	~~∞~~	4	~~∞~~	~~0~~
~~∞~~	~~7~~	~~3~~	~~0~~	~~∞~~

N≠n, 4=5 the least value in the uncrossed cell is 5

∞	10	~~0~~	~~1~~	~~∞~~
10	∞	~~0~~	~~∞~~	~~0~~
~~0~~	~~0~~	~~∞~~	~~7~~	~~6~~
8	∞	~~4~~	~~∞~~	~~0~~
∞	7	~~3~~	~~0~~	~~∞~~

N ≠ n, 4≠5 the least value is 7

∞	3	~~0~~	1	~~∞~~
3	∞	~~0~~	∞	~~0~~
~~0~~	~~0~~	~~∞~~	~~14~~	~~13~~
1	∞	~~4~~	∞	~~0~~
~~∞~~	~~0~~	~~3~~	~~0~~	~~∞~~

N≠n, 4≠5 the least value is 1

∞	2	0	0	∞
2	o	0	∞	0
0	0	∞	2	2
0	o	4	∞	0
∞	0	4	0	∞

N=n,

Using our optimal assignment table we can assign

∞	2	~~0~~	[0]	∞
2	∞	[0]	∞	~~0~~
[0]	~~0~~	∞	2	2
~~0~~	∞	4	∞	[0]
∞	[0]	4	0	∞

1 -----4----- 5---- 2---- 3--- 1

Cost = 10+5+4+20+10 = Rs. 49/-

Using penalty method

A salesman has to visit 5 cities ABCD &E the distance (miles) between the 5 cities are as follows

	A	B	C	D	E
A	∞	7	6	8	4
B	7	∞	8	5	6
C	6	8	∞	9	7
D	8	5	9	∞	8
E	4	7	7	8	∞

Row minimization

∞	3	2	4	0
4	∞	3	0	1
0	2	∞	3	1
3	0	4	∞	3
0	3	3	1	∞

Column minimization

~~∞~~	~~3~~	~~0~~	~~4~~	~~0~~
~~4~~	~~∞~~	1	~~0~~	1
~~0~~	~~2~~	∞	~~3~~	1
~~3~~	~~0~~	2	~~∞~~	3
~~0~~	~~3~~	1	~~1~~	∞

N≠n, 4≠5, least value is 1

∞	3	0	4	0
4	∞	0	0	0
0	2	0	3	0
3	0	1	∞	2
0	3	0	1	∞

N=n 5=5

The optimal assignment

∞	3	[0]	4	~~0~~
4	∞	~~0~~	[0]	~~0~~
0	2	∞	3	[0]
3	[0]	1	∞	2
[0]	3	~~0~~	1	∞

A----C---E---- A, B D B COST=6 +7+4 +5+5 =27------

Using penalty method we can get one sequence

∞	~~3~~	$0_{0+0=0}$	4	$0_{0+0=0}$
4	~~∞~~	$0_{0+0=0}$	$0_{0+1=1}$	$0_{0+0=0}$
$0_{0+0=0}$	~~2~~	∞	3	$0_{0+0=0}$
~~3~~	$\cancel{0_{2+1=3}}$	~~1~~	~~∞~~	~~2~~
$0_{0+0=0}$	~~3~~	$0_{0+0=0}$	1	∞

D --- B

	A	C	D	E
A	∞	$0_{0+0=0}$	4	$0_{0+0=0}$
B	4	$0_{0+0=0}$	----	$0_{0+0=0}$
C	$0_{0+0=0}$	∞	3	$0_{0+0=0}$
E	$0_{0+0=0}$	$0_{0+0=0}$	1	∞

since, the above table in the column d there are no zeros, so select the least value in the that row which is 1 and subtract with all other elements in that row.

	A	C	~~D~~	E
A	∞	$0_{0+0=0}$	~~3~~	$0_{0+0=0}$
B	4	$0_{0+0=0}$	~~----~~	$0_{0+0=0}$
C	$0_{0+0=0}$	∞	~~2~~	$0_{0+0=0}$
~~E~~	~~$0_{0+0=0}$~~	~~$0_{0+0=0}$~~	~~$0_{2+0=2}$~~	~~∞~~

E------D and the resulting matrix is

	A	C	E
A	∞	$0^{0+0=0}$	$0^{0+0=0}$
B	4	$0^{0+0=0}$	$0^{0+0=0}$
~~C~~	~~$0^{0+4=0}$~~	~~∞~~	~~$0^{0+0=0}$~~

C-----A and the resulting matrix

	C	E
A	----	$0^{0+0=0}$
B	$0^{0+0=0}$	$0^{0+0=0}$

B ---------C

A ---------E

The Optimal Assignment

A-----E-----D------ B-----C --- A

COST=4+8+5+8+6=31

Reference Books:

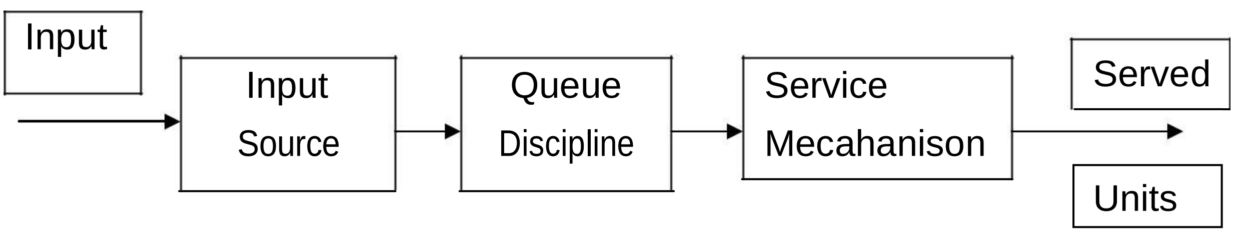
1. Taha H A, Operation Research - An Introduction, Prentice Hall of India, 7^{th} edition, 2003
2. Ravindran, Phillips and Solberg, Operations Research : Principles and Practice, John Wiely & Sons, 2^{nd} Edition
3. D.S.Hira, Operation Research, S.Chand & Company Ltd., New Delhi, 2004
4.

The Basic structure of queuing model

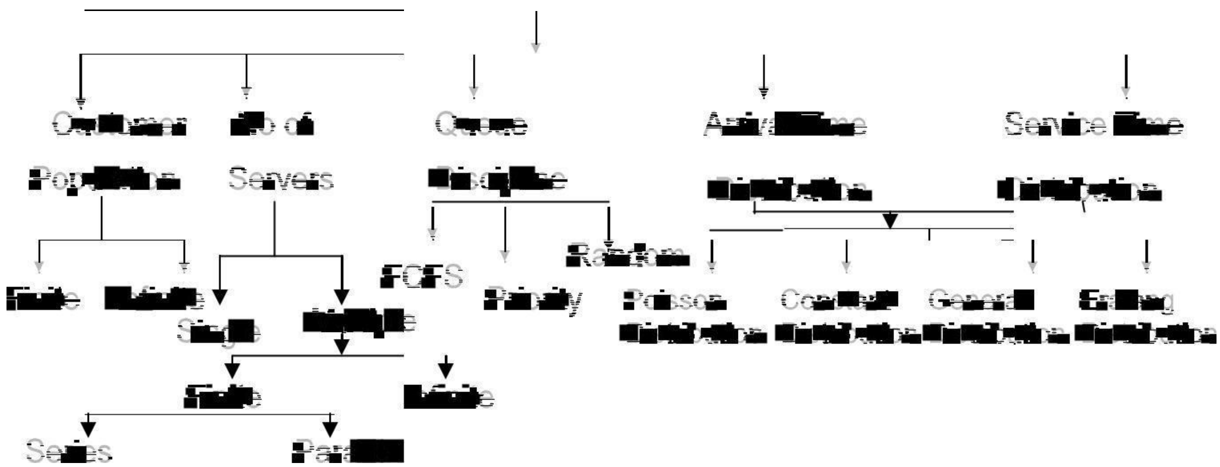
Introduction

Queues are a part of everyday life. We all wait in queues to buy a movie ticket, to make bank deposit, pay for groceries, mail a package, obtain a food in a cafeteria, to have ride in an amusement park and have become adjustment to wait but still get annoyed by unusually long waits.

The Queuing models are very helpful for determining how to operate a queuing system in the most effective way if too much service capacity to operate the system involves excessive costs. The models enable finding an appropriate balance between the cost of service and the amount of waiting.



Information required to solve the queuing problem:



Characteristics of the queuing system-:

(a) Input source

(b) Queue discipline

(c) Service mechanism

(a) Input source

One characteristic of the inp ut source is the size. The size is the total number of units that might require service from time to time. It may be assumed to be finite or infiniite.

The customer assumption is that they generate according to ‘Poisson Distribution’ at a certain average rate

Therefore, the equivalent assumption is that they generate according to exponential distribution between consecutive arrivals. To solve the problems use & assume customer population as ∞

(b) Queue Discipline

A queue is characterized by maximum permissible number of units that it contains. Queues are called finite or infinite, according to whether number is finite or infinite. The service discipline refers to the order in which number of queues are selected for service.

Ex: It may be FIFO, random or priority; FIFO is usually assumed unless stated otherwise.

(c) Service mechanism

This consists of one or more service facilities each of which contains one or more parallel service channel. If there is more than one service facility, the arrival unit may receive the service from a sequence of service channels.

At a given facility, the arrival enters at the service facility and is completely served by that server. The time elapsed from the commencement of the service to its completion for an unit at the service facility is known as service time usually, service time follows as exponential distribution.

Classification of queuing models using kendal & Lee notations

Generally, any queuing models may be completely specified in the following symbolic form a / b/ c : d / e

a $\rightarrow$ Type of distribution of inter – arrival time b

$\rightarrow$ Type of distribution of inter – service time

c $\rightarrow$ Number of servers

d $\rightarrow$ Capacity of the system

e $\rightarrow$ Queue discipline

M $\rightarrow$ Arrival time follows Poisson distribution and service time follows an exponential distribution.

Model I : M / M / 1 : / FCFS

Where M Arrival time follows a Poisson distribution

M → Service time follows a exponential distribution

1 → Single service model

□ → Capacity of the system is infinite

FCFS → Queue discipline is first come first served

Model II : M / M / 1 : N / FCFS

Where N → Capacity of the system is finite

Model III : M / M / 1 : / SIRO

Where SIRO → Service in random order

Model IV : M / O / 1 : / FCFS

Where D → Service time follows a constant distribution or is deterministic

Model V : M / G / 1 : / FCFS

Where G → Service time follows a general distribution or arbitrary distribution

Model VI : M / E_k / 1 : / FCFS

Where E_k → Service time follows Erlang distribution with K phases.

Model VII : M / M / K : / FCFS

Where K → Multiple Server model

Model VIII : M / M / K : N / FCFS

Model I: M / M /1 : /FCFS

Formulas: 1. Utilization factor traffic intensity /

Utilization parameter / Busy period

$$\rho = \frac{\lambda}{\mu}$$

Where λ = Mean arrival rate ; μ= mean service rate

Note : μ > λ in single server model only

2. Probability that exactly zero units are in the system

$$P_o = 1 - \frac{\lambda}{\mu}$$

3. Probability that exactly 'n' units in the system

$$P_n = P_o \left(\frac{\lambda}{\mu}\right)^n$$

4. Probability that n or more units in the system

$$P_{n \text{ or more}} = \left(\frac{\lambda}{\mu}\right)^n$$

more then 'n' means n should be n+1

5. Expected number of units in the queue / queue length

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

6. Expected waiting time in the queue

$$W_q = \frac{L_q}{\lambda}$$

7. Expected number of units in the system

$$L = L_q + \frac{\lambda}{\mu}$$

8. Expected waiting time in the system

$$W = W_q + \frac{1}{\mu}$$

9. Expected number of units in queue that from time to time – (OR) non - empty queue size

$$D = \frac{\mu}{\mu - \lambda}$$

10. Probability that an arrival will have to wait in the queue for service

Probability = 1 - P_o

11. Probability that an arrival will have to wait in the queue more than w (where w > o), the waiting time in the queue

$$\text{Probability} = \frac{\lambda}{\mu} e^{(\lambda - \mu) w}$$

12. Probability that an arrival will have to wait more than v(v > o) waiting time in the system is

$$= e^{(\lambda - \mu) v}$$

13. Probability that an arrival will not have to wait in the queue for service = P_o

Model 1 - Problems

1. Arrivals at a telephone both are considered to be Poisson at an average time of 8 min between our arrival and the next. The length of the phone call is distributed exponentially, with a mean of 4 min.

Determine

(a) Expected fraction of the day that the phone will be in use.

(b) Expected number of units in the queue Expected waiting time in the queue.

(c) Expected number of units in the system.

(e) Expected waiting time in the system

(f) Expected number of units in queue that from time to time.

(g) What is the probability that an arrival will have to wait in queue for service?

(h) What is the probability that exactly 3 units are in system

(i) What is the probability that an arrival will not have to wait in queue for service?

(j) What is the probability that there are 3 or more units in the system?

(k) What is the probability that an arrival will have to wait more than 6 min in queue for service?

(l) What is the probability that more than 5 units in system

(m) What is the probability that an arrival will have to wait more than 8 min in system?

(n) Telephone company will install a second booth when convinced that an arrival would have to wait for attest 6 min in queue for phone. By how much the flow of arrival is increased in order to justify a second booth.

Solution:

The mean arrival rate = λ = 1/8 x 60 = 7.5 / hour.

The mean service = μ = x 60 = 15 / hour.

a) Fraction of the day that the phone will be in use

$$\rho = \frac{\lambda}{\mu} = \frac{7.5}{15} = 0.5$$

(b) The expected number pf units in the queue

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{7.5^2}{15(15 - 7.5)}$$

$$L_q = 0.5\ (units)\ person$$

(c) Expected waiting time in the queue

$$W_q = \frac{L_q}{\lambda}$$

$$= \frac{0.5}{7.5} = 0.066\ hrs$$

(d) Expected number of units in the system:-

L= L_q+ λ / μ

= 0.5+0.5

L = 1 person

(e) Expected waiting time in the system

$$W = Wq + \frac{1}{\mu}$$

$$= 0.066 + \frac{1}{15} = 0.133$$

(f) Expected number of units in the queue that form from time to time:-

$$D = \frac{\mu}{\mu - \lambda}$$

$$= \frac{15}{15 - 7.5} = 2\ persons$$

(g) Probability that an arrival will have to wait in the system:-

$$P_{ro} = 1 - P_o$$

$$P_o = 1 - \frac{\lambda}{\mu}$$

$$= 1 - 1 - \frac{\lambda}{\mu}$$

$$P_{ro} = \frac{\lambda}{\mu} = \frac{7.5}{15} = 0.5$$

(h) The Probability that exactly zero waits in the system:-

$$P_o = 1 - \frac{\lambda}{\mu}$$

$$= 1 - 0.5 = 0.5$$

(i) The probability that exactly 3 units in the system:-

$$P_{no} = P \left(\frac{\lambda}{\mu}\right)^n \qquad n = 3$$

$$P_3 = 0.5(0.5)^3 = 0.0625$$

(j) Probability that an arrival will not have to wait for service:-

$$P_o = 1 - \frac{\lambda}{\mu}$$

$$= 0.5$$

(k) Probability that 3 or more units in the system:-

$$P_{n\ or\ more} = \left(\frac{\lambda}{\mu}\right)^n \qquad n = 3$$

$$P_{n\ or\ more} = 0.5^3 = 0.125$$

(l) Probability that an arrival will have to wait more than 6mins in queue for service

$$P_{ro} = \frac{\lambda}{\mu} e^{(\lambda - \mu)\omega}$$

$$\omega = 6 \min = \frac{6}{60} hrs$$

$$P_{ro} = 0.5\, e^{(7.5-15)\frac{6}{60}}$$

$$P_{ro} = 0.236$$

(m) Probability that more than 5 units in the system

$$P_{ro} = \left(\frac{\lambda}{\mu}\right)^n \qquad n = 6$$

$$P_{ro} = 0.5^6 = 0.015$$

(n) Probability that an arrival will directly enter for service P_o= 0.5

(O) Probability that arrival will have to wait more than 8mins in the system. V = 8 / 60 hrs

$$P_{ro} = e^{(\lambda \square \mu)v}$$

$$= e^{(7.5 \square 15)\frac{8}{60}}$$

$$= 0.367$$

(p)

$$W_q = \frac{6}{60} hrs = 0.1\, hr$$

$$4 \quad W_q = \frac{L_q}{\lambda} = \frac{\lambda^2}{\mu(\mu \square \lambda)\lambda}$$

$$0.1 = \frac{\lambda}{15(15 \square \lambda)}$$

$$4 \quad \lambda = 9 \text{ per hour}.$$

To justify a second booth should be increased from 7.5 to 9 per hour

2) In a self service store with one cashier, 8 customers arrive on an average of every 5 mins. and the cashier can serve 10 in 5 mins. If both arrival and service time are exponentially distributed, then determine

a) Average number of customer waiting in the queue for average. (3.2)

b) Expected waiting time in the queue (0.033)

c) What is the probability of having more than 6 customers In the system

(0.209)

Solution:

Mean arrival rate =λ= =1.6 x 60

= 96 / hour

Mean service rate = μ = x 60

= 120 / hour.

(a) Average number of customers waiting in queue for service

$$L_q = \frac{\lambda^2}{\mu(\mu \square \lambda)} = \frac{96^2}{120 \ (120 \square 96)}$$

$$L_q = 3.2\, customers$$

(b) Expected waiting time in the queue

$$W_q = \frac{L_q}{\lambda} = \frac{3.2}{96} = 0.033$$

(c) Probability of having more than 6 customers in the system

$$P_{6\,or\,more} = \frac{\lambda}{\mu}^{n} \qquad where\ n = 7$$

$$= \frac{96}{120}^{7} = 0.209$$

3) Consider a box office ticket window being manned by a single server. Customer arrives to purchase ticket according to Poisson input process with a mean rate of 30/hr. the time required to serve a customer has an ED with a mean of 90 seconds determine:

(a) Mean queue length. (2.25)

(b) Mean waiting time in the system. (0.1)

(c) The probability of the customer waiting in the queue for more than 10min. (0.1416)

(d) The fraction (0.75)

The mean arrival rate = $\lambda = 30/hr$

The mean service rate $\mu = \frac{1}{90} \cdot 60 \cdot 60$

$= 40/hr$

(a) Mean queue length

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{30^2}{40(40-30)} = 2.25 \; customers$$

(b) Mean waiting time in the system

$$W = W_q + \frac{1}{\mu}$$

$$= \frac{L_q}{\lambda} + \frac{1}{\mu}$$

$$= \frac{2.25}{30} + \frac{1}{40}$$

$$= 0.1hr$$

(c) Probability of the customer waiting in queue for more than 10min.

$$W\frac{10}{60} = 1/6\,hour$$

$$P_{ro} = \frac{\lambda e^{(\lambda-\mu)w}}{\mu}$$

$$= \frac{30e^{(30-40)1/6}}{40}$$

$$P_{ro} = 0.1416$$

(d) Fraction of time the serve is busy

$$\rho = \frac{\lambda}{\mu}$$

$$= \frac{40}{30}$$

$$= 0.75hr$$

4) A T.V repairman repair the sets in the order in which they arrive and expects that the time required to repair a set has an ED with mean 30mins. The sets arrive in a Poisson fashion at an average rate of 10/8 hrs a day.

(a) What is the expected idle time / day for the repairman? (0.375x8)

b) How many TV sets will be there awaiting for the repair? (1.04)

Solution

Mean arrival rate = λ = $\frac{10}{8}$ hours

Mean service rate = μ = $\frac{1}{30}$ x 60 = 2 hours

(a) Expected idle time / day of the repair

Busy Period = $\frac{\lambda}{\mu} = \frac{1.25}{2}$ = 0.625 hour

$$4\ idle\ time = P_o = 1 - \frac{\lambda}{\mu} = 1 - 0.625 = 0.375$$

$$4\ idle\ time / day = 0.375 \cdot 8 = 3\ hrs / day$$

(b) Number of T.V sets awaiting for the repair:-

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{1.25^2}{2(2 - 1.25)} = 1.04$$

5) In a bank there is only on window. A solitary employee performs all the service required and the window remains continuously open from 7am to 1pm. It has discovered that an average number of clients is 54 during the day and the average service time is 5mins / person. Find

a) Average number of clients in the system (3)
b) Average waiting time (0.25)
c) The probability that a client has to spend more than 10mins in a system. (0.60)

Solution

The mean arrival rate = $\lambda = \dfrac{54}{6} = 9 \; clients / hour$

The mean service rate $= \dfrac{1}{5} \cdot 60 = 12 \; clients / hour$

(a) Average number of customer in the system

$$L = L_q + \frac{\lambda}{\mu}$$

$$= \frac{\lambda}{\mu(\mu - \lambda)} + \frac{\lambda}{\mu}$$

$$= \frac{9^2}{12(12 - 9)} + \frac{9}{12}$$

$$L = 3 \; clients$$

(b) Average waiting time:-

$$W_q = \frac{L_q}{\lambda} = \frac{\lambda^2}{\mu(\mu - \lambda)\lambda} = \frac{9}{12(12 - 9)} = 0.25$$

(c) Probability that a customer has to spend more than 10min in a system.

$$\nabla = \frac{10}{60} = 1/6 hr$$

$$P_{ro} = e^{(\lambda - \mu)v} = e^{(9 - 12) 1/6} = 0.606$$

6) A departmental Secretary receive an average of 8 job / hr. many are short jobs, while other are quiet long. Assume however, that the time to perform a job has an ED mean of 6mins determine

a) The average elapsed time from the time the secretary receives a job, until it is completed. (0.5)

b) Average number of jobs in a system (4)

c) The probability that the time in the system is greater than ½hr. (0.3621)

d) Probability of more than 5 jobs in the system. (0.2628)

Solution

Mean arrival rate = λ = 8 jobs / hrs

Mean service rate = μ = x 60

= 10 jobs / hrs.

(a) Average elapsed time from the time the secretary receives a job on till it is completed

$$W = Wq + \frac{1}{\mu}$$

$$= \frac{Lq}{\lambda} + \frac{1}{\mu}$$

$$= \frac{\lambda^2}{\mu(\mu - \lambda)\lambda} + \frac{1}{\mu}$$

$$= \frac{8}{10(10 - 8)} + \frac{1}{10} = 0.5$$

(b) Average number of jobs in the system:-

$$L = L_q + \frac{\lambda}{\mu}$$

$$= \frac{\lambda^2}{\mu(\mu - \lambda)} + \frac{\lambda}{\mu}$$

$$= \frac{8^2}{10(10-8)} + \frac{8}{10}$$

$$L = 4\ jobs$$

(c) Probability that the customer spends time in the system is greater than ½ hr.

$$v = 0.5\ hr$$

$$P_{ro} = e^{(\lambda - \mu)v}$$

$$= e^{(8-10)0.5}$$

$$= 0.367$$

(d) Probability of more than 5 jobs in the system:-

$$P_{ro} = \left(\frac{\lambda}{\mu}\right)^n \qquad n = 6$$

$$P_{ro} = \left(\frac{8}{10}\right)^6 = 0.262$$

7) At public telephone booth in a post – office arrivals are considered to be Poisson fashion with an average inter arrival time of 12mins. The length of the phone call is ED with a mean of 4mins. Determine:

(a) The probability that the fresh arrival will not have to wait for the phone. (0.66)

(b) What is the probability that the an arrival will have to wait for more than 10mins before the phone is free (0.0629)

(c) What is the average length of the queue that forms from time to time (1.5)

Solution:

Mean arrival rate = λ = $\frac{1}{12}$ x 60 = 5 / hr

Mean service rate = μ = $\frac{1}{4}$ x 60 = 15 / hr

(a) Probability that fresh arrivals will not have to wait for the phone:

$$P_{ro} = P_o$$

$$= 1 \square \frac{\lambda}{\mu}$$

$$= 1 \square \frac{5}{15}$$

$$= 0.66$$

(b) Probability that an arrival will have to wait more than 10 min before the phone is free :

$$W = \frac{10}{60} = \frac{1}{6} hr$$

$$P_{ro} = \frac{\lambda}{\mu} e^{(\lambda \square \mu)\omega}$$

$$= \frac{5}{15} e^{(5 \square 15)\frac{1}{6}}$$

$$= 0.629$$

(c) Average length of the queue that form from time to time:

$$D = \frac{\mu}{\mu \square \lambda} = \frac{15}{15 \square 5}$$
$$= 1.5$$

8) There is congestion on the platform of a railway station. The trains arrive at a rate of 30/days. The service time for any train is ED with an average of 36mins. Calculate:

(a) Mean queue size (2.25)

(b) Probability that there are more than 10 trains in the system. (0.0422)

Solution

Mean arrival rate $=\lambda = 30 / days$

$$= \frac{30}{24} = 1.25 / hr$$

Mean service rate $= \mu = \frac{1}{36} \cdot 60 = 1.66 / hr$

(a) Mean queue size:-

$$L_q = \frac{\lambda^2}{\mu(\mu \square \lambda)} = \frac{1.25^2}{1.66(1.66 \square 1.25)}$$

$$L_q = 2.295 \; per \; hr$$

(b) Probability that than 10 trains in the system n = 11

$$P_{ro} = \left(\frac{\lambda}{\mu}\right)^n = \left(\frac{1.25}{1.66}\right)^{11}$$
$$= 0.0422$$

9) The arrival rate for a waiting line system obeys a P.D with a mean of 0.5 units/hr. it is required that the probability of one or more units in the system does not exceed 0.25. what is the minimum service rate that must be provided if the service duration will be distributed exponentially? (2/hr)

Solution

$\lambda = 0.5\ units / hr$

$$P_{n\ or\ more} = 0.25 = \left(\frac{\lambda}{\mu}\right)^n$$

$n = 1$

$4\ \frac{0.5}{\mu} = 0.25 \qquad 4\ \mu = 2 / hr$

10) In a municipality hospital patients arrival are considered to be Poisson with an arrival interval time of 10mins. The doctors (examination and dispensing) time many be assumed to be ED with an average of 6mins find :

a) What is the chance that a new patient directly sees the doctor? (0.4)

b) For what proportion of the time the doctor is busy? (0.6)

c) What is the average number of patients in the system? (1.5)

d) What is the average waiting time of the system? (1.5)

e) Suppose the municipality wants to recruit another doctor, when an average waiting time of an arrival is 30mins in the queue. Find out hose large should be to justify a 2nd doctor? (λ = 8.33)

Solution

$$\lambda = \frac{1}{10} \cdot 60 = 6 / hr$$

$$\mu = \frac{1}{6} \cdot 60 = 10 / hr$$

(a) Probability that a new patient straight away sees the doctor:-

$$P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{6}{10} = 0.4$$

(b) Proportion of time the doctor is busy:-

$$\rho = \frac{\lambda}{\mu} = \frac{6}{10} = 0.6hr$$

(c) Average number of patients in the system

$$L = L_q + \frac{\lambda}{\mu}$$

$$= \frac{\lambda^2}{\mu(\mu - \lambda)} + \frac{\lambda}{\mu} = \frac{6^2}{10(10-6)} + \frac{6}{10} \qquad L = 1.5$$

(d) Average waiting in the system:-

$$W = W_q + \frac{1}{\mu} = \frac{L_q}{\lambda} + \frac{1}{\mu}$$

$$= \frac{\lambda^2}{\mu(\mu - \lambda)\lambda} + \frac{1}{\mu}$$

$$= \frac{6}{10(10-6)} + \frac{1}{10} \qquad W = 0.25$$

$$W_q = \frac{30}{60} = 05hr$$

$$W_q = \frac{L_q}{\lambda} = \frac{\lambda^2}{\mu(\mu \square \lambda)\lambda} = 0.5$$

(e)

$$= \frac{\square\lambda}{10(10 \square \lambda)\square 6} = 0.5 \quad 4 \quad \lambda = \frac{50}{} = 8.33 / hr$$

The value of λ has to be increased from 6 to 8.33 justify a second doctor.

11) At a one man barber shop customers arrive according to P.D with a mean arrival rate of 5/hr. The hair cutting time is ED with a hair cut taking 10 min on an average assuming that the customers are always willing to wait find:

a) Average number of customer in the shop [5]

b) Average waiting time of a customer [0.833]

c) The percent of time an arrival Can walkright with out having to wait [16.66%]

d) The probability of a customer waiting more than 5mins [0.766]

Solution

$$\lambda = 5 / hr$$

Mean service rate= $\mu = \frac{1}{10} \cdot 60$

$$= 6 / hr$$

(a) Average number of customer's in the shop.

$$L = L_q + \frac{\lambda}{\mu}$$

$$= \frac{\lambda^2}{\mu(\mu - \lambda)} + \frac{\lambda}{\mu}$$

$$= \frac{5^2}{6(6 - 5)} + \frac{5}{6}$$

L= 5 customers

(b) Average waiting time of a customer.

$$W_q = \frac{L_q}{\lambda}$$

$$= \frac{\lambda^2}{\mu(\mu - \lambda)\lambda}$$

$$= \frac{5}{6(6 - 5)} = 0.833hr$$

(c) Percent of time arrival can walk right without having to wait.

$$p_o = 1 - \frac{\lambda}{q} \cdot 100$$

$$= 1 - \frac{5}{6} \cdot 100$$

$$= 16.66\%$$

d) Probability of a customer waiting more than 5mins.

$$W = \frac{5}{60} = 1/12$$

$$\therefore P_{ro} = \frac{\lambda}{\mu} e^{(\lambda - \mu)w}$$

$$= \frac{5}{6} e^{(5-6)1/2}$$

$$\therefore P_{ro} = 0.766$$

12) At a stamp vender window of a post office 20 customers arrive on an average every 10 min. the vender clerk can serve 5 customers in 2 min. Determine

a) Average number of customer in the System [4]

b) Average waiting time of a customer [0.0266]

c) Probability of a customer waiting more than 3mins before being served[0.1785]

d) Idle time of the vender clerk in a shift of 8hrs [1.6]

Solution

$$\lambda = \frac{20}{10} \cdot 60 = 120 / hr$$

$$\mu = \frac{5}{2} = 2.5 / \text{min} = 2.5 \cdot 60 = 150 / hr$$

(a) Average number of customers in the system:-

$$L = L_q + \frac{\lambda}{\mu} = \frac{\lambda^2}{\mu(\mu - \lambda)} + \frac{\lambda}{\mu}$$

$$= \frac{120^2}{150(150 - 120)} + \frac{120}{150} = 4 \; customers$$

(b) Average waiting time of a customer:-.

$$W_q = \frac{L_q}{\lambda} = \frac{3.2}{120} = 0.028hr$$

(e) Probability of a customer waiting more than 3mins before being received.

$$W = \frac{3}{60} = 1/20$$

$$P_{ro} = \frac{\lambda}{\mu} e^{(\lambda - \mu)w} = \frac{120}{150} e^{(120-150)1/20} \qquad P_{ro} = 0.1785$$

(d) Idle time of the vendor clerk in a shift of 8hours.

$$= P_o \cdot 8$$

$$= \left(1 - \frac{\lambda}{\mu}\right) \cdot 8$$

$$= \left(1 - \frac{120}{150}\right) \cdot 8$$

$$= 1.6hr$$

13) Arrivals of machinist at a tool crib are considered to be P.D at an average rate of 6/hr. the length of time the machinist must remain at the tool crib is ED with an average time being 0,05 hrs.

a) what is the problem that the machinist arriving at the tool crib will have to wait? [0.3]

b) What is the average number of machinist at the tool crib? [0.428]

c) The company will install a 2nd tool crib when convinced that a machinist would expect to spent at least 6mins waiting and being served at the tool crib. By how much the flow of machinist to the tool crib increase to justify the 2nd tool crib.

[x = 10]

Solution

$$\lambda = 6 / hr$$

$$\mu = \frac{1}{0.05} = 20 / hr$$

(a) Probability that the machinist arriving at the tool crib will have to wait.

$$= 1 - P_o$$

$$= \frac{\lambda}{\mu} = \frac{6}{20} = 0.3$$

(b) Average number of machinist at the tool crib.

$$L = L_q + \frac{\lambda}{\mu}$$

$$= \frac{\lambda^2}{\mu(\mu - \lambda)} + \frac{\lambda}{\mu}$$

$$= \frac{6^2}{20(20 - 6)} + 0.3$$

$$= 0.42$$

(C)

$$W = \frac{6}{60} = 0.1hr$$

$$0.1 = \quad W = \frac{\lambda}{\mu(\mu - \lambda)} + \frac{1}{\mu}$$

$$0.1 = \frac{\lambda}{20(20 - \lambda)} + \frac{1}{20}$$

$$4\ \lambda = 10 / hr$$

14) Jobs arrive at an inspection station according to Poisson process at a mean rate of 2/hr and are inspect one at a time on a FIFO basis. The quality control engineer both inspects and makes minor adjustments. The total service time for the job appears to be ED with a mean of 25mins. Jobs that arrive but cannot be inspected immediately by the engineer must be stored until the engineer is free to take them. Each job requires 1 sq mts space determine

a) The waiting line length [4.16]
b) The waiting time [2.08]
c) % of idle time of the engineer [16.66%]
d) The floor space to be provided in the quality control room. [5]

Solution

$\lambda = 2 / hr$

$\mu = \frac{1}{25} \cdot 60 = 2.4 / hr$

(a) $$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{2^2}{2.4(2.4 - 2)} = 4.16$$

(b) $$W_q = \frac{L_q}{\lambda} = \frac{4.16}{2} = 2.08$$

c) Idle time of the engineer:-

$$P_o = 1 - \frac{\lambda}{\mu} \cdot 100 = 16.66\ \%$$

(d) Floor space to be provided in the quality control room

$$L = L_q + \frac{\lambda}{\mu} = 4.16 + \frac{2}{2.4}$$

$$L = 4.993$$

$$L \approx 5 \cdot 1m^2$$

15) The arrival of aircraft at an international tends to follow a Poisson fashion, in spite of schedule flight time, due to high operating variability in the schedule time. It can be assumed that the aircraft arrives at an average rate of 6/hr. The landing service is provided through a single runway by a control tower according to ED with an average service time of 6mins/flight:

(a) Find the prob. that will more than 10mins all together to wait for landing and to land an air-craft. [0.513]

(b) What is prob. that the runway will be free for an incoming flight? [0.4]

Solution

$$\lambda = 6/hr$$

$$\mu = \frac{1}{6} \cdot 60 = 10/hr$$

$$V = \frac{10}{60} = \frac{1}{6}$$

$$P_{ro} = e^{(\lambda - \mu)v}$$

(a) $$= e^{(6-10)1/6}$$

$$= 0.513$$

b) Probability that the runway will be free for an incoming flight.

$$P_o = 1 - \frac{\lambda}{\mu}$$

$$= 1 - \frac{6}{10}$$

$$P_o = 0.4$$

16) At what rate must the clerk of a super market work in order to ensure a prob. Of 0.9 that the Customer will not have to wait longer than 12mins in the system. It is assumed that the arrivals follows a Poisson fashion at the rate of 15/hr. The length of service by the clerk has an ED.

(a) Also find the average number of customers queuing for service. [0.738]

(b) The Prob. of having more than 10 customers in the system [1.9×10^{-3}]

Solution:

$$v = \frac{12}{60} = \frac{1}{5} = 0.2$$

$$\lambda = 15 / hr$$

$$P_{ro} = e^{(\lambda - \mu)}$$

$$1 - 0.9 = e^{(15 - \mu)0.2}$$

$$0.1 = e^{(15 - \mu)}$$

4 $\mu \; 0.1 = (15 - \mu)0.2$

4 $\mu = 26.5$

(a) Average number of customer’s queuing for service

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

$$= \frac{15^2}{26.5(26.5 - 15)}$$

$$L_q = 0.738$$

(b) Probability of having more than 10 customers in the system: $n = 11$

$$P_{ro} = \left(\frac{\lambda}{\mu}\right)^n$$

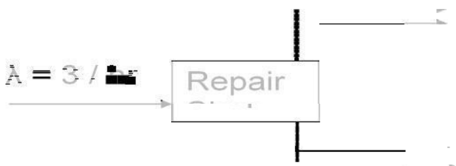
$$P_{ro} = \left(\frac{15}{26.5}\right)^{11}$$

$$P_{ro} = 0.00191$$

17) A mechanic is to hired to repair a machine which breaks down at an average rate of 3/hr. breakdowns are distributed in time in a manner that may be regarded as Poisson. The non-productive time on any machine is considered to cost the Company. Rs. 5/hr. The Company has the choice to 2 mechanics A & B.

The mechanic A repairs the machines at an average rate of 4/hr and he will demand Rs. 3/hr. The mechanic B costs Rs. 5/hr and can repair the machines exponentially at a n average rate of 6/hr. Decide which mechanic should be hired.

Solution



$Mechanic \rightarrow A$

$\mu = 4 / hr \rightarrow \cos t\ Rs\ 3 / hr$

$Mechanic \rightarrow B$

$\mu = 6 / hr \rightarrow \cos t\ Rs\ 5 / hr$

Consider the mechanic A

$\lambda = 3$ / hr, $\mu = 4$ / hr

The number of break down machine in the system

$$L = L_q + \frac{\lambda}{\mu}$$

$$= \frac{\lambda^2}{\mu(\mu - \lambda)} + \frac{\lambda}{\mu}$$

$$= \frac{3^2}{4(4-3)} + \frac{3}{4}$$

$$L = 3\ m/c's$$

The non-productive time the company / hr = 3 x 5 =15 Rs

The amount paid to the mecha nic A per hour = 3 Rs

The total expected cost per hour = 15+3 = 18 Rs

Consider the mechanic B

λ = 3 / hr, μ = 6 / hr

The number of break down machine in the system

$$L = L_q + \frac{\lambda}{\mu}$$

$$= \frac{\lambda^2}{\mu(\mu - \lambda)} + \frac{\lambda}{\mu}$$

$$= \frac{3^2}{6(6 - 3)} + \frac{3}{6}$$

$$L = 1\ m/c's$$

The non-productive time cost of the company / hr = 1 x 5 =
5 Rs Amount paid to the mechanic / hr = 5

Total expected cost per hour = 5+5 = 10 Rs

Selected mechanic B as the total expected cost per hour of mechanic B is less than the total expected cost per hour of mechanic A.

QUEUEING THEORY PROBLEMS : MODEL – 2

Multiple server Model

M / M / K : ∞ / FCFS

Formulae

Multiple server Model

(1)

$$P_o = \frac{1}{\sum_{n=0}^{K-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{K!}\left(\frac{\lambda}{\mu}\right)^K \frac{k\mu}{(k\mu - \lambda)}}$$

(2)

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n . P_o$$

(3)

$$L_q = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^K}{(k-1)!(k\mu - \lambda)^2} \cdot P_o$$

(4)

$$L = L_q + \frac{\lambda}{\mu}$$

(5)

$$W_q = \frac{L_q}{\lambda}$$

(6) $W = Wq + \frac{1}{\mu}$

(7) $\frac{\lambda}{k\mu}$

1) A Commercial bank has 3 cash paying assistants customers are found to arrive in a Poisson fashion at an average rate of 6/hr for business transaction. The service time is found to have an with a mean of 18 mins. The customers are processed on FCFS basis. Calculate

a) Average number of customers in the system

b) Average time a customer spends in the system

c) Average queue length

d) How many hours a week can a cash paying assistant spend with the customers.

Solution:

$K = 3$

$\lambda = 6 / hr$

$\mu = \frac{1}{18} \cdot 60 = 3.33 / hr$

$$P_o = \frac{1}{\sum_{n=0}^{K-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{K!}\left(\frac{\lambda}{\mu}\right)^K \frac{k\mu}{k\mu - \lambda}}$$

$$= \frac{1}{\frac{1}{0!}\left(\frac{6}{3.33}\right)^0 + \frac{1}{1!}\left(\frac{6}{3.33}\right)^1 + \frac{1}{2!}\left(\frac{6}{3.33}\right)^2 + \frac{1}{3!}\left(\frac{6}{3.33}\right)^3 \frac{3 \cdot 3.33}{3 \cdot 3.33 - 6}}$$

$P_o = 0.145$

(a) Average number customers in the system:-

$$L_q = \frac{\lambda \mu \left(\frac{\lambda}{\mu}\right)^K}{(k-1)!\,(k\mu - \lambda)^2} \cdot P_o$$

$$= \frac{6 \cdot 3.33 \left(\frac{6}{3.33}\right)^3}{2!\,(3 \cdot 3.33 - 6)^2} \cdot 0.145$$

$$L_q = 0.532$$

$$4\ L = L_q + \frac{\lambda}{\mu}$$

$$= 0.532 + \frac{6}{3.33}$$

$$L = 2.334$$

(b) Average time a customer spends in the system

$$W = Wq + \frac{1}{\mu} = \frac{Lq}{\lambda} + \frac{1}{\mu}$$

$$= \frac{0.532}{6} + \frac{1}{3.33} = 0.388$$

(c) Average queue length

$$Lq = 0.532$$

(d) Assuming 5 days a week and 8 hrs a day the number of hrs in a week the cash paying assistant spends with the customers

$$= \rho \cdot 5 \cdot 8$$

$$= \frac{\lambda}{k\mu} \cdot 5 \cdot 8$$

$$= \frac{6}{3 \cdot 3.33} \cdot 5 \cdot 8$$

$$= 24.02 \; hrs$$

(2) A telephone exchange has two long distant operators. The telephone company finds that during the fashion at an average rate of 15/hr. The length mean of 5 mins.

(a) what is the probability that a subscriber will have to wait for his long distant dials on the peak hour of the day.

(b) what is the average waiting time for the customers.

Solution:

$K = 2$

$\lambda = 15 / hr$

$$\mu = \frac{1}{5} \cdot 60 = 12 / hr$$

$$P_o = \frac{1}{\frac{1}{0!}\left(\frac{15}{12}\right)^0 + \frac{1}{1!}\left(\frac{15}{12}\right)^1 + \frac{1}{2!}\left(\frac{15}{12}\right)^2 \frac{2 \cdot 12}{2 \cdot 12 - 15}}$$

$$P_o = 0.230$$

(a) Probability that a subscriber will have to wait for his long distant call is

$$P_{ro} = 1 \square P_o \square P_1$$

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n .P_o$$

$$P_1 = \frac{1}{1!}\left(\frac{15}{12}\right)^1 \cdot 0.230$$

$$P_1 = 0.2875$$

$$4 = 1 \square 0.230 \square 0.2875$$

$$P_{ro} = 0.48$$

(b) average waiting time for the customer

$$W_q = \frac{L_q}{\lambda}$$

$$= \frac{\lambda \mu \left(\frac{\lambda}{\mu}\right)^k}{\lambda (k \square 1)!(k\mu \square \lambda)^2} \cdot P_o$$

$$= \frac{12 \left(\frac{15}{12}\right)^2 \cdot 0.23}{\square 12 \; 1!(24 \square 15)^2} = 0.053 \; hr$$

3) An insurance company has 3 clerks in its branch office. People arrive with claims against the company are found to arrive in a Poisson fashion at an average of 20 per 8 hours a day. The amount of time that a clerk spends with the client is found to have ED with a mean time of 40 mins. The clients are processed in the order of their appearance.

(a) How many hours a week can a clerk expect to spend with the clients.

(b) How much time an average does a client spend in the branch office.

Solution:

$$K = 3$$

$$\lambda = \frac{20}{8} = 2.5 / hr$$

$$\mu = \frac{1}{40} \cdot 60 = 1.5 / hr$$

$$P_o = \frac{1}{\sum_{n=0}^{K-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{K!}\left(\frac{\lambda}{\mu}\right)^K \frac{k\mu}{k\mu - \lambda}}$$

$$= \frac{1}{\frac{1}{0!}\left(\frac{2.5}{1.5}\right)^0 + \frac{1}{1!}\left(\frac{2.5}{1.5}\right)^1 + \frac{1}{2!}\left(\frac{2.5}{1.5}\right)^2 + \frac{1}{3!}\left(\frac{2.5}{1.5}\right)^3 \frac{3 \cdot 1.5}{3 \cdot 1.5 - 2.5}}$$

$$P_o = 0.173$$

(a) The number of hours per week a clerk expects to spend with the client

= ρ x 5 x 8 assuming 5 days a week and 8 hrs / day

$$= \frac{\lambda}{\mu k} \cdot 5 \cdot 8$$

$$= \frac{2.5}{1.5 \cdot 3} \cdot 5 \cdot 8 = 22.22\, hrs$$

(b) Average time a clerk spends in the branch office

$$W = Wq + \frac{1}{\mu}$$

$$= \frac{Lq}{\lambda} + \frac{1}{\mu}$$

$$= \frac{\dfrac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^k \cdot P_o}{(k-1)!\,(k\mu-\lambda)^2}}{\lambda} + \frac{1}{\mu}$$

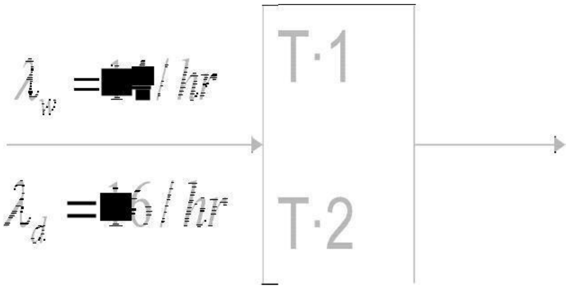
$$= \frac{1.5\left(\frac{2.5}{1.5}\right)^3 \cdot 0.173}{2!\,(3\cdot 1.5 - 2.5)^2} + \frac{1}{1.5}$$

$$= 0.816\ hrs.$$

4) A bank has 2 tellers working on saving accounts. The 1st tellers handles withdrawal's only and the 2nd teller handles deposits only, it has been found that service time distribution for depositors and withdrawal's. Both are E.D with a mean service time of 3min /customer. Depositor are found to arrive in a Poisson fashion with a arrival rate of 16/hr and withdrawal's also drive in a Poisson with a mean rate of 14/hr.

What would be the effect on the average waiting time for the depositors and withdrawal's if each teller would handle both withdrawal's and deposits. What would be the effect if the time would only be accomplished by increasing the service time to 3.5mins.

Solution:



Waiting time in the queue for the depositors

$$Wq_D = \frac{L_q}{\lambda}$$

$$= \frac{\lambda^2}{\lambda\mu(\mu \square \lambda)} = \frac{\square 16}{20(20 \square 16)} = 0.2 hrs$$

waiting time in the queue for the withdrawal's

$$Wq_w = \frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu \square \lambda)}$$

$$= \frac{14}{20(20 \square 14)} = 0.1160 hrs$$

Consider it as a multi server moddel.

K = 2

$$\lambda = \lambda_W + \lambda_D = 30 / hrr$$

$$\mu = 20 / hr$$

$$P_o = \frac{1}{\frac{1}{0!}\left(\frac{30}{20}\right)^0 + \frac{1}{1!}\left(\frac{30}{20}\right)^1 + \frac{1}{2!}\left(\frac{30}{20}\right)^2 \frac{2 \cdot 20}{2 \cdot 20 - 30}}$$

$$P_o = 0.143$$

Waiting time in the queue

$$w_q = \frac{L_q}{\lambda}$$

$$= \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^k \cdot P_o}{\lambda\,(K-1)!\,(K\mu-\lambda)^2} = \frac{20\left(\frac{30}{20}\right)^2 \cdot 0.143}{1!\,(2 \cdot 20 - 30)^2}$$

= 0.064hrs is the waiting time in the queue

If the service time is increased from 3 to 3.5mins

$$P_o = \frac{1}{\frac{1}{0!}\left(\frac{30}{17.14}\right)^0 + \frac{1}{1!}\left(\frac{30}{17.14}\right)^1 + \frac{1}{2!}\left(\frac{30}{17.14}\right)^2 \frac{2 \cdot 17.14}{2 \cdot 17.14 - 30}}$$

$$P_0 = 0.065$$

$$W_q = \frac{17.14\left(\frac{30}{17.14}\right)^2 \cdot 0.065}{1!\,(2 \cdot 17.14 - 30)^2}$$

$$W_q = 0.192hrs$$

By increasing the service time from 3 to 3.5 min the waiting time in the queue for the depositors is decreased from 0.2 to 0.19hrs.

But in the case of withdrawal's the waiting time in the queue increased from 0.116hrs to 0.19hrs

5) A tax consulting firm has 3 counters in its offices to receive the people who have problems concerning their income and the sales tax. On an average 48 persons arrive in 8hrs a day. Each tax advisor spends 15 min an on average for a arrival of the arrival time follows a Poisson distribution and the service time follows a E.D.

(a) Find the average number of customer in the system.

(b) Average waiting time of the customer in the system.

(c) Average number of customers waiting the queue for service.

(d) Average waiting time of the customers in the queue.

(e) How many hours each week a tax advisor spends performing his job.

(f) Probability that a customer has to wait before he gets service.

(g) Expected number of idle tax advisors at any specified time

Solution:

$$K = 3$$

$$\lambda = \frac{48}{8} = 6hrs$$

$$\mu = \frac{1}{15} \cdot 60 \quad 4 / hrs$$

$$P_0 = \frac{1}{\frac{1}{0!}\left(\frac{6}{4}\right)^0 + \frac{1}{1!}\left(\frac{6}{4}\right)^1 + \frac{1}{2!}\left(\frac{6}{4}\right)^2 + \frac{1}{3!}\left(\frac{6}{4}\right)^3 \frac{3 \cdot 4}{3 \cdot 4 - 6}}$$

$$4 P_0 = 0.210$$

a) Average number of customers in the system:

$$L = L_q + \frac{\lambda}{\mu}$$

$$= \frac{\lambda\mu(\lambda/\mu)^K}{(K-1)!(K\mu-\lambda)^2} \cdot P_0 + \frac{\lambda}{\mu}$$

$$= \frac{6 \cdot 4 \left(\frac{6}{4}\right)^3 \cdot 0.21}{2!(3 \cdot 4 - 6)^2} + \frac{6}{4}$$

$$L = 1.73 \; customers$$

b) Average waiting time of the customer in the system.

$$W = W_q + \frac{1}{\mu}$$

$$= \frac{L_q}{\mu} + \frac{1}{\mu}$$

$$= \frac{\frac{\lambda\mu(\lambda/\mu)^K \cdot P_0}{(K-1)!(K\mu-\lambda)^2}}{\lambda} + \frac{1}{\mu}$$

$$= \frac{4\left(\frac{6}{4}\right)^3 \cdot 0.21}{2!(3 \cdot 4 - 6)^2} + \frac{1}{4}$$

$$W = 0.289 \; hour$$

(c) Average number of customer in the queue

$$L_q = \frac{\lambda\mu(\lambda/\mu)^K \cdot P_0}{(K-1)!(3 \cdot 4 - 6)}$$

$$L_q = 0.23$$

(d) Average time of the customers in the queue.

$$W_q = \frac{L_q}{\lambda} = \frac{0.23}{6} = 0.038$$

(e) Assuming 5 days a week and 8 hours per day the number of hours the tax advisors spents with customers during the week.

$$= \rho \cdot 5 \cdot 8$$

$$= \frac{\lambda}{K\mu} \cdot 5 \cdot 8 = \frac{6}{3 \cdot 4} \cdot 5 \cdot 2$$

$$= 5 hours$$

(f) Probability that a customer has to wait before he gets service

$$= 1 \square P_0 \square P_1 \square P_2$$

$$P_1 = \frac{1}{1!} \left(\frac{\lambda}{\mu K}\right)^1 \cdot P_0 = 0.105$$

$$= \frac{1}{11} \left(\frac{6}{3 \cdot 4}\right)^1 \cdot 0.21 = 0.02625$$

$$P_2 = \frac{1}{2!} \left(\frac{6}{3 \cdot 9}\right)^2 \cdot 0.21$$

$$= 0.02625$$

$$41 \square 0.21 \square 0.105 \square 0.02625$$

$$= 0.239$$

g) Expected number of idle tax advisors at any specified time

$$= 3P_0 + 2P_1 + 1P_2 + 0P_3$$

$$= 3\cdot 0.21 + 2 \cdot 0.105 + 1\cdot 0.02625$$

$$= \quad 1.5\ \%$$

Reference Books:

1. **Taha H A**, Operation Research - An Introduction, Prentice Hall of India, 7th edition, 2003
2. **Ravindran, Phillips and S olberg**, Operations Research : Principles and Practice, John Wiely & Sons, 2nd Edition
3. **D.S.Hira**, Operation Research, S.Chand & Company Ltd., New Delhi, 2004

LEARNING OBJECTIVES OF THE MODULE:

By the end of this module we will learn about:

- Projects, their Management and Terminology used
- Construction of project networks
- Project management techniques: CPM and PERT
- CPM - Determination of minimum project duration
  - Flexibility in executing project activities
  - Shortening (Crashing) the project duration
- PERT- Handling probabilistic activity-time estimates

'Project' is not a new word for any of us. We read about new technology development (3G mobiles, vaccine for H1N1), Implementation of new transport system (Metro, Petronet), Quality improvement program (TEQIP), New civil construction (stadium, factory), etc. The common features among these entitle them to be called as Projects.

Definition of Project: A project is a temporary endeavor with a collection of interrelated activities with each activity consuming time and resources. It is designed to achieve a specific and unique outcome and is subject to time, cost and quality constraints.

Project Management (PM):

Proper planning, scheduling, executing and controlling of project activities is required to ensure that the projects are completed within the stipulated time and budget, complying to all quality and safety requirements. A good PM utilizes the resources most effectively.

Several analytical techniques such as PERT, CPM, PEP, RAMPS have been developed to aid real time project management.

The phases that comprise PM can be identified as (i) Project planning, (ii) Project scheduling and (iii) Project executing and Controlling.

Project Planning phase involves :

- Set ting project objectives and scope
- Preparing Work Breakdown Structure (WBS)
- Estimating activity time and resource requirements
- Identifying the interrelationships between activities
- Arranging activities for network analysis

Project Scheduling involves:

- Determining the start and finish times for each activity
- Determining the critical path and flexibility in each activity

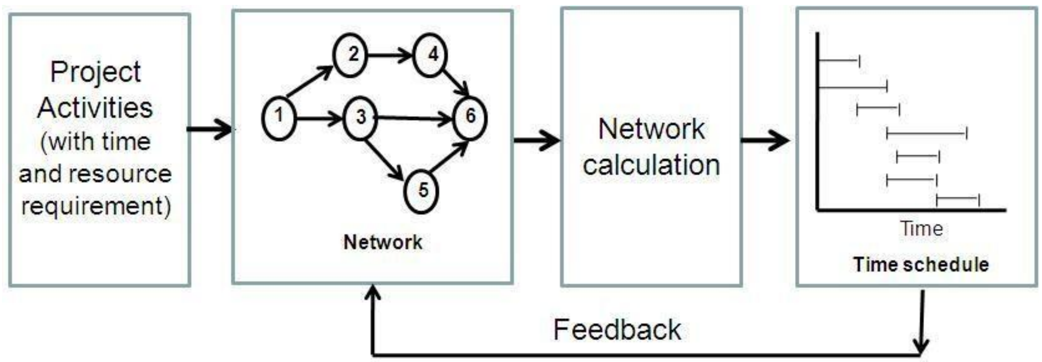
- Communicating schedule using visual aids (Gantt chart) Project

Executing and Controlling involves:

- Carry out the project activities as per plan and schedule
- Periodically evaluate and correct the project progress
- Crash project duration if necessary (resource reallocation)

An effective feedback system should be present to take into account the realities on site and incorporate the changes, if any, into the project plan updating it.

Pictorial representation of Phases of Project Planning and Scheduling



Terms used in network analysis:

WBS: Break down the project into constituent activities such that each activity is clearly identifiable and manageable

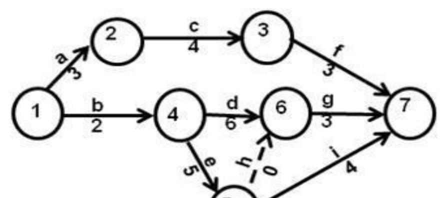
Activity: This is a physically identifiable part of the project that consumes time and resources. It is represented by an arrow (in AOA diagrams)

Events (node): These are the beginning or end points of an activity. Event is a point in time and does not consume any time or resource and is represented by a circle.

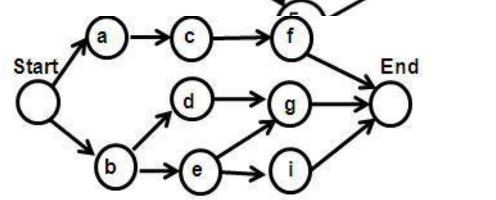
Path: This is a continuous chain of activities from the beginning to the end of the project

Network: A graphical representation of logically and sequentially arranged arrows and nodes of a project. It indicates the interrelationships between the activities of a project

AOA (Activity-On-Arrow) Diagram: A network with activities represented on arrows and event on nodes. Often dummy arrow is needed to establish precedence relationship which makes the network a little cumbersome and requires greater computation. But it is easily understandable.



AON (Activity-On-Node) Diagram: A network with activities represented on Nodes. Arrows indicate only the interdependencies between them. The use of dummy activities can be avoided



Network Construction:

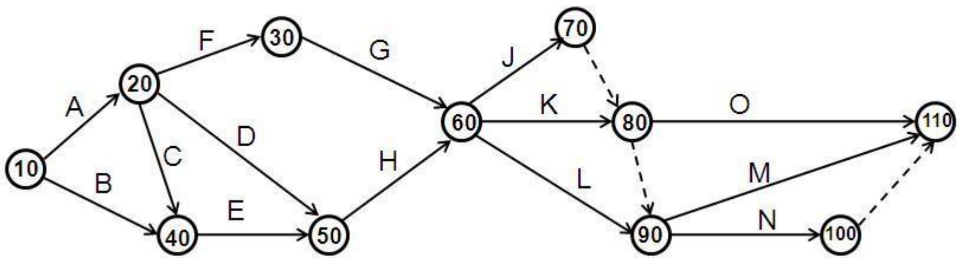
Know the Interrelationships: For a network representation of a project, first we need to know the interrelationships between activities such as (i) Which activities follow a given activity (successors) ?, (ii) Which activities precede a given activity (predecessors) ? and (iii) Which activities can be executed concurrently with a given activity ?

Know the Guidelines for Construction: Then we need to follow the rules given below. (i) Activities progress from left to right (Time flows rightwards), (ii) Each activity is represented by only one straight and solid arrow, (iii) No two activities can be identified by the same end events, (iv) An activity which shows the logical relationship between its immediate predecessor and successor activities, but does not consume time and resources is represented by a dummy activity (dashed-line arrow), (v) Arrows should not cross each other as far as possible and (vi) Avoid Curved arrows, Dangling arrows, and Looping of network

Know to Number the events: (i) The initial event with only outgoing arrows and no incoming arrows is numbered as 1, (ii) Delete all arrows going out of event 1. This will convert in some more events into initial ones. Number them 2, 3, …, (iii) Delete all arrows going out of these events too. This will yield some more initial events. Numbering them further as was done previously and (iv) Continue until the final node is reached which has only incoming arrows and no outgoing arrow.

Problem 1. Draw an economical AOA n/w using the following data

Job	Predecessor	Job	Predecessor	Job	Predecessor
A	-	F	A	L	G, H
B	-	G	F	M	J, K, L
C	A	H	D, E	N	J, K, L
D	A	J	G, H	O	K, J
E	B, C	K	G, H		



Problem 2. (solved in ppt). Represent the following set of activities on a network:

Activity Task	Symbol	Imm. Predecessor
Open the petrol tank cap	A	-
Add petrol (Paid service)	B	A

Close the petrol tank cap	C	B
Open bonnet	D	-
Check and add engine oil (Paid service)	E	D
Check radiator coolant	F	D
Check battery	G	D
Close bonnet	H	E,F,G
Clean the wind shield	I	-
Check air pressure in tires	J	-
Prepare bill	K	B,E
Collect payment	L	K

Critical Path Method (CPM):

CPM was developed by E I du Pont de Nemours & Co. (1957) for construction of new chemical plant and maintenance shut-down. CPM does not incorporate uncertainties in job-times, thus usable for projects with activities having single time estimates, which can be arrived at with prior experience on similar projects. It assumes activity time is proportional to the resources allocated to it (within a certain limit). CPM is mostly suitable for construction type projects. The objective of using CPM is to determine Critical path, Minimum project duration and Floats available with each activity.

Project Evaluation and Review Technique (PERT):

This technique was developed by the U S Navy (1958) for the POLARIS missile program. The project involved coordination of thousands of contractors and agencies. With the help of PERT, the project got completed 2 years ahead of schedule. PERT is suitable for Non-repetitive projects (ex. R & D work), where job-times are not estimable with certainty *a priori.* PERT uses multiple estimates of activity-time (probabilistic nature). The technique emphasizes on the completion of various stages of a project. Jobs that may cause delays are known in advance in terms of their variability.

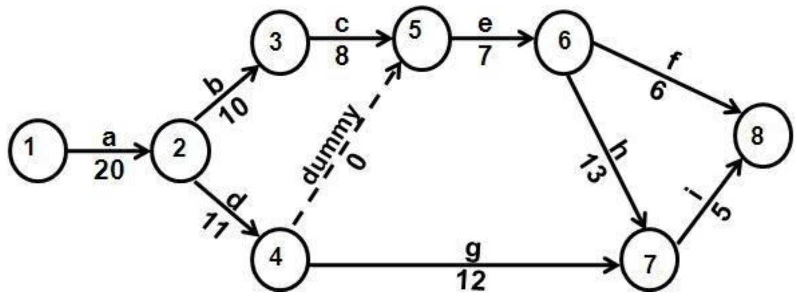
Critical Path Analysis:

Problem 3. Suppose a robot building firm plans the following project. Draw the n/w and find the Critical path

Project Activity		Immediate predecessor	Activity duration in days
a	Design a new robot	-	20
b	Build prototype units	a	10
c	Test prototypes	b	8

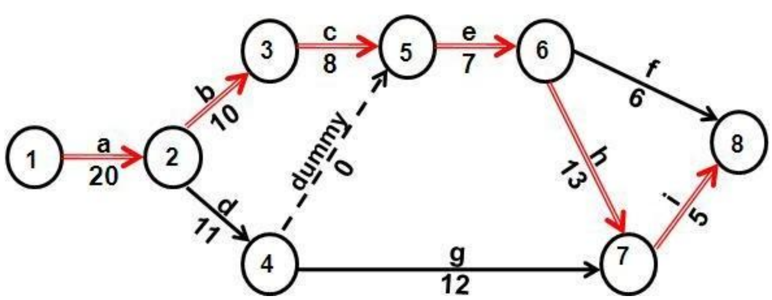
d	Estimate material costs	a	11
e	Refine Robot design	c, d	7
f	Demonstrate Robot to customer	e	6
g	Estimate labor costs	d	12
h	Prepare technical proposal	e	13
i	Deliver proposal to customer	g, h	5

Project Network:



List of all possible sequences (chain/path) of activities (Enumeration method):

Path / Sequence			**Total duration**
a-b-c-e-f	1-2-3-5-6-8	20+10+8+7+6	= 51 days
a-b-c-e-h-i	1-2-3-5-6-7-8	20+10+8+7+13+5	= 63 days (Critical Path)
a-d-e-f	1-2-4-5-6-8	20+11+0+7+6	= 44 days
a-d-e-h-i	1-2-4-5-6-7-8	20+11+0+7+13+5	= 56 days
a-d-g-i	1-2-4-7-8	20+11+12+5	= 48 days



Problem 4. (solved in ppt). A small maintenance project consists of the following 10 jobs whose precedence relationships are identified by their node numbers. Draw an AOA diagram and identify the CP by enumeration method.

Initial Node	Final Node	Duration (days)
1	2	2
2	3	3
2	4	5
3	5	4
3	6	1
4	6	6
4	7	2
5	8	8
6	8	7
7	8	4

Limitations of enumeration method: (i) Very difficult to use when the complexity of network increases, (ii) No information on flexibility available with respect to activities, (iii) Difficult to schedule activities in complex networks

Hence, we use a Structured method for Network analysis

Structured approach:

Here, CP calculations involve TWO passes

Forward pass: To determine the Earliest Occurrence (EO) times of events. The computations start at Node 1 and advance recursively to the last Node *n*

Initial Step: Set EO(*1*) = 0, as project starts at time 0

General Steps for determine EO(*j*):

1. Consider separately, each node (say, *i*) from where an activity is directly converging into node *j*
2. Add EO(*i*) and the corresponding activity time t_{ij}
3. Select the maximum of them as EO(*j*)

The forward pass is complete when EO(*n*) is computed

Backward Pass: To determine the Latest Occurrence (LO) times of events. The computations start at the last Node *n* and end at Node 1

Backward pass starts after the Forward pass is completed

Initial Step: Set LO(*n*) = EO(*n*), as acceptable project delay is 0

General Steps for determine LO(*i*):

1. Consider separately, each node (say, j) to which an activity is directly reaching from node i
2. Subtract activity time t_{ij} from the corresponding LO(j)
3. Select the minimum of them as LO(i)

The Backward pass is complete when LO(1) is computed

An event j will be critical, if EO(j) = LO(j)

An activity ij will be Critical, if it satisfies the following three conditions:

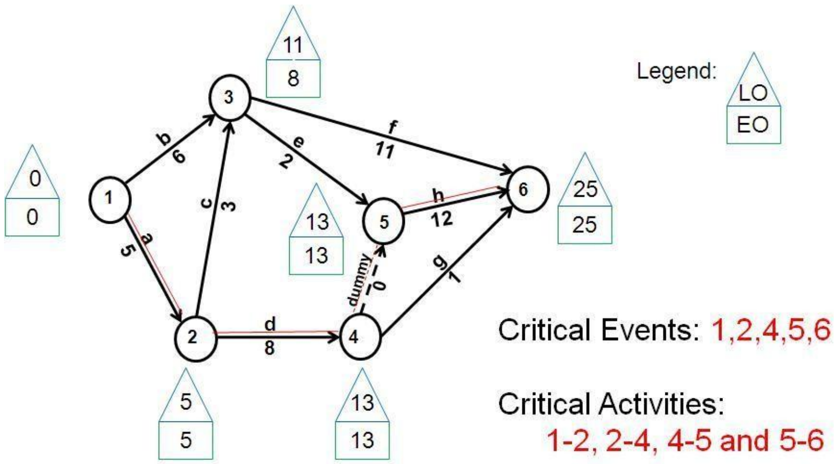
1) EO(i) = LO(i)
2) EO(j) = LO(j)
3) LO(j) – LO(i) = EO(j) - EO(i) = t_{ij}

Otherwise the activity is Noncritical

For an activity ij , LO(j) - EO(i) gives the maximum span during which the activity may be scheduled

Problem 5. Determine the Critical Path for the activity data given below. All durations are in days.

Initial Node	Final Node	Duration (days)
1	2	5
1	3	6
2	3	3
2	4	8
3	5	2
3	6	11
4	5	0
4	6	1
5	6	12



Calculate the floats associated with each activity using the above information

Calculation of floats associated with activities:

Total Float: The time by which an activity *ij* can be delayed without affecting the project duration

TF(*ij*) = { [LO(*j*) – EO(*i*)] - t_{ij} }

Free Float: The time by which an activity *ij* can be delayed without affecting the EO(*j*) FF(*ij*) = { [EO(*j*) – EO(*i*)] - t_{ij}

Independent Float: The time by which an activity *ij* can be delayed without affecting the floats of any other activity

IF(*ij*) = { [EO(*j*) – LO(*i*)] - t_{ij} }

All floats are zeros for a Critical activity.

The information about the occurrences of events can be used to develop time schedule and calculate floats for all activities

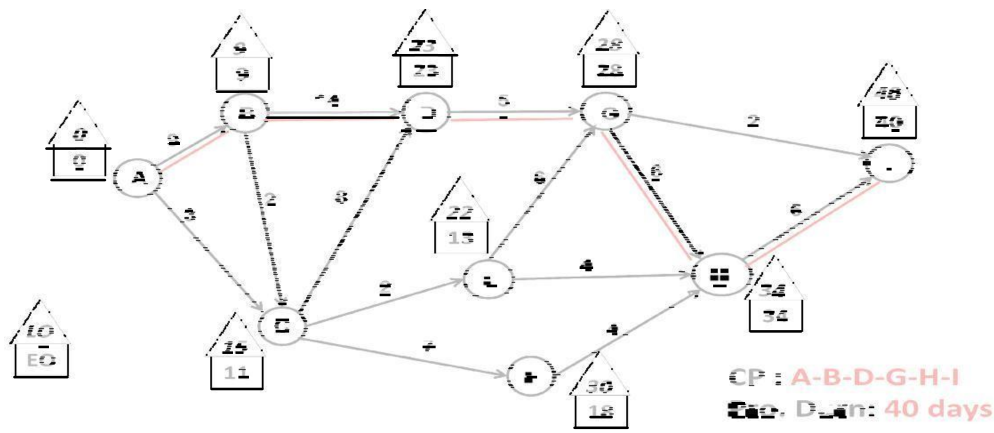
Activity *i-j*	**t** $_{ij}$	**ES(*ij*)**	**EF(*ij*)**	**LS(*ij*)**	**LF(*ij*)**	**TF(*ij*)**	**FF(*ij*)**	**IF(*ij*)**
1-2	5	0	5	0	5	0	0	0
1-3	6	0	6	5	11	5	2	2
2-3	3	5	8	8	11	3	0	0
2-4	8	5	13	5	13	0	0	0
3-5	2	8	10	11	13	3	3	0
3-6	11	8	19	14	25	6	6	3
4-5	0	13	13	13	13	0	0	0
4-6	1	13	14	24	25	11	11	11
5-6	12	13	25	13	25	0	0	0

Problem.6. For the given project activity data, compute:

1) Critical Path
2) Early Start
3) Early Finish
4) Late Start
5) Late Finish
6) Total Float
7) Free Float and
8) Independent Float

Initial node	Final Node	Duration (days)
A	B	9
A	C	3
B	C	2
B	D	14

C	D	8
C	E	2
C	F	7
D	G	5
E	G	6
E	H	4
F	H	4
G	H	6
G	I	2
H	I	6



Node i	Node j	Durn.	ES	EF	LS	LF	TF	FF	IF
A	B	9	0	9	0	9	0	0	0
A	C	3	0	3	12	15	12	8	8
B	C	2	9	11	13	15	4	0	0
B	D	14	9	23	9	23	0	0	0
C	D	8	11	19	15	23	4	4	0
C	E	2	11	13	20	22	9	0	- 4
C	F	7	11	18	23	30	12	0	- 4
D	G	5	23	28	23	28	0	0	0
E	G	6	13	19	22	28	9	9	0
E	H	4	13	17	30	34	17	15	8
F	H	4	18	22	30	34	12	12	0
G	H	6	28	34	28	34	0	0	0

G	I	2	28	30	38	40	10	10	10
H	I	6	34	40	34	40	0	0	0

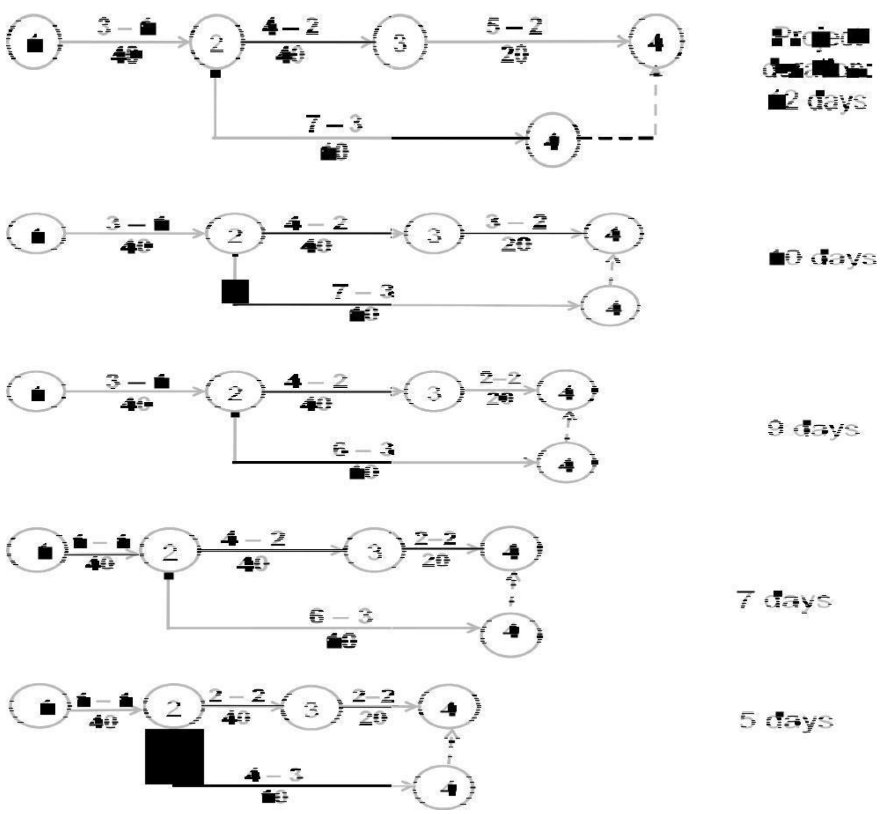
Crashing Project Duration:

In many situations it becomes ne cessary to cut down the project duration. How c an it be done? Activities that are critical need to be crashed in order to reduce the projec t duration as it is these activities that determin e the project duration. But this has got its own c ost implications. Reduction in proje ct duration calls for more resources to be pumpe d in and hence, the direct costs increase. Whereas indirect costs such as equipment rent, s upervision charges, etc. reduce. Thus, it becomes necessary to identify a project duration up to which the project can be crashed so that over all project costs are minimum.

Problem.7. Find the lowest cost schedule of the following project given the ove rhead expenses as Rs.45,000/- per day.

Activity	Normal duration	Crash duration	Cost of crashing (x1000 Rs/day)
1-2	3	1	40
2-3	4	2	40
2-4	7	3	10
3-4	5	2	20

Solution: Draw a squared netwo rk placing the CP at the centre.



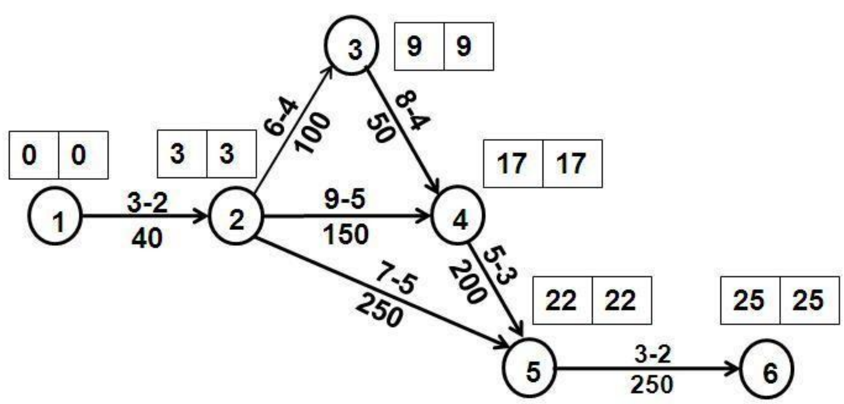
Activity crashed	Days saved	Proj. duration	Cost of crashing	Total Cost of crashing	Over-heads	Total Cost
None	0	12	0	0	540	540
3-4	2	10	2x20=40	40	450	490
3-4 & 2-4	1	9	(20+10)x1=30	70	405	475
1-2	2	7	40x2=80	150	315	465
2-3 &2-4	2	5	(40+10)x2=100	250	225	475

The lowest cost schedule is the plan corresponding to project duration of 7 days.

Problem.8. Table below gives the time and cost data with respect to normal and crash periods of a project. (a) Draw the n/w of the project, (b) What is the normal duration and cost of the project? (c) Determine the project cost if all activities are crashed indiscriminately (d) Determine the optimum project duration, if the indirect cost is Rs. 150/day

Activity	Normal time (days)	Normal cost (Rs.)	Crash time (days)	Crash cost (Rs.)
1-2	3	360	2	400
2-3	6	1400	4	1600
2-4	9	2000	5	2600
2-5	7	1000	5	1500
3-4	8	400	4	600
4-5	5	1600	3	2000
5-6	3	500	2	750

a)



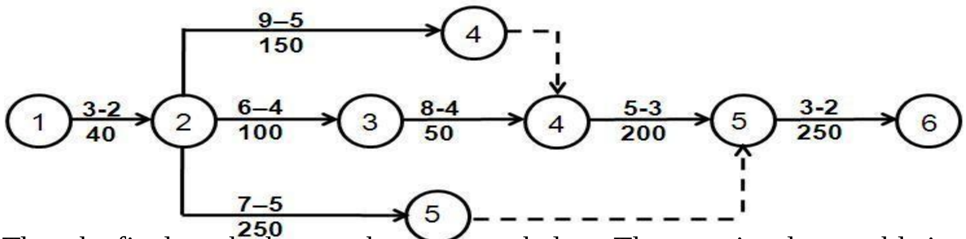
Critical path : 1 – 2 – 3 – 4 – 5 – 6

b) Normal Project Duration : 25 days
Normal cost of the project is 7,260 + 3,750 = 11,010

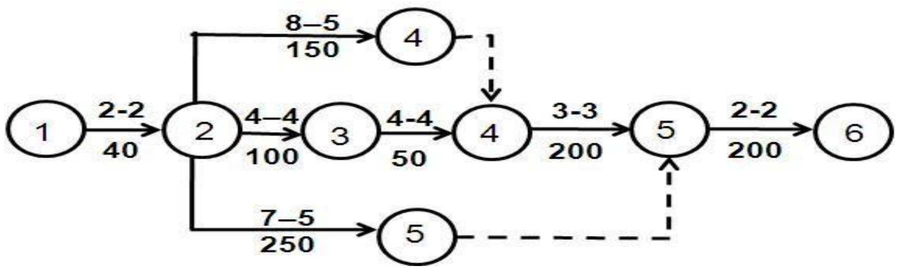
c) Cost of the project if all activities are crashed indiscriminately:

= Rs.9,450 + 2,250 = 11,700

d) Now draw a squared network as shown below. Choose the activities on the critical path to crash such that the present critical path continues to remain as (at least one of) the critical path. Also the cost of crashing/day shall be the least among available options at any stage.



Thus the final crashed network appears as below. The associated cost table is also shown.



Activity crashed	**Days saved (days)**	**Project durn (days)**	**Direct cost (Rs.)**	**Overheads (Rs.)**	**Total project cost (Rs.)**
None	0	25	7260	3750	11,010
1-2	1	24	7300	3600	10,900
3-4	4	20	7500	3000	10,500
2-3	1	19	7600	2850	10,450
4-5	2	17	8000	2,550	10,550
2-3 & 2-4	1	16	8,250	2,400	10,650
5-6	1	15	8,500	2,250	10,750

The table shows that the project duration of 19 days is most economical and optimum.

Program Evaluation and Review Technique (PERT)

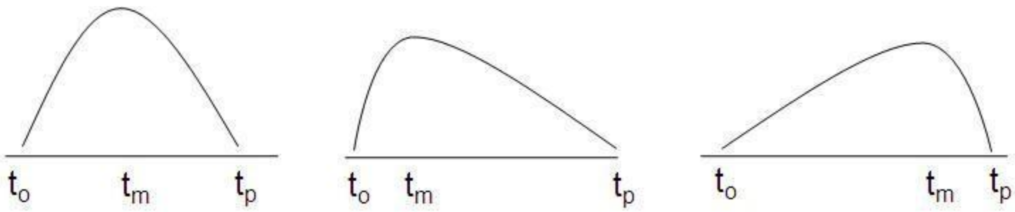
Used in such projects where the activities are, to some extent, governed by States-of-nature and the organization does not have full control over the activity duration and hence the time estimates of activities are probabilistic. For example, R&D projects, projects exposed to seasonal variations.

In PERT, we use three time estimates for an activity which reflect the degrees of uncertainty t_o = Optimistic time: When environment is very favorable

t_m = Most likely time: When environment is just normal

t_p = Pessimistic time: When environment is very unfavorable There can be a number of time estimates between t_o and t_p

The frequency distribution curve of activity time 't' is assumed to be a β distribution which is unimodal at t_m and has extremes at t_o and t_p. But t_m need not be the mid point between t_o and t_p.



The expected (average) time t_e of an activity is calculated as a weighted average of these three estimates.

$t_e = (t_o + 4\,t_m + t_p) / 6$ where t_e need not be equal to t_m

How reliable t_e is? How confident are we that the activity gets over in time t_e? This depends upon the spread of the distribution of 't' or variability of 't'. Larger the variation, lesser would the confidence. Example: Distributions with same mean but different spreads

Variance (σ^2)' and 'Standard deviation (σ)' are two important measures of variability. In PERT, variance of an activity is calculated as: $\sigma^2 = [(t_p - t_o)/6]^2$

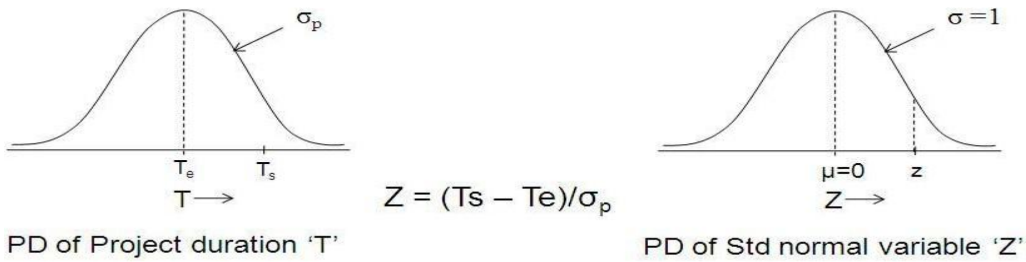
Using t_e s, the earliest and latest occurrences of each event are calculated from which the CP is determined

$T_e = \Sigma$ (t_e of all critical activities)

$\sigma_p^2 = \Sigma$ (σ^2 of all critical activities)

There is a possibility of alternate Critical paths. In such cases we need to consider the critical path with greater variance. As the analysis is based on the probabilistic time estimates even near Critical paths should be carefully examined.

When activity times ('t_j' for j= 1, 2,…n) are independent and identically distributed random variables, then (by Central limit theorem) project duration (T) will follow normal distribution with expected project duration T_e and project variance σ_p^2 . With T_e and σ_p we can compute the probability of completing the project within a due date using normal PD tables.



Problem 9. Given the list of activities in a project and their time estimates (in days):

a) Draw the project network,
b) Determine the critical path(s) and the expected project duration
c) What is the probability that project will be completed in 35 days?
d) What due date has 90% chance of being met?

Activity	t_o	t_m	t_p
1-2	6	12	30
1-3	3	6	15
1-4	3	9	27
2-6	4	19	28
3-5	3	9	27
3-6	2	5	8
4-5	1	4	7
5-6	6	12	30

a)

b)

Activity	t_e
1-2	12
1-3	7
1-4	11
2-6	18
3-5	11
3-6	5
4-5	4
5-6	14

Critical path = 1 – 3 – 5 – 6

Expected project duration T_e = 32 days

c)

Activity	t_o	t_m	t_p	CA	σ^2
1-2	6	12	18		4
1-3	3	6	15	1-3	4
1-4	3	9	27		16
2-6	4	19	28		16
3-5	3	9	27	3-5	16
3-6	2	5	8		1
4-5	1	4	7		1
5-6	6	12	30	5-6	16

Variance σ_p^2 = 36 days

Standard deviation σ_p = $\sqrt{36}$ = 6 days

Expected duration of the project T_e = 32 days Std. deviation of project duration σ_p = 6 days Scheduled duration T_s= 35 days

We know, $Z = (Ts – Te)/\sigma_p$

$= (35-32)/6 = +0.5$

The area under the normal curve (from standard normal PD table) up to Z = +0.5 = 0.6915 (i.e. 69.15 % chance)

The probability that project will be completed in 35 days is 0.6915

d)

We know, $Z = (T_s – T_e)/\sigma_p$

For 90% chance (probability = 0.9) area under the std. normal curve, we have Z = +1.28 $+1.28 = (T_s - 32)/6$

Hence, T_s = 39.68 days

Project duration of 39.68 days has 90% chance of being met

Advantages and Limitations of PERT/CPM

Advantages:

- Simple to understand and use
- Show whether the project is on schedule; or behind/ ahead of the schedule
- Identify the activities that need closer attention (critical)
- Determine the flexibility available with activities
- Show potential risk with activities (PERT)
- Provide good documentation of the project activities
- Help to set priorities among activities and resource allocation as per priority

Limitations:

- Demand clearly defined and stable activities
- Precedence relationship should be known in advance
- Overemphasis on Critical path
- Activity time estimates are subjective
- Activity times in PERT may not follow Beta PD in reality
- Cost of crashing an activity may not be linear

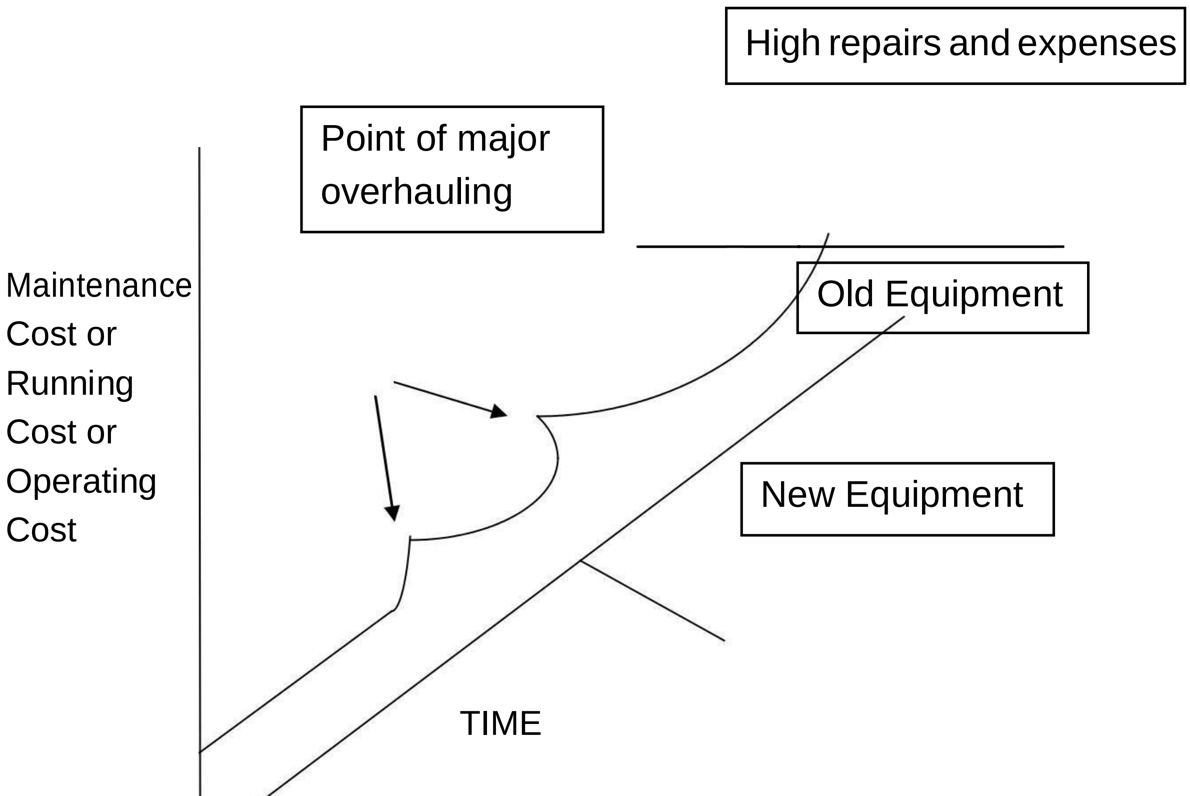
Some Computer Software Available for Network Analysis

- Microsoft Project (Microsoft Corp.)
- PowerProject (ASTA Development Inc.)
- Primavera Project Planner (Primavera)

- MacProject (Claris Corp.)

Reference Books:

1. **Taha H A**, Operation Research - An Introduction, Prentice Hall of India, 7^{th} edition, 2003
2. **Ravindran, Phillips and Solberg**, Operations Research : Principles and Practice, John Wiely & Sons, 2^{nd} Edition
3. **D.S.Hira**, Operation Research, S.Chand & Company Ltd., New Delhi, 2004
4. **L.S.Srinath**, PERT and CPM- Principles and Applications, 3^{rd} Edition, Affiliated East-West Press, 2000
5. **Gaither and Frazier**, Operations Management, Thomson, South Western Publishers, 2002



Reasons for replacement

1. Technological development
2. Poor performance over years
3. Unable to meet the required demands

Capital equipment that deteriorates with time:

It is concerned with the equipment an machinery that deteriorates with time. Many people feel that equipment should not be replaced until it is physically worn off. But, it is not correct, preferable equipment must be constantly renewed and updated otherwise it will be in the risk of failure or it may become obsolete.

Reasons for replacement

1. Deterioration
2. Obsolescence
3. Technological development
4. Inadequacy

Deterioration is the decline in the performance of the equipment as compared to the new equipment. It may occur due to wear and tear. Due to this

a) Increase in the maintenance cost.
b) Reduces the product quality
c) Decreases the rate of production
d) Increases the labor cost
e) Reduces the efficiency of the equipment

Models:

Model1:

"Replacement of items whose maintenance Cost increases with time and the value of the money remains constant during the period"

Model 2: "replacement of items whose maintenance cost increases with time and value of money also changes with time".

Model 3: "Group Replacement policy"

Model1: Notation and symbols

C- Purchase cost of the machinery or equipment

S- Salvage value or resale value or scrap value of the machinery or equipment

T_C total cost increased on the item or equipment during the period y

$T_c = C + m(Y) – S$

Where M(Y) is the cumulative maintenance cost in that period.

G(Y) $\rightarrow$ Average cost incurred on the equipment or item during the period.

$G(Y) = T_c / y$

Problem 1

The cost of the machine is Rs 6100/- and its scrap value is Rs 100 at the end of every year. The M.C. found from experience are as follows:

Year	1	2	3	4	5	6	7	8
M.C	100	250	400	600	900	1200	1600	2000

When should the machine be replaced?

Solution:

Given: C=6100

S=100

Year (Y)	Maintenance	Cumulative Maintenance cost m(y)	Total Cost C-S + m(y)	Average Cost g(y)
1	100	100	6100	6100
2	250	350	6350	3175
3	400	750	6750	2250
4	600	1350	7350	1837.5
5	900	2250	8250	1650
6	1200	3450	9450	1575
7	1600	5050	11050	1578.5
8	2000	7050	13050	1631.25

It's clear from the above analysis that the machine has to replaced at the end of 6^{th} year or at the beginning of 7^{th} year because the maintenance cost of 7^{th} year is more than the average cost of the machine i.e 1578.5 > 1575.5

Problem 2

A fleet owner finds from his past experience records that the cost of the machine is Rs 6000/- and the running cost are given below. At what age the replacement is due;-

Year	1	2	3	4	5	6	7	8
Maintenance Cost	1000	1200	1400	1800	2300	2800	3400	4000
Resale Value	3000	1500	750	375	200	200	200	200

Solutions:

Year (Y)	Maintenance	Cumulative Maintenance cost m(y)	C-S	Total Cost C-S + m(y)	Average Cost g(y)
1	1000	1000	3000	4000	4000
2	1200	2200	4500	6700	3350
3	1400	3600	5250	8850	2950
4	1800	5400	5625	11025	2756.25
5	2300	7700	5800	13500	2700
6	2800	10500	5800	16300	2716.67
7	3400	13900	5800	19700	2814.28
8	4000	17900	5800	23700	2962.5

The machine should be replaced at the end of 5^{th} year of at the beginning of 6^{th} year because the maintenance cost of the 6 year is more than the average cost of the machine i.e 2716 > 2700

Problem 3:

A Fleet owner finds from this past experience that the cost/year of running the truck whose purchase price rises to Rs 60000/- are given below Solutions; given C= 60000/-

Year	1	2	3	4	5	6	7	8
Maintenance	10000	12000	14000	18000	23000	28000	34000	40000
Depreciation	30000	45000	52500	56250	58000	58000	58000	58000

At what age the truck is to be replaced?

Solution:

Year (Y)	Maintenance	Cumulative Maintenance cost m(y)	Depreciation C-S	Total Cost C-S + m(y)	Average Cost g(y)
1	10000	10000	30000	40000	40000
2	12000	22000	45000	67000	33500
3	14000	36000	52500	88500	29500
4	18000	54000	56250	110250	27562.5
5	23000	77000	58000	135000	27000
6	28000	108000	58000	163000	27166.66
7	34000	139000	58000	197000	28142.85
8	40000	179000	58000	237000	29625

The machine should be replaced at the end of 5^{th} year or at the beginning of 6^{th} year because the maintenance cost of the 6^{th} year is more than the average cost of the machine. i.e 27166.66 > 27000.

Problem 4:

An auto rickshaw driver finds from his previous records that the cost/year of running an autorickshaw whose purchase the cost/year of running an autorickshaw whose purchase price is Rs. 7000/- is as follows:-

Year	1	2	3	4	5	6	7	8
Maintenance cost	1100	1300	1500	1900	2400	2900	3500	4100
Resale value	3100	1600	850	475	300	300	300	300

Solution:

Year (Y)	Maintenance	Cumulative Maintenance cost m(y)	C-S	Total Cost C-S + m(y)	Average Cost g(y)
1	1100	1100	3900	5000	5000
2	1300	2400	5400	7800	3900
3	1500	3900	6150	10050	3350
4	1900	5800	6525	12325	3081.25
5	2400	8200	6700	14900	2980
6	2900	11100	6700	17800	2966.6
7	3500	14600	6700	21300	3042.85
8	4100	18700	6700	25400	3175

The machine should be replaced at the end of 6^{th} year or at the beginning of 7^{th} year because the maintenance cost of the 7^{th} year is more than the average cost of the machine i.e 3042.85 > 2966.6

Problem 5:

Machine A costs Rs 9000/- annual operating cost is Rs 200/- for the first year and then increases by Rs 2000/- every year. Determine the best age at which the machine should be replaced and what would be the average cost of owning is operating cost of the machine?

Machine B costs Rs 10000/- annual operating cost is Rs 400/- for the first year and then increased by Rs. 800/- every year you now have a machine A which is one year old. Should you replace it with B? If so when? Assume machines have no resale value and that the future costs are not discounted.

Solution:-

Machine A:- cost price C = 9000/- ; S = 0

Year (Y)	Maintenance Cost	Cumulative Maintenance cost m(y)	Total cost (T_c) C-S + m(y)	Average Cost g(y)
1	200	200	9200	9200
2	2200	2400	11400	5700
3	4200	6600	15600	5200
4	6200	12800	21800	5450
5	8200	21000	30000	6000

Machine B:- cost price C = 10,000/- ; S = 0

Year (Y)	Maintenance Cost	Cumulative Maintenance cost m(y)	Total cost (T_c) C-S + m(y)	Average Cost g(y)
1	400	400	10400	10400
2	1200	1600	11600	5800
3	2000	3600	13600	4533.33
4	2800	6400	16400	4100
5	3600	10000	20000	4000
6	4400	14400	24400	4066.67

Machine A should be replaced at the end of the 3rd year and the average cost is 5200/- . machine B should be replaced at the end of 5th year and the average cost is Rs 4000/-.

The I year maintenance cost of machine A which is a year old is Rs2200/- < which is less than the average cost of machine B i.e. 4000.

The II year maintenance cost of machine A (1 year old) is 4200 > avg cost of machine B (i.e. 4000/-)

The III year maintenance cost of machine A is 6200/- which is greater than the average cost of machine B. i.e. 4000/-.

It is clear from above analysis that machine A that is one year old can be used one more year from now and then replaced with machine B.

Case 2 or Model II

"Replacement of items whose maintenance cost increases with time and the value of money also changes with time"

The maintenance cost varies with time and we want to find out the optimum time period at which the items will be replaced value of money decreases with a constant rate which is known as depreciation ratio or discounted factor which is given by $V= 1/ (1+i)^n$ for the value of 1 rupee where $i \longrightarrow$ rate of interest , $n \longrightarrow$ no. of years

Problem 6:-

A company buys a machine for Rs 6000/-. The maintenance cost are expected to be Rs 300/- in each year for the first 2 years and go up annually as follows 700, 1000, 1500, 2000, and 2500. Assume the money is worth of 20% per year. When the machine should be replaced.

Solution:-

C = 6000/-

I = 20%

Assumption:- in this to solve the problem we assume that the maintenance cost is spent on the machine at the beginning of each year as 1.

Year	Maintenance cost	Present value of 1 Re	Present value of Maintenance cost	Cumulative Present value Maintenance cost m (y)	Total cost T_C = C-S + m(y)	Cumulative Present value of 1 Re	Weighted Average cost
1	300	1	300	300	6300	1	6300
2	300	0.833	249.9	549.9	6549.9	1.833	3573.32
3	700	0.694	486.1	1036	7036	2.527	2784.32
4	1000	0.578	578	1614	7614	3.105	2452.17
5	1500	0.482	723	2337	8337	3.587	2324.22
6	2000	0.401	802	3139	9139	3.988	2291.62
7	2500	0.334	835	3974	9974	4.322	2307.72

Specimen calculations for 3rd year by $V= 1/ (1+i)^n$

$=1/(1.2)^2$

$=0.694$

$=0.694*700 = 486.1$

The machine should be replaced at the end of 6th year or at the beginning of 7th year because the maintenance cost in the 7th year is more than the average cost of the machine 2500 > 2291.62

Problem 7:

A machine cost Rs 10000/- the operating cost is Rs 500/- for the first 5 years and then increased by Rs 100/- every year subsequently from the 6th year onwards. Assuming the money is worth of 10% per year. Find the optimal length of time to hold the machine before replacement.

Solution:-

C=10,000, i =

10% S = 0

Assumption:

To solve the problem, we assume that maintenance cost spent on the equipment is at the beginning of each year.

Specimen Calculations for 3rd year

$V= 1/(1+i)^n = 1/(1.1)^3 = 0.826$

$= 0.826 \times 500$

$= 413$

Year	Maintenance cost	Present value of 1 Re	Present value of Maintenance cost	Cumulative Present value Maintenance cost m (y)	Total cost T_C=C-S+m(y)	Cumulative Present value of 1 Re	Weighted Average cost
1	500	1	500	500	10500	1	10500
2	500	0.909	454.5	954.5	10954.5	1.909	5738.34
3	500	0.826	413	1367.5	11367.5	2.735	4156.31
4	500	0.751	375.5	1743	11743	3.486	3368.62
5	500	0.683	341.5	2084.5	12084.5	4.169	2898.66
6	600	0.621	372.6	2457.1	12457.1	4.79	2600.65
7	700	0.564	394.8	2851.9	12851.9	5.354	2400.43
8	800	0.513	410.4	3262.3	13262.3	5.867	2260.49
9	900	0.466	419.4	3681.7	13681.7	6.333	2160.38
10	1000	0.424	424	4105.7	14105.7	6.757	2087.57
11	1100	0.385	423.5	4529.2	14529.2	7.142	2034.33

12	1200	0.350	420	4949.2	14949.2	7.492	1995.36
13	1300	0.318	413.4	5362.6	15362.6	7.81	1967.04
14	1400	0.289	404.6	5767.2	15767.2	8.099	1946.81
15	1500	0.263	394.5	6161.7	16161.7	8.362	1932.76
16	1600	0.239	382.4	6544.1	16544.1	8.601	1923.51
17	1700	0.217	368.9	6913	16913	8.818	1918.01
18	1800	0.198	356.4	7269.4	17269.4	9.016	1915.42
19	1900	0.180	342	7611.4	17611.4	9.196	1915.11
20	2000	0.163	326	7937.4	17937.4	9.359	1916.59

Conclusion:

The machine should be replaced at the end of 19th year or at the beginning of 20th year because the maintenance cost in the 20th year is more than the average cost of 19th year i.e. 1916.59 > 1915.11

Problem 8:

As Rs 6000/- for the first 4 years and increasing by Rs 2000/- per year in the 5th and subsequent years. If money is worth of 10% per year, when should the truck be replaced? Assume the truck has no resale value.

(Ans: end of 10th year Rs 18000.40/-)

Problem 9:

A manufacturer is offered two machines A & B. A is priced at Rs 5000/- and running cost are estimated at Rs. 800/- in each of the 1st 5 years, then increasing by Rs 200/- per year in the 6th and subsequent years. The machine B which has the same capacity as A, costs. Rs 2500/- but will have a running cost of Rs 1200/-per year for the 1st 6 years and then increasing by Rs 200/- per year thereafter. If money is worth of 10% per year which machine should be purchased? Assume the scrap value is zero at the end of year. (Ans : M/c A :- 9th year end : Rs 1750

M/c B:- 8th Year end : Rs 1680.39)

Problem 10:

A lorry is priced at Rs250, 000. The operating cost is estimated at the rate of Rs 25000 for each year for the first 5 years and increasing by Rs 5000/- per year in the 6th and subsequent years. Decide when the lorry should be replaced. Assume the scrap value is negligible at the end. The worth of money is 25% per year.

(Ans : End of 16th year : Rs.82664.6)

Problem 11:

Assume that the present value of Rs 1/- to be spent in a year time is Rs 0.9 and purchase cost of the machine is Rs 3000/-. The running costs are given below.

Year	1	2	3	4	5	6	7
Running cost	500	600	800	1000	1300	1600	2000

When should the machine be replaced.

(Ans : End of 6^{th} year : Rs 1539.85) Solution: Assumption maintenance cost spent on the machinery is at the beginning of each year

$V= 1/ (1+i)^n$

$0.9 = 1/ (1+i)^1 =11\%$

Year	Maintenance cost	Present value of 1 Re	Present value of Maintenance cost	Cumulative Present value Maintenance cost m (y)	Total cost T_C=C-S+m(y)	Cumulative Present value of 1 Re	Weighted Average cost
1	500	1	500	500	3500	1	3500
2	600	0.9	540	1040	4040	1.9	2126.31
3	800	0.811	648.8	1688.8	4688.8	2.711	1729.54
4	1000	0.731	731	2419.8	5419.8	3.442	1574.6
5	1300	0.658	855.4	3275.2	6275.2	4.1	1530.53
6	1600	0.593	948.8	4224	7224	4.693	1539.31
7	2000	0.534	1068	5292	8292	5.227	1586.37

Specimen Calculations for 3^{rd} year

$V= 1/ (1+i)^n$

$= 1/ (1.11)^2 =0.811$

=0.811 X 800

= 648.8

Conclusion: machinery should be replaced at the end of 5^{th} year and at the beginning of 6^{th} year because the maintenance cost in the 6^{th} year is more than the average cost of machine in 5^{th} year i.e. 1539.37 > 1530.53

Problem 12

If you wish to have a return of 10% on your investment which of the following plans would you prefer

Particulars	Plan A	Plan B
First cost in Rs or fixed cost or capital or initial investment	2,00,000	2,50,000
Scrap value after 15 years in Rs.	1,50,000	1,80,000
Excess of annual revenue over annual disbursement or cash inflow	25,000	30,000

Solution:

Using the present value analysis

Consider the plan A initial investment =

200000/-Present value of Initial investment

=200000/-Scrap value after 15 years =150000/-

Present value of scrap value is = $1/(1.1)^{15}$ X 150000 = 35,908.8 Average annual cash inflow =26000/-

Present value of cash flow

P.V = $1/(1.1)^{1}$ X 25000+$1/(1.1)^{2}$ X 25000 + $1/(1.1)^{3}$ X25000 + ------- +$1/(1.1)^{15}$ X25000

Present worth factor = $(1+i)^n - 1/ i (1+i)^n$ = 7.606 when n=15, i=0.1

The present value of cash flow = 25000X 7.606 =190150/-

Net present value = P V of inflows - PV of outflows

=[(PV of salvage) + (PV of inflow)] – initial investment =35908 + 190150 -200000

= 26058/-

Consider Plan B:

Consider the plan A initial investment = 250000/-Present value of Initial investment =250000/-Scrap value after 15 years =160000/-

Present value of scrap value is = 1/ $(1.1)^{15}$ X180000 = 43,090.56 Average annual cash inflow =30000/-

Present value of cash flow

P.v = 1/ $(1.1)^{1}$ X30000+1/ $(1.1)^{2}$ X30000 + 1/ $(1.1)^{3}$ X30000 + ------- +1/ $(1.1)^{15}$ X30000

Present worth factor = $(1+i)^{n}$ - 1/ i $(1+i)^{n}$ = 7.606 when n=15, i=0.1

The present value of cash flow = 30000X 7.606 =228180/-

Net present value = P V of inflows - PV of outflows

=[(PV of salvage) + (PV of inflow)] – initial investment =43090 + 228180 -250000

= 21270/-

Since the net present value is more for plan A, so Select Plan A

Problem 13

Using the present value analysis which crane is to be selected. Assume the interest as 10%

Particulars	Crane A	Crane B
Capital	25000	18000
Maintenance cost per year	1000	700
Scrap value at the end of 10 years	25000	30000

Solution : Using the present value analysis:-

Consider Plan A:

Capital = 25000/-

Present value of Initial investment =25000/-

Scrap value after 10 years =8000/-

Present value of scrap value is = $1/(1.1)^{10}$ X8000 = 3084.35

Average annual cash outflow (machine and labour cost)

Present value of cash flow

P.V = $1/(1.1)^{1}$ X1000+ $1/(1.1)^{2}$ X1000 + $1/(1.1)^{3}$ X1000 + ------- + $1/(1.1)^{10}$ X1000

Present worth factor = $(1+i)^{n}$ - $1/ i (1+i)^{n}$ = 6.145 when n=10, i=0.1

The present value of cash flow = 1000X 6.145 = 6145 /-

Net present value = P V of inflows - PV of outflows

=PV of salvage - (PV of inflow+ initial investment) =3084.35 – (25000 + 6145)

= 28060.65/-

Consider Plan B:

Capital = 18000/-

Present value of Initial investment =18000/-

Scrap value at the end of 10 years = 3000/-

Present value of scrap value is = $1/(1.1)^{10}$ X 3000 = 1156.63/-Average annual cash outflow (machine and labour cost) Present value of cash flow

P.V= $1/(1.1)^{1}$ X 700+ $1/(1.1)^{2}$ X 700 + $1/(1.1)^{3}$ X 700 +---------+ $1/(1.1)^{10}$ X 700

Present worth factor = $(1+i)^{n}$ - $1/ i (1+i)^{n}$ = 6.145 when n=10, i=0.1

The present value of cash flow = 700X 6.145 = 4301.5 /-

Net present value = P V of inflows - PV of outflows

=PV of salvage - (PV of inflow+ initial investment) =1156.63 – (18000 + 4301.5)

= 21144.87/-

Since, the net present value is more for machine B, so select machine B

Group Replacement Policy:-

Replacement of items that fail completely

We always come across situation where the probability of failure in any s/m increases with time. The nature of the s/m may be such that if any item fails then it may result in complete breakdown of the s/m. this breakdown implies loss of production, work-in-progress.

Individual replacement policy:-

Whenever an item fails it should be replaced immediately. Group replacement policy:- 2 steps

In the first step, it consists of individual replacement at the time of failure of any unit in the s/m.

In the second step, there is a group replacement of some live units at some suitable time.

In the group replacement we decide that all items in a s/m should be replaced after a certain interval of time irrespective of the fact that the items have failed or not with a provision that if any item fails before this time it should be replaced immediately. It requires

a) Rate of individual replacement during the period

b) Total cost incurred for individual and group replacement during the chosen period. The period for which the total cost incurred is minimum will be the optimum period for replacement.

Problem 14

The following mortality rate have been observed from a certain type of light bulbs

Week	1	2	3	4	5
% of bulbs failing by the end of week.	10	25	50	80	100
Individual failure at the end of week.	0.1	0.15	0.25	0.3	0.2

There are 1000 bulbs in use, the cost is Rs 1/- is represented an individual bulb which has burnt out if all the bulbs are replaced simultaneously. It would cost 25 paise per bulb. It is proposed to replace all bulbs at fixed intervals determine optimum replacement policy for the bulbs.

At what group replacement prize for bulb would a policy of strictly individual replacement becomes preferable to the adopted policy.

Solution:-

Assumption: in this to solve the problems we assume that the bulbs that fail during the week are replaced at the end of the week.

Let Ni denote the number of replacements made at the end of the ith week.
Let N_0 is initially the no. of items in the s/m.
If it is not given assume it as 100 or 1000

N_1 = expected no. of bulbs being replaced at the end of I
week $N_1 = N_0 XP_1$

$= 1000X0.1$

$N_1 = 100$

$N_2 = N_0 XP_2 + N_1 XP_1$

$= 1000X0.15 + 100x0.1$

$= 281$

$N_3 = N_0 XP_3 + N_1 XP_2 + N_2 X \ P_1$
=1000X0.25 + 100X0.15 +
160X0.1 = 281

Expected life of the bulbs $=\sum_{I=1}^{n} X_i P_i$

Where X_i no of weeks &
P_i is the corresponding probability
= (1X0.1) + (2X0.15) + 3X0.25) + (4X0.3) + (5X0.2)
= 3.35 weeks.

Average no. of failures per week = (initial no. of items in the s\m) / (expected life)
= No. of failure / expected life
= 1000 /3.35
= 298.6
≈ 299

Cost of individual replacement per bulb =1 Rs

Total cost of individual replacement per week =299 x1
Rs =299

Consider the group replacement

End of the week	Total cost of group replacement	Average cost per week
1	(1000X0.25) + (100X1) = 350/-	350/1 =350/-
2	1000X0.25) + (100X1) + (160X1) = 510/-	510/2 = 255/-
3	1000X0.25) + (100X1) + (160X1) +(281X1)= 791/-	791/3 = 263.66/-

Conclusion: with reference to above data, it is preferred the group replacement once in two weeks and Avg cost per week is 255/-

Where in the case of individual replacement Average cost is 299/-

So prefer group replacement

b) (1000 x X) + (100x 1) +(160x1)/2 >
299 x >0.338
i.e. 34 paise

Problem 15:-

A automatic safety electric switches attached to a press has the following probability.

No of Years	1	2	3	4	5	6	7
Probability of failure	0.05	0.1	0.15	0.2	0.35	0.1	0.05

If the average cost to replace the single switch is Rs 15/- but, this comes down to Rs. 3/- when the replacement is carried out on the group basis. Find the optimum replacement plan.

Solution:

Week	1	2	3	4	5	6	7
Probability of failure	0.05	0.1	0.15	0.2	0.35	0.1	0.05

Assumption: - here to solve the problem assume that the no. of switches that fail during the year are replaced just before the end of the year.

Let N_i Denote the no of replacement mode at the end of the ith year N_1 - expected no of switched being replaced at the end of the 1st year Let N_0 is initially the no. of items in the system.
If it is not given assume it as 100

$N_1 = N_0 X P_1$

$= 100X0.05$

$N_1 = 5$

$N_2 = N_0 \ \ XP_2 + N_1 \ XP_1$

$= 100 \ X \ 0.1 + 5x0.05$

$= 10.25$

$N_3 = N_0 \ XP_3 + N_1 \ XP_2 + N_2 \ X$
$P_1 = 100X0.15 + 5X0.1 +$
$10.25X0.05 = 16.0125$

$N_4 = N_0 X P_4 + N_1 X P_3 + N_2 X P_2 + N_3 X P_1$

=100x0.2+5x0.15+10.25x0.1+16.0125x0.05

=22.5

Expected life of the switches

=1x0.05+2x0.1+3x0.15+4x0.2+5x0.35+6x0.1+7x0.05

=4.2 years

Avg. no failures per years = No. of items in s/m initially/expected life = 100/4.2 =24

Avg cost of individual replacement / switch = Rs 15/-. Total cost of individual replacement per year =24x15 =360/-

Consider Group Replacement

End of the year	Total cost of group replacement	Average cost per Year
1	100x3+5x15 =375	375/1=375
2	100x3+5X15+10.25x15=528.75	528.75/2=264
3	100x3+5X15+10.25x15+16x15=768.75	768.75/3=256.3
4	100x3+5X15+10.25x15+16x15+22.5x15=1106.25	1106.25/4=276.5

Prefer the group replacement once in three years and the avg cost /year =Rs256.3/-. Whereas, in case of individual replacement. The avg cost per year is Rs 360/-. So prefer group replacement.

Problem 16:

The following mortality list rates have been observed for a certain type of tube lights

Week	1	2	3	4	5	6	7
Probability of failure to date	0.04	0.18	0.29	0.46	0.68	0.92	1

The cost of replacing a tube light individually is Rs.2.35. the cost of group replacement is Rs. 0.65/tube. Determine the optimum replacement policy.

Solution:

Let N_0 =100

Here to solve the problem, assume that the no. of tube light that fail during the week are replaced at the end of the week.

N_i - expected no of switched being replaced at the end of the 1^{st} week N_1 - expected no of switched being replaced at the end of the 1^{st} week $N_1 = N_0 XP_1$

$= 100X0.04$

$N_1 = 4$

$N_2 = N_0 XP_2 + N_1 XP_1$

$= 100 X 0.14 + 4x0.04$

$= 14.16$

$N_3 = N_0 XP_3 + N_1 XP_2 + N_2 X\ P_1$

$=100X0.11 + 4X0.14 + 14.16X0.04 = 12.1264$

$N_4 = N_0 XP_4 + N_1 XP_3 + N_2 X P_2 + N_3 XP_1$

$=100x0.17+4x0.11+14.16x0.14+12.1264x0.04$

$=19.9$

Expected life of the tube light

$=1x0.04+2x0.14+3x0.11+4x0.17+5x0.22+6x0.24+7x0.08$

=4.43 years

Avg. no failures per week = No. of items in s/m initially/expected life = 100/4.43

$=22.57 \approx 23$

Avg cost of individual replacement / tube = Rs 2.35/-. Total cost of individual replacement per week =23x2.35 =54.05/-

Consider Group Replacement

End of the week	Total cost of group replacement	Average cost per week
1	100x0.65+4x2.35 =74.4	74.4/1=74.4
2	100x0.65+4x2.35+14.16x2.35=107.67	107.67/2=53.83
3	100x0.65+4X2.35+14.16x2.35+12.1264x2.35=136.17	136.17/3=45.39
4	100x0.65+4X2.35+14.16x2.35+12.1264x2.35+19.9x2.35= 182.93	182.93/4=45.73

Conclusion: prefer group replacement once in three week where the avg cost/year =Rs45.39/-. In case of individual replacement, the avg cost / week =54.05/-

So prefer group replacement

Problem 17:

A computer contains 10000 resistors. When a resistor fails it is replaced. The cost of replacing a resistor individually is Rs 1/-. If all the resistors are at the same time the cost / resistor is reduced by 65 paise the probability of serving at the end of the month is given below. What is the optimum replacement plan?

Month	0	1	2	3	4	5	6
Probability of survival	1	0.97	0.9	0.7	0.3	0.15	0

Solutions:

Month	0	1	2	3	4	5	6
Probability of failure cumulative	0	0.03	0.1	0.3	0.7	0.85	1
Probability of failure in each month	0	0.03	0.07	0.2	0.4	0.15	0.15

Given N_0 =10000

Assume that the no of resistors are that fail during the month are replaced at the end of the month.

Let N_i denote the no of replacements at the end of ith month.

N_1 = expected no of resistors being replaced at the end of 1st month.

$N_1 = N_0 XP_1$

$= 10000X0.03$

$N_1 = 300$

$N_2 = N_0 XP_2 + N_1 XP_1$

$= 10000 X 0.07 + 300x0.03$

$= 709$

$N_3 = N_0 XP_3 + N_1 XP_2 + N_2 X P_1$

$=10000X0.2 + 300X0.07 +$

$709X0.03 = 2042.27$

$N_4 = N_0 XP_4 + N_1 XP_3 + N_2 X P_2 + N_3 XP_1$

$=10000x0.4+300x0.2+709x0.07+2042.27x0.03$

$=4170.89$

Expected life of the tubes

=1x0.03+2x0.07+3x0.2+4x0.4+5x0.15+6x0.15

=4.02 years

Avg. no failures per month = No. of items in s/m initially/expected life = 10000/4.02

=2487.56 ≈2488

Avg cost of individual replacement resistor = Rs 1/-. Total cost of individual replacement per month =2488x1 =2488/-

Consider Group Replacement

End of the Month	Total cost of group replacement	Average cost per Month
1	10000x0.35+300x1 =3800/-	3800/1= 3800
2	10000x0.35+300x1+709x1=4509/-	4509/2= 2254.5
3	10000x0.35+300X1+709x1+2042.27x1=6551.27	6551.27/3= 2183.7
4	10000x0.35+300X1+709x1+2042.27x1+4170.89x1=10722.16	10722.16/4= 2680.54

Conclusion: prefer group replacement once in three months where the avg cost/month =Rs 2183.7/-. In case of individual replacement, the avg cost / month =2488/-

So prefer group replacement

Problem 18:

Suppose a special purpose type of light never last longer than 2 weeks. There is a chance of 0.3 that a bulb will fail at the end of 1^{st} week cost per for individual replacement is Rs 1.25 is the cost/bulb in the group replacement is Rs 0.50. it is cheapest to replace bulbs.

a) Individually
b) Every week
c) Every 2^{nd} week

Solution:

Week	1	2
Prob. 0f Failure	0.3	0.7

Let N_0 =100

Assumption:- No of bulbs that fail during the week are replaced at the end of week

Let N_i denote the no of replacements at the end of ith week.

N_1 = expected no of resistors being replaced at the end of 1^{st} week

$N_1 = N_0 XP_1$
$= 100X0.3$

$N_1 = 300$

$N_2 = N_0 XP_2 + N_1 XP_1$
$= 100 X 0.7 + 30x0.3$
$= 79$

Expected life of the tubes =1x0.3+2x0.7

=1.7 weeks

Avg. no failures per week = No. of items in s/m initially/expected life = 100/1.7 =58.82 ≈59

Avg cost of individual replacement / bulb = Rs 1.25/-. Total cost of individual replacement per week =59x1.25 =73.75/-

Consider Group Replacement

End of the week	Total cost of group replacement	Average cost per week
1	100x0.5+30x1.25 =87.5/-	87.5/1=87.5
2	100x0.5+30x1.25+79x1.25=186.25/-	186.25/2= 93.125

Conclusion:

Avg cost if replaced individually is 73.75/-

Whereas if replaced after every week will cost Rs 87.5/- and if replaced after two weeks it will cost Rs 93.125/-

So prefer individual replacement.

Problem 19:

The following mortality rates have been observed for a certain type of light bulbs

Weeks	0	1	2	3	4	5
% of bulbs surviving by the end of week	100	90	75	50	20	0
	1	0.9	0.75	0.5	0.2	0

It costs Rs. 2/- to replace an individual bulb which has burnt out. If all bulbs are replaced simultaneously the cost would e reduced by Rs 1.5/- per bulb. Determine the optimum replacement policy.

Solution:

Week	1	2	3	4	5	6
Prob. of failure cumulative	0	0.1	0.25	0.5	0.8	1
Prob. of failure in each month	0	0.1	0.15	0.25	0.3	0.2

Assumption: - to solve the problem we assume that the no of bulbs that fail during the week are replaced at the end of the week.

Let N_i denote the no of replacements made at the end of the ith week. Let N_0 is initially the no. of Items in the system.

Let N_0 =100

$N_1 = N_0 XP_1$

= 100X0

$N_1 = 0$

$N_2 = N_0 XP_2 + N_1 XP_1$

= 100 X 0.1 + 0x0

= 10

$N_3 = N_0 XP_3 + N_1 XP_2 + N_2 X P_1$

=100X0.15 + 0X0.1 +10X0

= 15

$N_4 = N_0 XP_4 + N_1 XP_3 + N_2 X P_2 + N_3$

XP_1 =100x0.25+0x0.15+1x0.1+15x0

=26

Expected life of the tubes

=1x0+2x0.1+3x0.15+4x0.25+5x0.3+6x0.2

=4.35 weeks

Avg. no failures per week = No. of items in s/m initially/expected life = 100/4.35

=22.98 ≈23 bulbs

cost of individual replacement bulb = Rs 2/-. Total cost of individual replacement per week =2x23 =46/-

Consider Group Replacement

End of the week	Total cost of group replacement	Average cost per week
1	100x0.5+0x2 =50/-	50/1= 50
2	100x0.5+0x2+10x2=70/-	70/2= 35
3	100x0.5+0X2+10x2+15x2=100	100/3= 33.33
4	100x0.5+0X2+10x2+15x2+26x2=152	152/4= 38

Conclusion: prefer group replacement once in three months where the avg cost/week =Rs 33.33/-. In case of individual replacement, the avg cost / week =46/-

So prefer group replacement

Problem 20:-

Supplies of bins are located throughout a plant when bin is empty. The foreman calls to the store room and the fill it. There are 10 bins. Located over the plant it has been suggested that it might be cheap to routine.

Individual bins as required it is estimated that the time required to service a single bin costs Rs3/-. But, it would cost only Rs 10/- to service all bins at one trip. The probability of bins being empty is the function of time. It is recognized that an empty bin must be filled whenever this occurs. What is the best policy for supply.

Time since refill (in shift)	1	2	3	4
Prob. of being empty	0.1	0.2	0.5	1

Solution:

Given N_0 =10

Time since refills (shifts)	1	2	3	4
Prob. Of failure	0.1	0.1	0.3	0.5

Assumption: no of bins that are full during the shift are serviced at the end of rack shift

Let N_i = no of replacement at the end of i^{th}

shift $N_1 = N_0 XP_1$

$= 10x0.1$

$N_1 = 1$

$N_2 = N_0 XP_2 + N_1 XP_1$

$= 10 X 0.1 + 1x0.1$

$= 1.1$

$N_3 = N_0 XP_3 + N_1 XP_2 + N_2 X P_1$

$=10X0.3 + 1X0.1 + 1.1X0.1 = 3.21$ $N_4 =$
$N_0 XP_4 + N_1 XP_3 + N_2 X P_2 + N_3 XP_1$

$=10x0.5+1x0.3+1.1x0.1+3.21x0.1$

$=5.731$

Expected life of bins

$=1x0.1+2x0.1+3x0.3+4x0.5$

$=3.2$

Avg. no failures per shift = No. of items in s/m initially/expected life = 10/3.2 =3.125 ≈3

Avg cost of individual replacement per bin = Rs 3/-. Total cost of individual replacement per shift

=3x3 =9/-

Consider Group Replacement:

End of To	al cost of group replacement Average cos	t per shift the shift
1	10x1+1x3 =13/-	13/1=13
2	10x1+1x3+1.1x31=16.3/-	16.3/2=8.15
3	10x1+1X3+1.1x3+3.21x3=25.93	25.93/3=8.54

Conclusion: prefer group replacement once in two months where the avg cost/shift =Rs 8.15/-. In case of individual replacement, the avg cost / shift =9/-So prefer group replacement.

Problem 21:-

There is a large no. of light bulbs all of which must be kept in working order. If a bulb fails in service it costs Rs 1/- to replace it. But, if all bulb fails in service in the same operation. It costs only 35 paise a bulb. The proportion of bulb failing in successive time internals is known.

Weeks	1	2	3	4	5	6
Proportion of failing	0.09	0.16	0.24	0.36	0.12	0.03

Problem 22:-

The life of street bulbs is normally distributed about a mean of 6 weeks with a standard deviation of 1 week. The cost of replacing an individual bulb is Rs 6/- per bulb. A decision is made to replace all bulbs simultaneously at fixed intervals and to replace the individual bulb as they fail .

If the cost of group replacement is Rs 2.35/- per bulb, find out the optimum group

replacement policy? Solution:

$\mu=6$

$\sigma=1$

$\sigma\sqrt{2\pi}$

$Z= x - \mu / \sigma$

Week (x)	1	2	3	4	5	6	7	8	9
Z	-5	-4	-3	-2	-1	0	1	2	3
Prob. (cu)	0	0	0	0.023	0.159	0.5	0.84	0.971	1
Prob. In each week	0	0	0	0.023	0.136	0.341	0.34	0.131	0.029

Assumption:-

In this to solve the problem we assume that the street bulbs that fail during the week are replaced at the end of the week.

Let N_i =denote the no of replacement at the end week

Let N_0 =100

$N_1 = N_0 X P_1$

$= 100x00$

$N_1 = 0$

$N_2 = N_0 X P_2 + N_1 X P_1$

$= 100 X 0 + 0x0$

$= 0$

$N_3 = N_0 X P_3 + N_1 X P_2 + N_2 X P_1 = 0$

$N_4 = N_0\ XP_4 + N_1\ XP_3 + N_2\ X\ P_2 +$
$N_3\ XP_1 = 100x0.023+0+0+0+0$
$=13.6$

$N_5 = N_0\ XP_5 + N_1\ XP_4 + N_2\ X\ P_3 + N_3\ XP_2 + N_4\ XP_1$
$= 100x0.136+0+0+0+0$
$= 13.6$

$N_5 = N_0\ XP_5 + N_1\ XP_4 + N_2\ X\ P_3 + N_3\ XP_2 + N_4\ XP_1$
$+ N_5\ XP_0 = 100x0.341+0+0+0+0+0$
$=34.1$

Expected life of bins
=1x0+2x0+3x0+4x0.023+5x0.136+6x0.341+7x0.34+8x0.131+9x0.029
=6.507 weeks

Avg. no failures per week = No. of items in s/m initially/expected life = 100/6.507
=15.36 ≈ 15

Avg cost of individual replacement per bulb = Rs 6/-. Total cost of individual replacement per week =6x15 =Rs90/-

Consider Group Replacement:

End of the week	Total cost of group replacement	Average cost per week
1	100x2.35+0=235/-	235/1=2353
2	100x2.35+0+0=235	235/2=117.5
3	100x2.35+0+0+0=235/-	235/3=78.33
4	100x2.35+0+0+0+2.31x6=248.86	248.86/4=62.215
5	100x2.35+0+0+0+2.31x6+13.6x6=330.46	330.46/5=66.092

Conclusion: with reference to above data, prefer group replacement once in four weeks where the average cost is Rs 62.215/-. In case of individual replacement avg. cost is =Rs90/-

So prefer group replacement.

Problem 23:-

A trucking company has kept records on tyre life. Although failure is a random process it is a function of time. The following data taken from 1000 failure have been summarized.

Thousands of kms	No. of failures
10	50
20	100
30	250
40	400
50	200

A scheduled tyre change costs Rs 3,500 including the tyre a failure results in a loss of travel time and required use of emergency tools which takes even longer. Resulting, a net loss of Rs 5000. In addition, this ruins tyre body which has a value of Rs 100 what would the least cost policy for tyre change?

Problem 24:-

The following failure rates have been observed for a certain type of light bulbs

Weeks	1	2	3	4	5
% failing by end of week	10	25	50	30	100

There are 1000 bulbs in use and its cost Rs2/- to replace an individual bulb which has burnt out. If all bulbs were replaced simultaneously, it cost 50 paise per bulb. It is proposed to replace all bulbs at fixed intervals at what intervals should all the bulbs be replaced?

At what group replacement price/bulb would a policy of strictly individual replacement becomes preferable to the adopted policy.

[end of 2^{nd} week, GR Rs 510/- > Rs 076/ bulb]

Reference Books:

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