# LECTURE NOTES

## ON

# STRUCTURAL ANALYSIS – II (A60131)

## III B. Tech - II Semester (JNTUH-R15)

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**JNTU Hyderabad - III Year B.Tech.**

**CE-II Sem (A60131) Structural**

**Analysis – II**

**SYLLABUS**

( L-T-P/D 4-0-0)

**UNIT – I**

**MOMENT DISTRIBUTION METHOD** – Analysis of single bay - single storey portal frames including side sway. Analysis of inclinedframes

**KANI’S METHOD:** Analysis of continuous beams including settlement of supports. Analysis of single bay single storey and single bay two storey frames by Kani’s method including side sway. Shear force and bending moment diagrams. Elastic curve.

**UNIT – II**

**SLOPE DEFLECTION METHOD** – Analysis of single bay - single storey portal frames by slope deflection method including side sway. Shear force and bending moment diagrams. Elastic curve.

**TWO HINGED ARCHES:** Introduction – Classification of two hinged arches – Analysis of two hinged parabolic arches – secondary stresses in two hinged arches due to temperature and elastic shortening of rib.

**UNIT-III**

**APPROXIMATE METHODS OF ANALYSIS:** Analysis of multi-storey frames for lateral loads: Portal method, Cantilever method and Factor method. Analysis of multi-storey frames for gravity (vertical) loads. Substitute frame method. Analysis of Mill bends.

**UNIT –IV**

**MATRIX METHODS OF ANALYSIS:** Introduction - Static and Kinematic Indeterminacy - Analysis of continuous beams including settlement of supports, using Stiffness method. Analysis of pin-jointed determinate plane frames using stiffness method – Analysis of single bay single storey frames including side sway, using stiffness method. Analysis of continuous beams up to three degree of indeterminacy using flexibility method. Shear force and bending moment diagrams. Elastic curve.

**UNIT – V**

**INFLUENCE LINES FOR INDETERMINATE BEAMS:** Introduction – ILD for two span continuous beam with constant and variable moments of inertia. ILD for propped cantilever beams.

**INDETERMINATE TRUSSES:** Determination of static and kinematic indeterminacies – Analysis of trusses having single and two degrees of internal and external indeterminacies – Castigliano’s second theorem.

**JNTU RECOMMENDED TEXT BOOKS:**

1. Structural Analysis Vol – I & II by Vazrani and Ratwani, KhannaPublishers
2. Structural Analysis Vol – I & II by Pundit and Gupta, Tata McGraw HillPublishers
3. Structural Analysis SI edition by AslamKassimali, Cengage Learning Pvt.Ltd

**JNTU RECOMMENDED REFERENCES:**

1. Matrix Analysis of Structures by Singh, Cengage Learning Pvt.Ltd
2. Structural Analysis byHibbler
3. Basic Structural Analysis by C.S. Reddy, Tata McGraw HillPublishers
4. Matrix Analysis of Structures by Pundit and Gupta , Tata McGraw HillPublishers
5. Advanced Structural Analysis by A.K. Jain, NemChandBros.
6. Structural Analysis – II by S.S. Bhavikatti, Vikas Publishing House Pvt.Ltd.

# UNIT I
# ANALYSIS OF PLANE FRAMES

## MOMENT DISTRIBUTION METHOD

### MOMENT DISTRIBUTION METHOD

This method of analyzing beams and frames was developed by Hardy Cross in 1930. Moment distribution method is basically a displacement method of analysis. But this method side steps the calculation of the displacement and instead makes it possible to apply a series of converging corrections that allow direct calculation of the end moments. This method of consists of solving slope deflection equations by successive approximation that may be carried out to any desired degree of accuracy. Essentially, the method begins by assuming each joint of a structure is fixed. Then by unlocking and locking each joint in succession, the internal moments at the joints are distributed and balanced until the joints have rotated to their final or nearly final positions. This method of analysis is both repetitive and easy to apply. Before explaining the moment distribution method certain definitions and concepts must beunderstood.

**Sign convention:** In the moment distribution table clockwise moments will be treated+ve and anticlockwise moments will be treated –ve. But for drawing BMD moments causing concavity upwards (sagging) will be treated +ve and moments causing convexity upwards (hogging) will be treated –ve.

**Fixed end moments:** The moments at the fixed joints of loaded member are called fixedend moment. FEM for few standards cases are given in previous chapter.

**Distribution factors:** If a moment 'M' is applied to a rigid joint 'o', as shown in figure, theconnecting members will each supply a portion of the resisting moment necessary to satisfy moment equilibrium at the joint. Distribution factor is that fraction which when multiplied with applied moment 'M' gives resisting moment supplied by the members. To obtain itsvalue imagine the joint is rigid joint connected to different members. If applied moment M cause the joint to rotate an amount '$\theta$' , then each member rotates by same amount.

From equilibrium requirement

$M = M_1 + M_2 + M_3 + \ldots\ldots$

$= K_1\theta + K_2\theta + K_3\theta = \theta\sum K$

$DF_1 = \frac{M1}{M} = \frac{K_1\theta}{\theta\sum K} = \frac{K_1}{\sum K}$

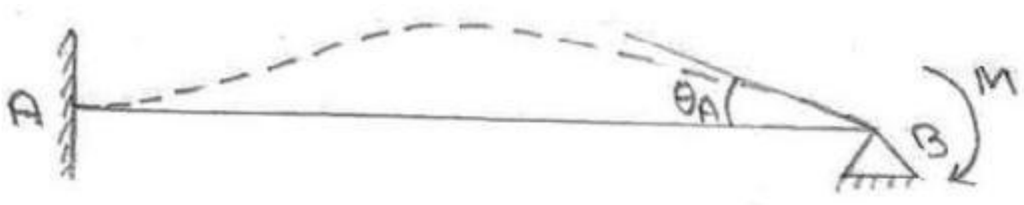
In general $DF = \frac{K}{\sum K}$ ....... .(6)

**Member relative stiffness factor:** In majority of the cases continuous beams and frames willbe made from the same material so that their modulus of electricity E will be same for all members. It will be easier to determine member stiffness factor by removing term 4E & 3E from equation (4) and (5) then will be called as relative stiffness factor.

$K_r = \frac{I}{L}$ for far end fixed

$K_r = \frac{3I}{4L}$ for far end hinged

**Carry over factors:** Consider the beam shown infigure




We have shown that

$$M = \frac{4EI}{L}\theta_A, R_B = \frac{3M}{2L}$$

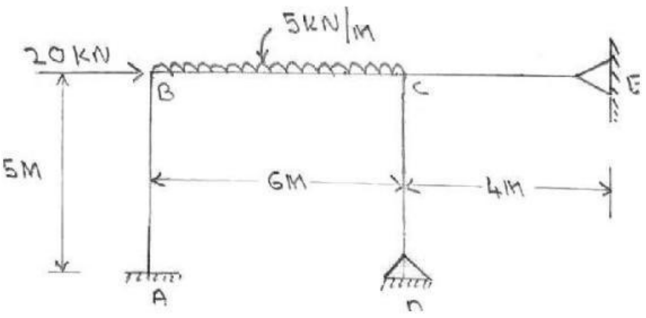
BM at A $\left(EI\frac{d^2y}{dx^2}\right)_{at\ x=L} = [(3M/2L)x - M]_{x=L} = \frac{M}{2}$

+ve BM of  at A indicates clockwise moment of  at A. In other words the moment 'M' at the pin induces a moment of  at the fixed end. The carry over factor represents the fraction of M that is carried over from hinge to fixed end. Hence the carry over factor for the caseof far end fixed is +. The plus sign indicates both moments are in the samedirection.

## MOMENT DISTRIBUTION FOR FRAMES WITH NO SIDE SWAY

The analysis of such a frame when the loading conditions and the geometry of the frame is such that there is no joint translation or sway, is similar to that given for beams.

Q. Analysis the frame shown in figure by moment distribution method and draw



BMD assume EI isconstant.

FEMS

$$M_{FAB=}\ M_{FBA=}\ M_{FCD=}\ M_{FDC=}\ M_{FCE=}\ M_{FEC=}\ 0$$

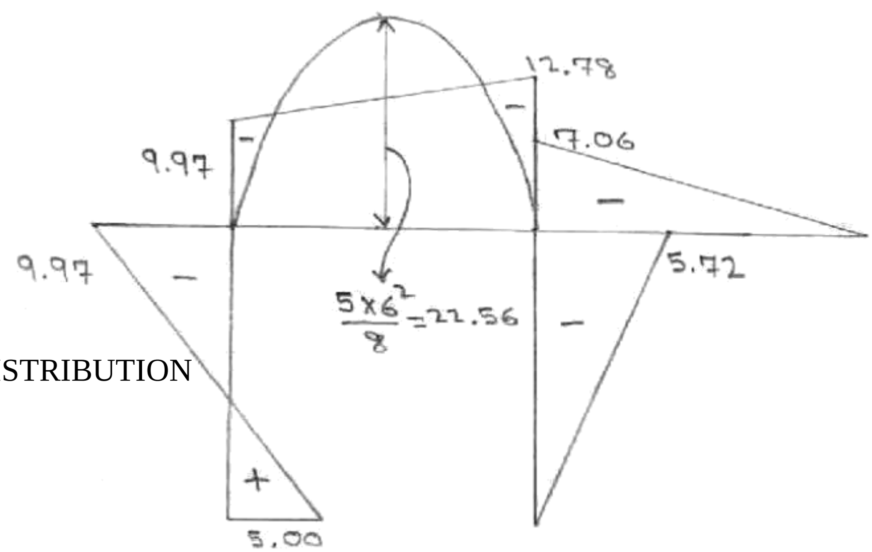
$$M_{FBC=} -\frac{5\times 6^2}{12} = -15KNm$$

$$M_{FCB=} \frac{5\times 6^2}{12} = 15KNm$$

| Jt. | Member | Relative stiffness (K) | ΣK | DF = $\frac{K}{\sum K}$ |
|---|---|---|---|---|
| B | BA | I/5 | $\frac{11}{30}I$ | 0.55 |
| | BC | I/6 | | 0.45 |
| C | CB | I/6 = 0.17 I | 0.51 I | 0.33 |
| | CD | $\frac{3}{4}$ I/5 = 0.15 I | | 0.3 |
| | CE | $\frac{3}{4} x \frac{I}{4}$ = 0.19 I | | 0.37 |

MOMENT DISTRIBUTION

| Jt | A | | B | | C | | D | E |
|---|---|---|---|---|---|---|---|---|
| Member | AB | BA | BC | CB | CD | CE | DC | EC |
| D.F | 0 | 0.55 | 0.45 | 0.33 | 0.3 | 0.37 | 1 | 1 |
| FEM | 0 | 0 | -15 | +15 | 0 | 0 | 0 | 0 |
| Balance | | 8.25 | 6.75 | -4.95 | -4.5 | -5.55 | | |
| C.O | 4.13 | | -2.48 | 3.38 | | | | |
| Balance | | 1.36 | 1.12 | -1.12 | -1.01 | -1.25 | | |
| C.O | 0.68 | | 0.56 | 0.56 | | | | |
| Balance | | 0.31 | 0.25 | -0.18 | -0.17 | -0.21 | | |
| C.O | 0.16 | | -0.09 | 0.13 | | | | |
| Balance | | 0.05 | 0.04 | -0.04 | -0.04 | -0.05 | | |
| C.O | 0.03 | | | | | | | |
| **Final moments** | **5** | **9.97** | **-9.97** | **12.78** | **-5.72** | **-7.06** | **0** | **0** |

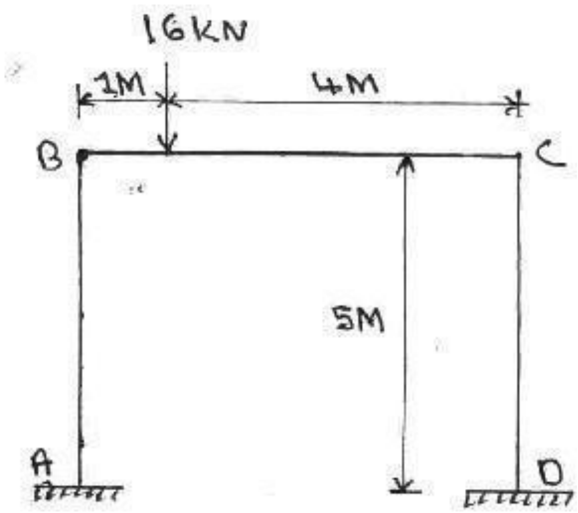



DISTRIBUTION

## MOMENT DISTRIBUTION METHOD FOR FRAMES WITH SIDE SWAY

Frames that are non symmetrical with reference to material property or geometry (different lengths and I values of column) or support condition or subjected to non-symmetrical loading have a tendency to side sway.

Analyze the frame shown in figure by moment distribution method. Assume EI is constant.




**A. Non SwayAnalysis:**

First consider the frame without side sway

$M_{FAB} = M_{FBA} = M_{FCD} = 0$

$M_{FBC} = -\frac{16\times1\times4^2}{5^2} = -10.24KNm$

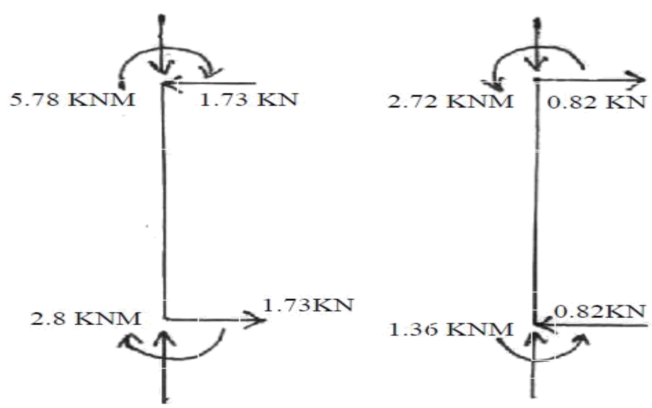
$M_{FCB} = \frac{16\times4\times1^2}{5^2} = 2.56KNm$

DISTRIBUTION FACTOR

| Jt. | Member | Relative stiffness K | ΣK | $DF = \frac{K}{\sum K}$ |
|---|---|---|---|---|
| B | BA<br>BC | I/5 = 0.2 I<br>I/5 = 0.2 I | 0.4 I | 0.5<br>0.5 |
| C | CB<br>CD | I/5 = 0.2I<br>I/5 = 0.2I | 0.4I | 0.5<br>0.5 |

DISTRIBUTION OF MOMENTS FOR NON-SWAY ANALYSIS

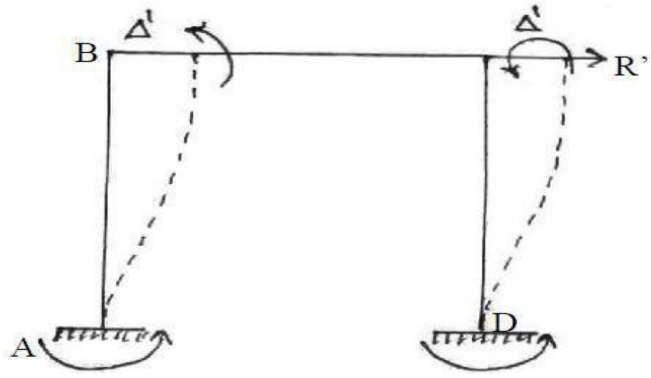
| Joint | A | | B | | C | D |
|---|---|---|---|---|---|---|
| Member | AB | BA | BC | CB | CD | DC |
| D.F | 0 | 0.5 | 0.5 | 0.5 | 0.5 | 0 |
| FEM | 0 | | -10.24 | 2.56 | 0 | 0 |
| Balance | | 5.12 | 5.12 | -1.28 | -1.28 | |
| CO | 2.56 | | -0.64 | 2.56 | | -0.64 |
| Balance | | 0.32 | 0.32 | -0.08 | -0.08 | |
| CO | 0.16 | | -0.64 | 0.16 | | -0.64 |
| Balance | | 0.32 | 0.32 | -0.08 | -0.08 | |
| C.O | 0.16 | | -0.04 | 0.16 | | -0.04 |
| Balance | | 0.02 | 0.02 | -0.08 | -0.08 | |
| C.O | 0.01 | | | | | -0.04 |
| **Final moments** | **2.89** | **5.78** | **-5.78** | **2.72** | **-2.72** | **-1.36** |




By seeing of the FBD of columns R = 1.73 – 0.82

(Using $F_x$ =0 for entire frame) = 0.91 KN ←

Now apply R = 0.91 KN acting opposite as shown in the above figure for the sway analysis. Sway analysis: For this we will assume a force R' is applied at C causing the frame to deflect as shown in the following figure.




Since both ends are fixed, columns are of same length & I and assuming joints B & C are temporarily restrained from rotating and resulting fixed end moment are

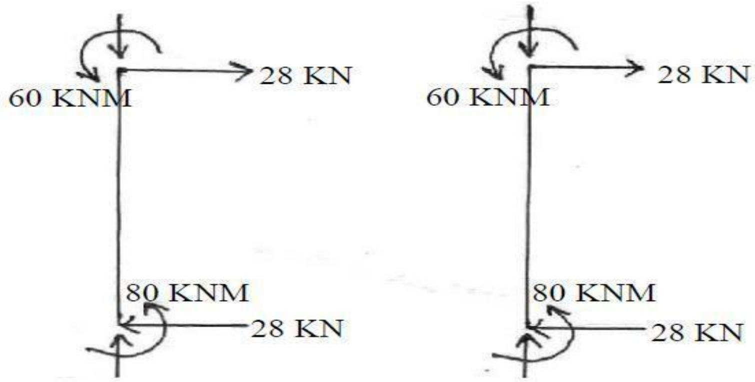
Assume

$$M'_{AB} = M'_{BA} = M'_{CD} = M'_{DC} = \frac{6EI}{L^2}\Delta$$
$$M'_{BA} = -100KNm$$
$$M'_{AB} = M'_{CD} = M'_{DC} = -100KNm$$

**Moment distribution table for sway analysis:**

| Joint | A | B | | C | | D |
|---|---|---|---|---|---|---|
| Member | AB | BA | BC | CB | CD | DC |
| D.F | 01 | 0.5 | 0.5 | 0.5 | 0.5 | 0 |
| FEM | −100 | −100 | 0 | 0 | −100 | −100 |
| Balance | | 50 | 50 | 50 | 50 | |
| CO | 25 | | 25 | 25 | | 25 |
| Balance | | −12.5 | −12.5 | −12.5 | −12.5 | |
| CO | -6.25 | | −6.25 | −6.25 | | −6.25 |
| Balance | | 3.125 | 3.125 | 3.125 | 3.125 | |
| C.O | 1.56 | | 1.56 | 1.56 | | 1.56 |
| Balance | | −0.78 | −0.78 | −0.78 | −0.78 | |
| C.O | -0.39 | | −0.39 | −0.39 | | 0.39 |
| Balance | | 0.195 | 0.195 | 0.195 | 0.195 | |
| C.O | 0.1 | | | | | 0.1 |
| **Final moments** | **- 80** | **- 60** | **60** | **60** | **– 60** | **– 80** |

Free body diagram of columns



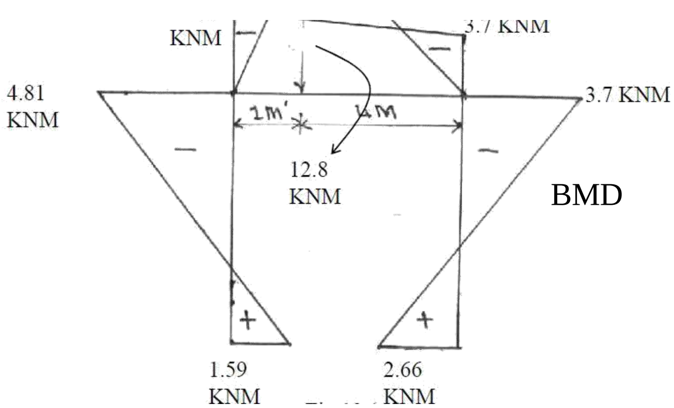

Using F,x = 0 for the entire frame R = 28 + 28 = 56 KN

Hence R' = 56KN creates the sway moments shown in above moment distribution table. Corresponding moments caused by R = 0.91KN can be determined by proportion. Thus final moments are calculated by adding non sway moments and sway.

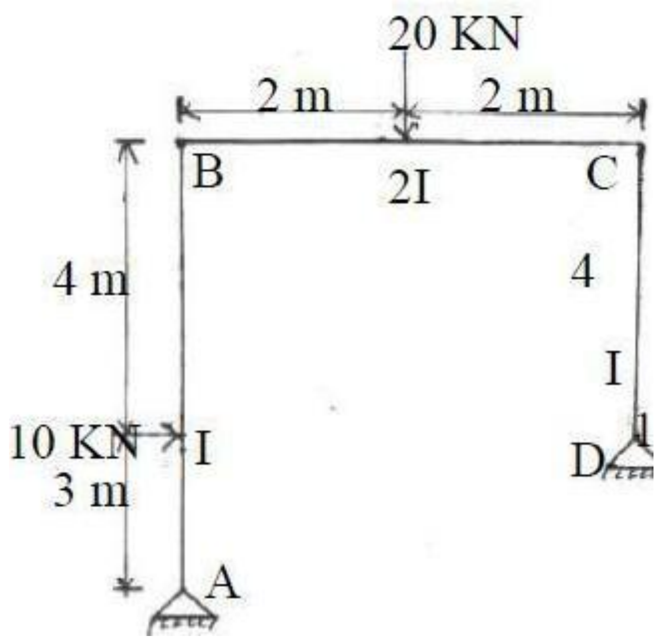
Moments calculated for R = 0.91KN, as shown below.

$$M_{AB} = 2.89 + \frac{0.91}{56}(-80) = 1.59KNm$$
$$M_{BA} = 5.78 + \frac{0.91}{56}(-60) = 4.81KNm$$

$$M_{BC} = -5.78 + \frac{0.91}{56}(60) = -4.81KNm$$
$$M_{CB} = 2.72 + \frac{0.91}{56}(60) = 3.7KNm$$
$$M_{CD} = -2.72 + \frac{0.91}{56}(-60) = -3.7KNm$$
$$M_{DC} = -1.36 + \frac{0.91}{56}(-80) = -2.66KNm$$



5.Q. Analysis the rigid frame shown in figure by moment distribution method and draw BMD




**A. Non SwayAnalysis:**

First consider the frame held from side sway

FEMS

$$M_{FAB=} -\frac{10 \times 3 \times 4^2}{7^2} = -9.8KNm$$

$$M_{FBA=} \frac{10 \times 4 \times 3^2}{7^2} = 7.3KNm$$

$$M_{FBC=} -\frac{20 \times 4}{8} = -10KNm$$

$$M_{FCB=} \frac{20 \times 4}{8} = 10KNm$$

$$M_{FCD=} M_{FDC=} 0$$

DISTRIBUTION FACTOR

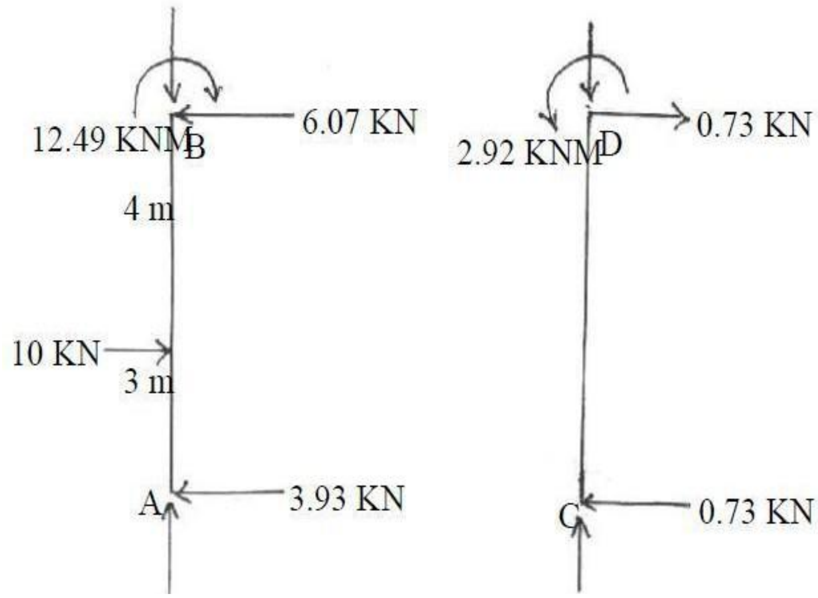
| Joint | Member | Relative stiffness k | Σk | DF = $\frac{K}{\sum K}$ |
|---|---|---|---|---|
| B | BA | $\frac{3}{4}\times\frac{I}{7} = 0.11I$ | 0.61 I | 0.18 |
| | BC | 2I/4 = 0.5I | | 0.82 |
| C | CB | 2I/4 = 0.5I | | 0.72 |
| | CD | $\frac{3}{4}\times\frac{I}{4} = 0.19\ I$ | 0.69 I | 0.28 |

DISTRIBUTION OF MOMENTS FOR NON-SWAY ANALYSIS

| Joint | A | B | | C | | D | |
|---|---|---|---|---|---|---|---|
| Member | AB | BA | BC | CB | CD | DC | |
| D.F | 1 | 0.18 | 0.82 | 0.72 | 0.28 | 1 | |
| FEM | -9.8 | 7.3 | -10 | 10 | 0 | 0 | |
| Release jt. 'D' | +9.8 | | | | | | |
| CO | | 4.9 | | | | | |
| Initial moments | 0 | 12.2 | -10 | 10 | 0 | 0 | |
| Balance | | -0.4 | -1.8 | -7.2 | -2.8 | | |
| CO | | | -3.6 | -0.9 | | | |
| Balance | | 0.65 | 2.95 | 0.65 | 0.25 | | |
| C.O | | | 0.33 | 1.48 | | | |
| Balance | | -0.06 | -0.27 | -1.07 | -0.41 | | |
| C.O | | | -0.54 | -0.14 | | | |
| Balance | | 0.1 | 0.44 | 0.1 | 0.04 | | |
| **Final moments** | **0** | **12.49** | **-12.49** | **2.92** | **-2.92** | **0** | |

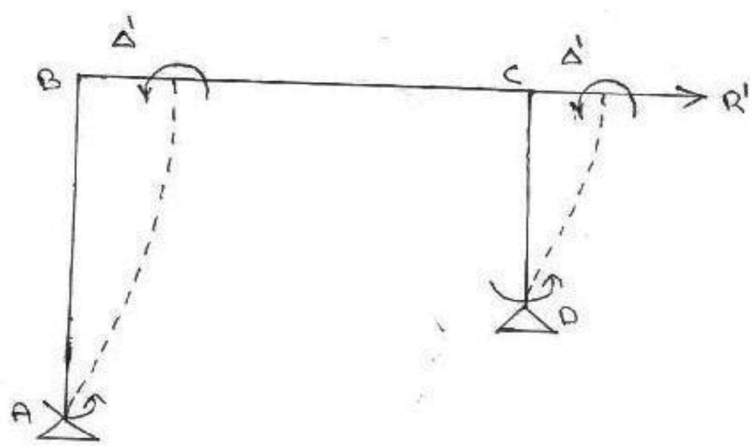
FREE BODY DIAGRAM OF COLUMNS

Applying Fx = 0 for frame as a Whole, R = 10 – 3.93 – 0.73 = 5.34 KN ←




Now apply R = 5.34KN acting opposite

**Sway analysis:** For this we will assume a force R' is applied at C causing the frame to deflect as shown in figure



Since ends A & D are hinged and columns AB & CD are of different lengths

$$M'_{BA} = -\frac{3EI}{L_1^2}\Delta', \quad M'_{CD} = -\frac{3EI}{L_2^2}\Delta',$$

$$\frac{M'_{BA}}{M'_{CD}} = \frac{\frac{3EI}{L_1^2}\Delta'}{\frac{3EI}{L_2^2}\Delta'} = \frac{L_2^2}{L_1^2} = \frac{4^2}{7^2} = \frac{16}{49}$$

Assume

$$M'_{BA} = -16KNm, M'_{AB} = 0$$
$$M'_{CD} = -49KNm, \; M'_{DC} = 0$$

MOMENT DISTRIBUTION FOR SWAY ANALYSIS

| Joint | A | B | | C | | D |
|---|---|---|---|---|---|---|
| Member | AB | BA | BC | CB | CD | DC |
| D.F | 1 | 0.18 | 0.82 | 0.72 | 0.28 | 1 |
| FEM | 0 | -16 | 0 | 0 | -49 | 0 |
| Balance | | 2.88 | 13.12 | 35.28 | 13.72 | |
| CO | | | 17.64 | 6.56 | | |
| Balance | | -3.18 | -14.46 | -4.72 | -1.84 | |
| CO | | | -2.36 | -7.23 | | |
| Balance | | 0.42 | 1.94 | 5.21 | 2.02 | |
| C.O | | | 2.61 | 0.97 | | |
| Balance | | -0.47 | -2.14 | -0.7 | -0.27 | |
| C.O | | | 0.35 | -1.07 | | |
| Balance | | 0.06 | 0.29 | 0.77 | 0.3 | |
| C.O | | | 0.39 | 0.15 | | |
| Balance | | -0.07 | -0.32 | -0.11 | -0.04 | |
| **Final moments** | **0** | **-16.36** | **16.36** | **35.11** | **-35.11** | **0** |

Using Fx = 0 for the entire frame R' = 11.12 kN ⟶

Hence R' = 11.12 KN creates the sway moments shown in the above moment distribution table. Corresponding moments caused by R = 5.34 kN can be determined by proportion. Thus final moments are calculated by adding non-sway moments and sway moments determined for R = 5.34 KN as shown below.

$M_{AB} = 0$

$$M_{BA} = 12.49 + \frac{5.34}{11.12}(-16.36) = 4.63KNm$$

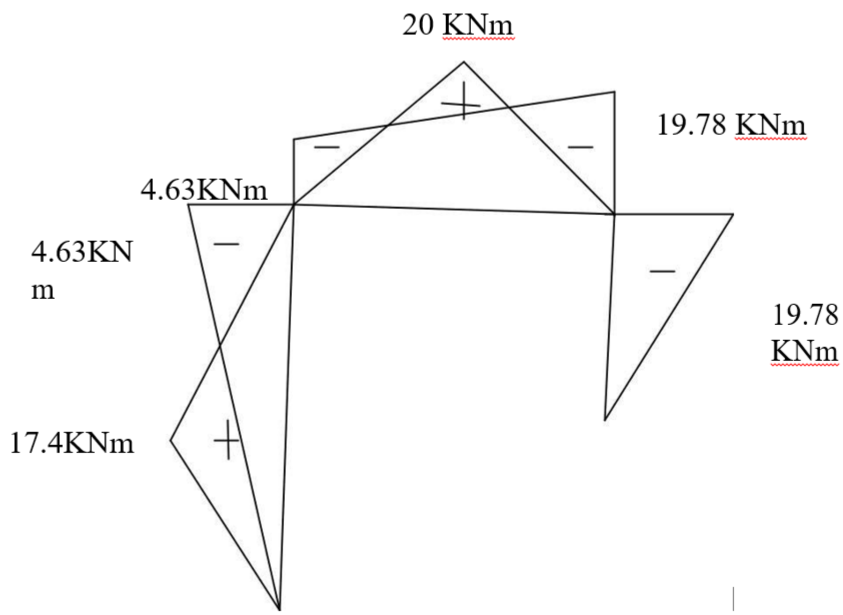
$$M_{BC} = -12.49 + \frac{5.34}{11.12}(16.36) = -4.63KNm$$

$$M_{CB} = 2.92 + \frac{5.34}{11.12}(35.11) = 19.78KNm$$

$$M_{CD} = -2.92 + \frac{5.34}{11.12}(-35.11) = -19.78KNm$$

$M_{DC} = 0$

20 KNm

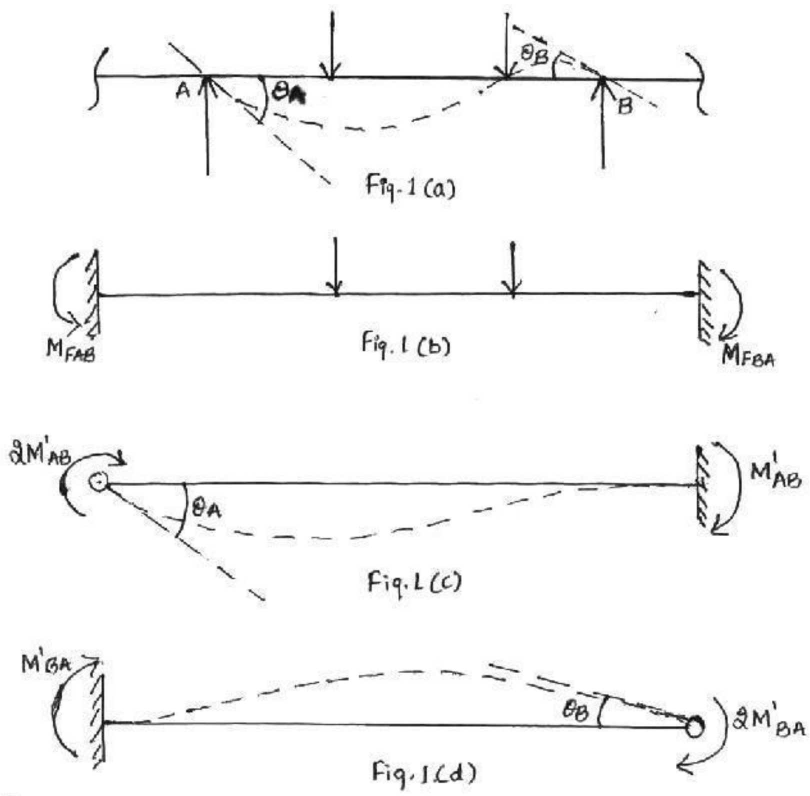



B.M.D

## PART B : KANI'S METHOD OF ANALYSIS

This method was developed by Dr. Gasper Kani of Germany in 1947. This method offers an iterative scheme for applying slope deflection method. We shall now see the application of Kani's method for different cases.

### BEAMS WITH NO TRANSLATION OF JOINTS:




Let AB represent a beam in a frame, or a continuous structure under transverse loading, as show in fig. 1 (a) let the $M_{AB}$ & $M_{BA}$ be the end moment at ends A & B respectively.

**Sign convention used will be:** clockwise moment +ve and anticlockwise moment –ve.

The end moments in member AB may be thought of as moments developed due to a superposition of the following three components of deformation.

Thus the final moment $M_{AB}$ & $M_{BA}$ can be expressed as super position of three moments

$$\left.\begin{aligned} M_{AB} &= M_{FAB} + 2M'_{AB} + M'_{BA} \\ M_{BA} &= M_{FBA} + 2M'_{BA} + M'_{AB} \end{aligned}\right\} \qquad \text{.........(1)}$$

For member AB we refer end 'A' as near end and end 'B' as far end. Similarly when we refer to moment $M_{BA}$, B is referred as near end and end A as far end.

1. The member 'AB' is regarded as completely fixed. (Fig. 1 b). The fixed end moments for this condition are written as $M_{FAB}$ & $M_{FBA}$, at ends A & B respectively.
2. The end A only is rotated through an angle $\theta_A$ by a moment $2M'_{AB}$ inducing a moment $M'_{AB}$ at fixed end B.
3. Next rotating the end B only through an angle $\theta_B$ by moment $2M'_{BA}$ while keeping end 'A' as fixed. This induces a moment $M'_{BA}$ at end A.

Hence above equations can be stated as follows. The moment at the near end of a member is the algebraic sum of (a) fixed end moment at near end. (b) Twice the rotation moment of the near end (c) rotation moment of the far end.

**Rotation factors:**

Fig. 2 shows a multistoried frame.

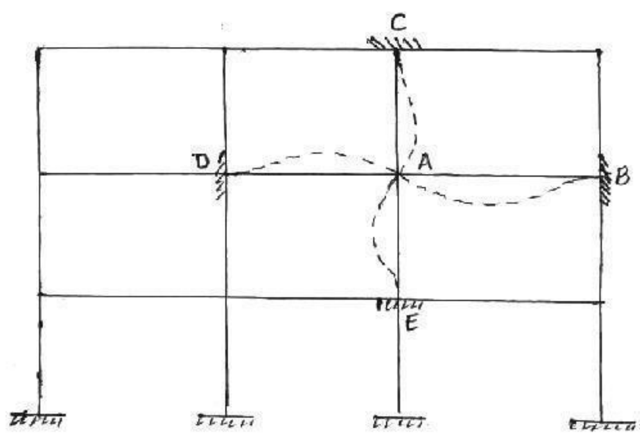



Fig. 2

Consider various members meeting at joint A. If no translations of joints occur, applying equation (1), for the end moments at A for the various members meeting at A are given by:

$$M_{AB} = M_{FAB} + 2M'_{AB} + M'_{BA}$$
$$M_{AC} = M_{FAC} + 2M'_{AC} + M'_{CA}$$
$$M_{AD} = M_{FAD} + 2M'_{AD} + M'_{DA}$$
$$M_{AE} = M_{FAE} + 2M'_{AE} + M'_{EA}$$

For equilibrium of joint A, $\Sigma M_A = 0$

$$\therefore \Sigma M_{FAB} + 2\Sigma M'_{AB} + \Sigma M'_{BA} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where ,

$\Sigma M_{FAB}$ =Algebraic sum of fixed end moments at A of all members meeting at A.

$\Sigma M'_{AB}$ = Algebraic sum of rotation moments at A of all member meeting at A.

$\Sigma M'_{BA}$ = Algebraic sum of rotation moments of far ends of the members meeting at A.

from equation (2)

$$\Sigma M'_{AB} = -\frac{1}{2}\left[\sum M_{FAB} + \sum M'_{BA}\right] \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

We know that $2M'_{AB} = \dfrac{4EI_{AB}}{L_{AB}}\theta_A = 4EK_{AB}\,\theta_A$

Where $K_{AB} = \dfrac{I_{AB}}{L_{AB}}$, relative stiffness of member AB

$$M'_{AB} = 2E\,K_{AB}\,\theta_A \quad \ldots\ldots\ldots\ldots\ldots(4)$$

$\therefore\ \Sigma M'_{AB} = 2E\theta_A\,\Sigma K_{AB}$ ……………(5) (At rigid joint A all the members undergo same rotation $\theta_A$ )

Dividing Equation (4)/(5) gives

$$\frac{M'_{AB}}{\Sigma M'_{AB}} = \frac{K_{AB}}{\Sigma K_{AB}}$$

$$\therefore\ M'_{AB} = \frac{K_{AB}}{\Sigma K_{AB}}\Sigma M'_{AB} \quad \ldots\ldots\ldots\ldots\ldots(5)$$

Substituting value of $\Sigma M'_{AB}$ from (3) in (5)

$$M'_{AB} = \left(-\frac{1}{2}\right)\frac{K_{AB}}{\Sigma K_{AB}}\left[\Sigma M_{FAB} + \Sigma M'_{BA}\right]$$

$$= U_{AB}\left[\Sigma M_{FAB} + \Sigma M'_{BA}\right] \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

where $U_{AB} = -\dfrac{1}{2}\,\dfrac{K_{AB}}{\Sigma K_{AB}}$ is called as rotation factor for member AB at joint A.

**Analysis Method:**

In equation (6) $\Sigma M_{FAB}$ is a known quantity. To start with the far end rotation moments $M'_{BA}$ are not known and hence they may be taken as zero. By a similar approximation the rotation moments at other joints are also determined. With the approximate values of rotation moments computed, it is possible to again determine a more correct value of the rotation moment at A from member AB using equation (6).

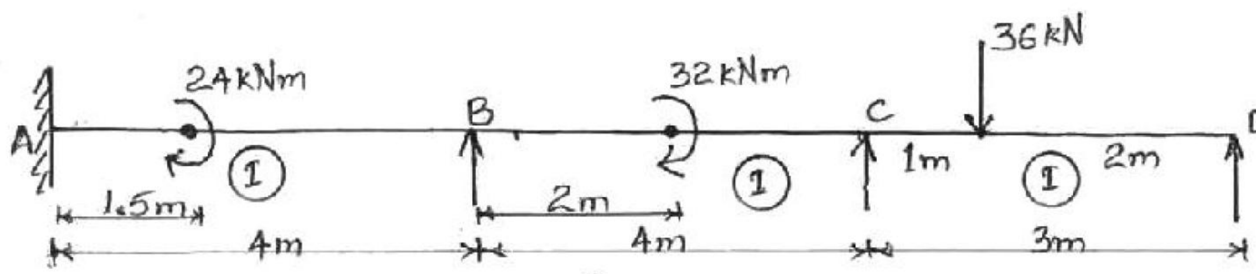
The process is carried out for sufficient number of cycles until the desired degree of accuracy is achieved.

The final end moments are calculated using equation (1).

**EXAMPLES:**

**Ex.1:**

Analyze the continuous beam shown in fig.




**Solution:**

a) <u>Fixed end moments:</u>

$$M_{FAB} = \frac{b(3a-l)}{l^2}M_o = \frac{2.5\,(3\times1.5-4)}{4^2}\times 24 = 1.88 \text{ kNm}$$

$$M_{FBA} = \frac{a(3b-l)}{l^2}M_o = \frac{1.5(3\times2.5-4)}{4^2}\times 24 = 7.88 \text{ kNm}$$

$$M_{FBC} = \frac{M_o}{4} = \frac{32}{4} = 8 \text{ kNm}$$

$$M_{FCB} = \frac{M_o}{4} = 8 \text{ kNm}$$

$$M_{FCD} = -\frac{36\times1\times2^2}{3^2} = -16 \text{ kNm}$$

$$M_{FDC} = \frac{36\times1^2\times2}{3^2} = 8 \text{ kNm}$$

b) Modification in fixed end moments:

Actually end 'D' is a simply supported. Hence moment at D should be zero. To make moment at D as zero apply –8 kNm at D and perform the corresponding carry over to CD.

Modified $M_{FDC} = 8 - 8 = 0$

Modified $M_{FCD} = -16 + \frac{1}{2}(-8) = -20$ kNm

Now joint D will not enter the iteration process.

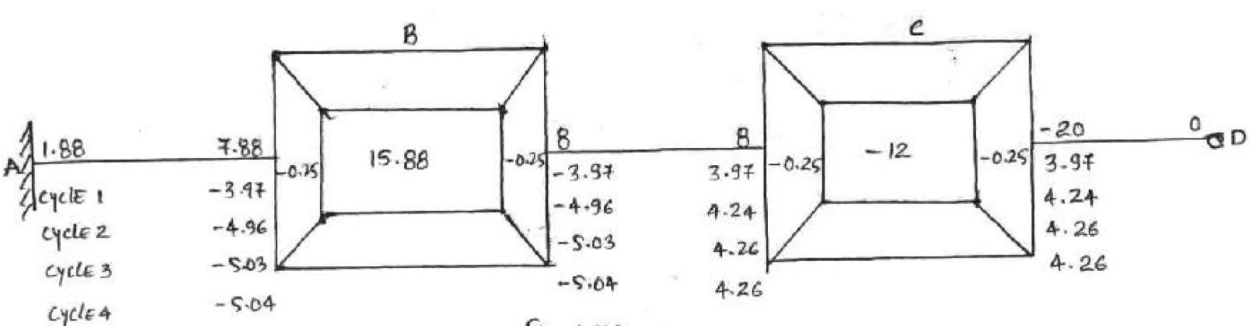
c) Rotation Factors:

| Joint | Member | Relative stiffness (K) | ΣK | Rotation Factor $U = -\frac{1}{2} \times \frac{K}{\sum K}$ |
|---|---|---|---|---|
| B | BA | I/4 = 0.25I | 0.5 I | – 0.25 |
| | BC | I/4 = 0.25I | | –0.25 |
| C | CB | I/4 = 0.25I | | –0.25 |
| | CD | $\frac{3}{4} \times \frac{I}{3} = 0.25I$ | 0.5I | –0.25 |

d) Sum of fixed end moments at joints:

$\Sigma M_{FB} = 7.88 + 8 = 15.88$ kNm

$\Sigma M_{FC} = 8 - 20 = -12$kNm



e) Iteration process

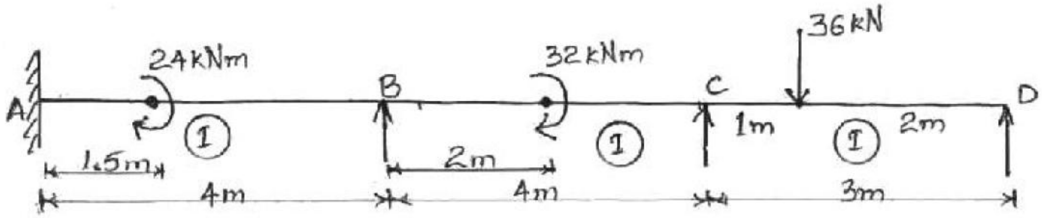
| Joint | B | | C | |
|---|---|---|---|---|
| Rotation Contribution | $M'_{BA}$ | $M'_{BC}$ | $M'_{CB}$ | $M'_{CD}$ |
| Rotation factor | –0.25 | –0.25 | –0.25 | –0.25 |
| Iteration 1 started at B assuming $M'_{CB}$ = 0 & taking $M'_{AB}$ = 0′ $M'_{DC}$ = 0. | –0.25 × (15.88 + 0 + 0) = -3.97 | –0.25 × (15.88 + 0 + 0) = -3.97 | –0.25 × (-12 –3.97 + 0) = 3.97 | –0.25 × (-12 –3.97 + 0) = 3.97 |
| Iteration 2 | –0.25 × (15.88 + 0 + 3.97) = –4.96 | –0.25 × (15.88 + 0 + 3.97) = – 4.96 | –0.25 × (-12 –4.96 + 0) = 4.24 | –0.25 (–12 –4.96 + 0) = 4.24 |
| Iteration 3 | –0.25 (15.88 + 0 + 4.24) = –5.03 | –0.25 (15.88 + 0 + 4.24) = –5.03 | –0.25 × (-12 –5.03+ 0) = 4.26 | –0.25 (–12 –5.03 + 0) = 4.26 |
| Iteration 4 | –0.25 (15.88 + 0 + 4.26) = –5.04 | –0.25 (15.88 + 0 + 4.26) = –5.04 | –0.25 × (-12 –5.03+ 0) = 4.26 | –0.25 (–12 –5.03 + 0) = 4.26 |

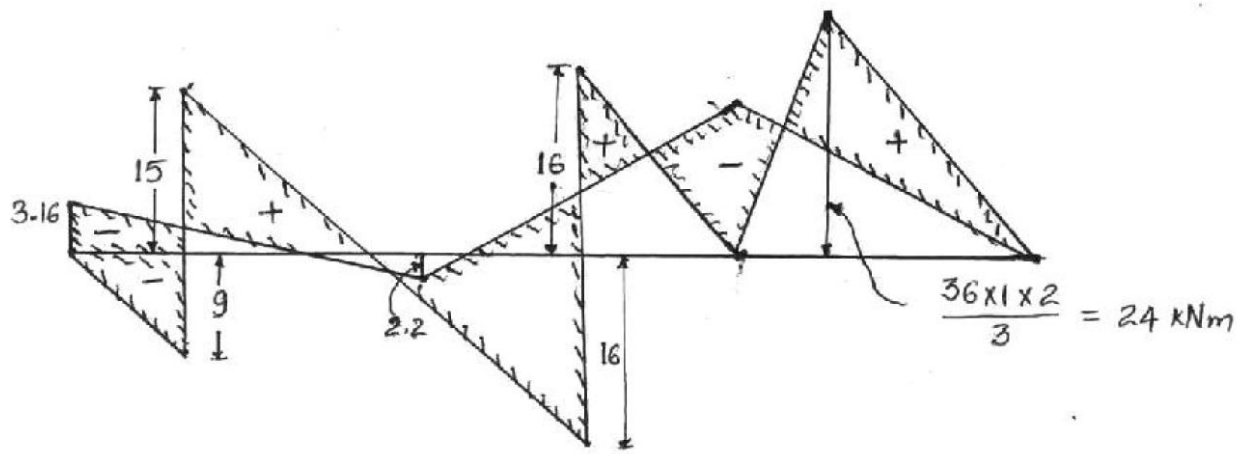
Iteration process has been stopped after 4th cycle since rotation contribution values are becoming almost constant. Values of fixed end moments, sum of fixed end moments, rotation factors along with rotation contribution values after end of each cycle in appropriate places has been shown in fig. 4 (b).

f) Final moments

| **Member (ij)** | **FEM $M_{Fij}$ (kNm)** | $2M'_{ij}$ **(kNm)** | $M^1_{ji}$ **(kNm)** | **Final moments** $(= M_{Fij} + 2M'_{ij} + M'_{ji})$ **(kNm)** |
|---|---|---|---|---|
| AB | 1.88 | 0 | -5.04 | -3.16 |
| BA | 7.88 | 2 (-5.04) = 10.08 | 0 | -2.2 |
| BC | 8 | 2 (-504) = –10.08 | 4.26 | +2.2 |
| CB | 8 | 2 × 4.26 = 8.52 | -5.04 | 11.48 |
| CD | -20 | 2 x 4.26 = 8.52 | 0 | -11.48 |

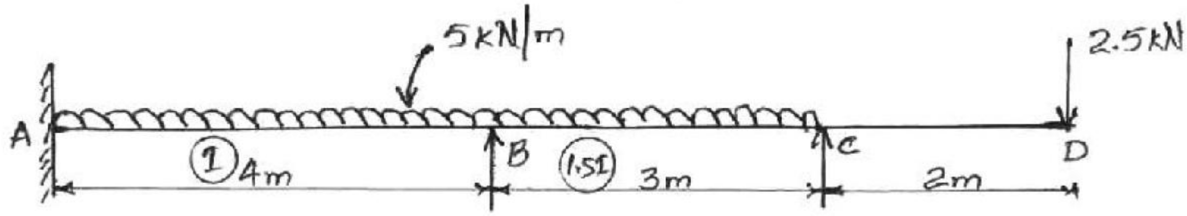
BMD is shown below:




**Ex.2:**

Analyze the continuous beam shown in fig.




Solution:

a) Fixed end moments:

$$M_{FAB} = -\frac{5\times4^2}{12} = -6.67 \text{ kNm}$$

$$M_{FBA} = +6.67 \text{ kNm}$$

$$M_{FBC} = \frac{-5\times3^2}{12} = -3.75 \text{ kNm}$$

$$M_{FCB} = +3.75 \text{ kNm}$$

$$M_{CD} = -2.5 \text{ x } 2 = -5 \text{ kNm}$$

b) Modification in fixed end moments:

Since $M_{CD}$ = − 5 kNm; $M_{CB}$ = + 5kNm, for this add 1.25 kNm to $M_{FCB}$ and do the corresponding carry over to $M_{FBC}$

∴ Now $M_{CB}$ = 5 kNm

Modified $M_{FBC} = -3.75 + \frac{1}{2}(1.25) = -3.13$ kNm

Now joint C will not enter in the iteration process.

c) Rotation factors:

| Jt. | Member | Relative stiffness (K) | ΣK | Rotation Factor $U = -\frac{1}{2} \times \frac{K}{\sum K}$ |
|---|---|---|---|---|
| B | BA<br>BC | I/4 = 0.25I<br>$\frac{3}{4} \times \frac{1.5I}{3} = 0.375I$ | 0.625I | − 0.2<br>−0.3 |
| C | CB<br>CD | 1.5I/3 = 0.5I<br>0 | 0.5I | −0.5<br>0 |

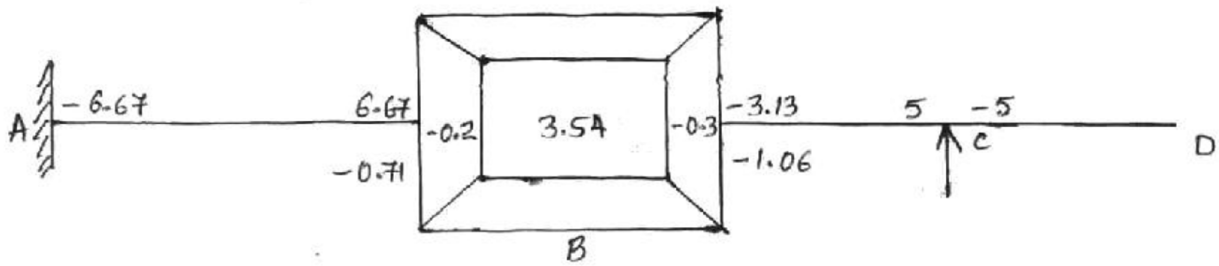
d) Sum of fixed end moments at joints:

$\Sigma M_{FB} = 6.67 - 3.13 = 3.54$ kNm

e) Iteration Process

| Joint | B | |
|---|---|---|
| Rotation Contribution | $M'_{BA}$ (kNm) | $M'_{BC}$ (kNm) |
| Rotation factor | − 0.2 | −0.3 |
| Iteration 1 started at B taking $M'_{AB} = 0$ & $M'_{CB} = 0$ | $-0.2 \times (3.54 + 0 + 0) = -0.71$ | $-0.3 \times (3.54 + 0 + 0) = -1.06$ |

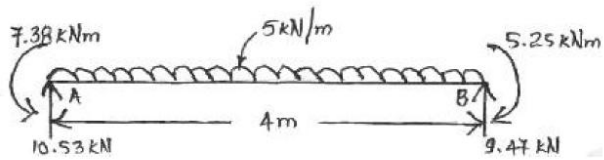
Since ‘B’ is the only joint needing rotation correction, the iteration process will stop after first iteration. Value of FEMs, sum of FEM at joint, rotation factors along with rotation contribution values in appropriate places is shown in fig. 5 (b)

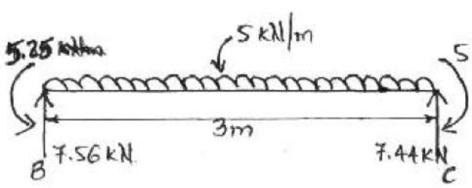


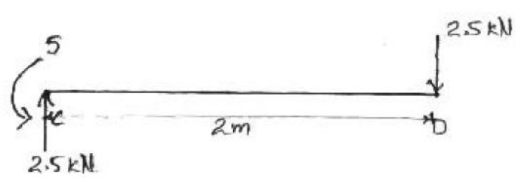

(f) Final moments:

| Member (ij) | FEM $M_{Fij}$ (kNm) | $2M'_{ij}$ (kNm) | $M^1_{ji}$ (kNm) | Final moments $\left(= M_{Fij} + 2M'_{ij} + M'_{ji}\right)$ (kNm) |
|---|---|---|---|---|
| AB | – 6.67 | 0 | –0.71 | –7.38 |
| BA | 6.67 | 2 × (-0.71) = | 0 | 5.25 |
| BC | –3.13 | 2 × (–1.06) | 0 | –5.25 |
| CB | | | | + 5 |
| CD | - | - | - | – 5 |
| DC | | | | 0 |

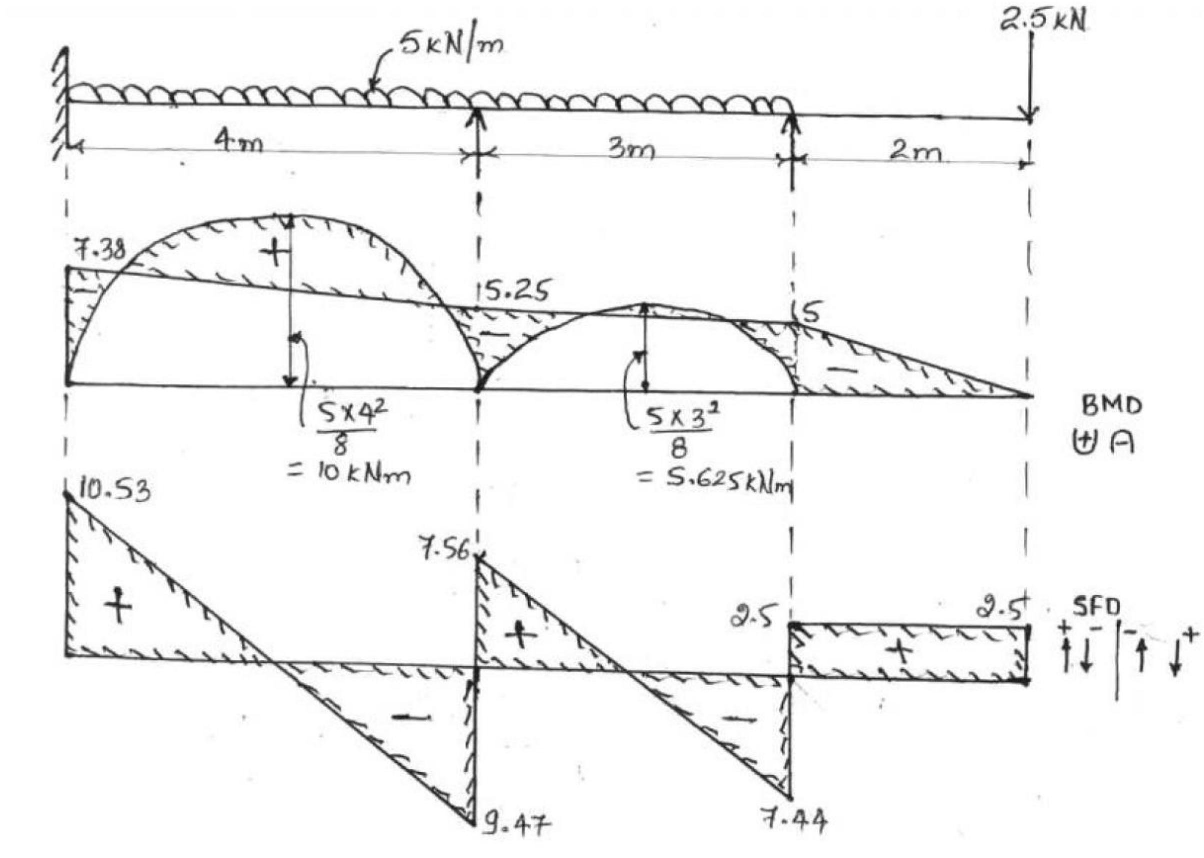
FBD of each span along with reaction values which have been calculated from statics are shown below:



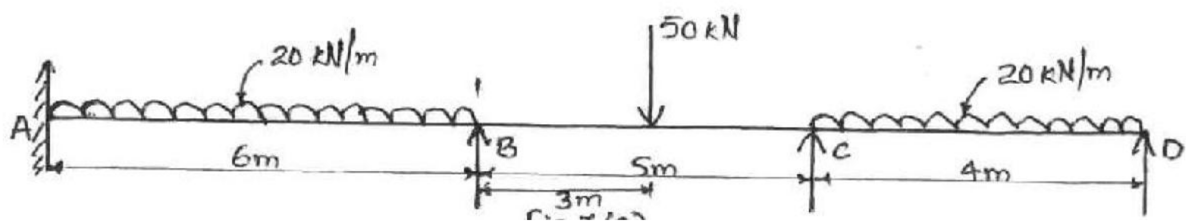



BMD and SFD are shown below




**Ex.2:**

In a continuous beam shown in fig. The support 'B' sinks by 10mm. Determine the moments by Kani's method & draw BMD.




Take $I = 1.2 \times 10^{-4}$ $mm^4$ & $E = 2 \times 10^5$ $N/mm^2$

Solution:

**(a) Calculation of FEM:**

$$M_{FAB} = -\frac{20\times 6^2}{12} - \frac{6\times 2\times 10^5 \times 1.2\times 10^{-4} \times 10^{12} \times 10}{(6000)^2 \times 10^6}$$

$= -60 - 40$

$= -100$ kNm

$M_{FBA}$ = +60 –40= 20 kNm

$$M_{FBC} = -\frac{50\times3\times2^2}{5^2} + \frac{6\times\ 2\times10^5\ \times1.2\times10^{-4}\times10^{12}\times10}{(5000)^2\times10^6}$$

$= -24 + 57.6$

$= 33.6$ kNm

$$M_{FCB} = +\frac{50\times3^2\times2}{5^2} + \frac{6\times2\times10^5\ \times1.2\times10^{-4}\times10^{12}\times10}{(5000)^2\times10^6}$$

$= 36 + 57.6$

$= 93.6$ kNm

C & D are at same level

$$\therefore M_{FCD} = -\frac{20\times4^2}{12} = -26.67 \text{ kNm}$$

$M_{FDC} = + 26.67$ kNm

b) Modification in fixed end moments:

Since end 'D' is a simply supported, moment at 'D' is zero. To make moment at D as zero apply a moment of – 26.67 kNm at end D and perform the corresponding carry over to CD.

$\therefore$ Modified $M_{FDC} = + 26.67 - 26.67 = 0$

Modified $M_{FCD} = -26.67 + \frac{1}{2}(-26.67)$

$= -40$ kNm

Other FEMs will be same as calculated earlier. Now joint 'D' will not enter the iteration process.

c) Rotation factors:

| Joint | Member | Relative stiffness (K) | ΣK | Rotation Factor $U = -\frac{1}{2} \times \frac{K}{\sum K}$ |
|---|---|---|---|---|
| B | BA | I/6 = 0.17 I | 0.37 I | −0.23 |
| | BC | I/5 = 0.2 I | | −0.27 |
| C | CB | I/5 = 0.2I | 0.39I | −0.26 |
| | CD | $\frac{3}{4}$ x I/4 = 0.19 I | | −0.24 |

d) Sum of fixed end moments:

$$\Sigma M_{FB} = 20 + 33.6 = 53.6 \text{ kNm}$$
$$\Sigma M_{FC} = 93.6 - 40 = 53.6 \text{ kNm}$$

e) Iteration process:

| Joint | B | | C | |
|---|---|---|---|---|
| Rotation Contribution | $M'_{BA}$ (kNm) | $M'_{BC}$ (kNm) | $M'_{CB}$ (kNm) | $M'_{CD}$ (kNm) |
| Rotation factor | −0.23 | −0.27 | −0.26 | −0.24 |
| Iteration 1 (Started at B by taking $M'_{AB} = 0$ and assuming $M'_{CB} = 0$ | −0.23 × (53.6 + 0 + 0) = −12.33 | −0.27 x (53.6 + 0+0)= −14.47 | −0.26×(53.6 – 14.47+ 0) = −10.17 | −0.24 (53.6 – 14.47) 10.96= −9.39 |
| Iteration 2 | −0.23 (53.6 – 10.17) = −10.00 | −0.27 (53.6 −10.17) = −11.73 | −0.26 (53.6 −11.73) = −10.89 | −0.24 (53.6 −11.73) =−10.05 |
| Iteration 3 | −0.23 (53.6−10.89) = −9.82 | −0.27 (53.6 −10.89) =−11.53 | −0.26 (53.6 −11.53) = −10.94 | −0.24 (53.6 −11.53) =−10.10 |
| Iteration 4 | −0.23 (53.6 −10.94) = −9.81 | −0.27 (53.6 −10.94) = −11.52 | −0.26 (53.6 −11.52) = −10.94 | −0.24 (53.6 −11.52) = −10.1 |

Iteration process has been stopped after fourth cycle since rotation contribution values are becoming almost constant. Values of FEMs, sum of fixed end moments, rotation factors along with rotation contribution values after end of each cycle in appropriate places has been shown in Fig. 7 (b).

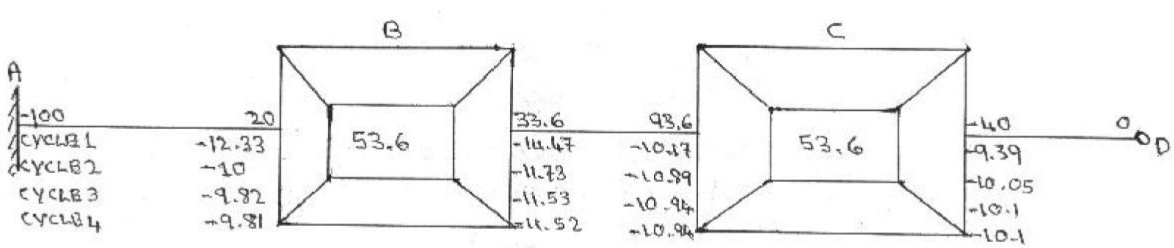
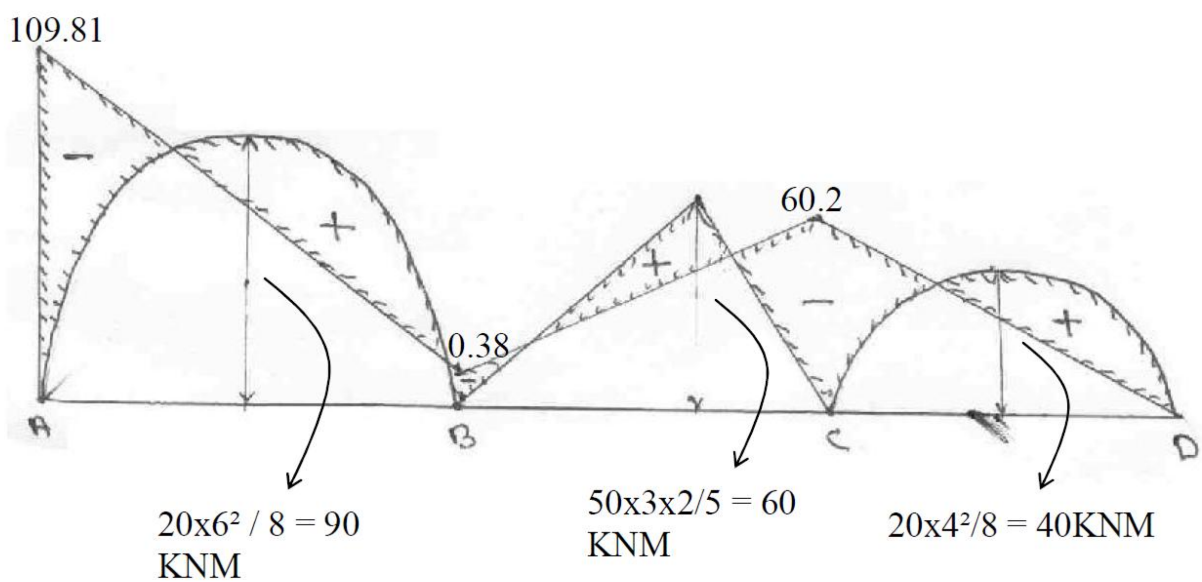



Fig: 7(b)

f) Final moments:

| **Member (ij)** | **FEM $M_{Fij}$ (kNm)** | $2M'_{ij}$ **(kNm)** | $M^{1}_{ji}$ **(kNm)** | **Final moments** $\left(= M_{Fij} + 2M'_{ij} + M'_{ji}\right)$ **(kNm)** |
|---|---|---|---|---|
| AB | − 100 | 0 | −9.81 | −109.81 |
| BA | 20 | 2 × (-9.81)= -19.62 | 0 | +0.38 |
| BC | 33.6 | 2 × (−11.52) = −23.04 | −10.94 | −0.38 |
| CB | 93.6 | 2 × (−10.94) = −21.88 | −11.52 | 60.2 |
| CD | −40 | 2 × (−10.1) = −20.2 | 0 | −60.2 |
| DC | 0 | 0 | 0 | 0 |

g) BMD is shown below:



## ANALYSIS OF FRAMES WITH NO TRANSLATION OF JOINTS

The frames, in which lateral translations are prevented, are analyzed in the same way as continuous beams. The lateral sway is prevented either due to symmetry of frame and loading or due to support conditions. The procedure is illustrated in following example.

Example-5. Analyze the frame shown in Figure 8 (a) by Kani's method. Draw BMD.

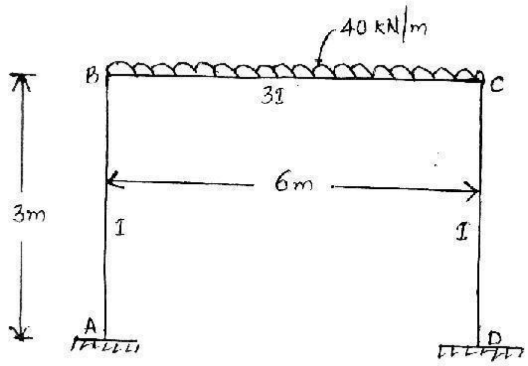



**Fig-8(a)**

**Solution:**

**(a) Fixed endmoments:**

$$M_{FAB} = M_{FBA} = M_{FCD} = M_{FDC} = 0$$

$$M_{FBC} = \frac{-40x6^2}{12} = -120\text{kNm.}$$

$$M_{FCB} = +120\text{kNm.}$$

**(b) Rotationfactors:**

| Joint | Member | Relative Stiffness (k) | Σk | Rotation factor = -½k/ Σ |
|---|---|---|---|---|
| B | BC | 3I/6 = 0.5I | 0.83I | -0.3 |
| | BA | I/3 = 0.33I | | -0.2 |
| C | CB | 3I/6 = 0.5I | 0.83I | -0.3 |
| | CD | I/3 = 0.33I | | -0.2 |

**(c) Sum of FEM:**

$$\Sigma M_{FB} = -120+0 = -120$$

$$\Sigma M_{FC} = 120+0 = +120$$

**(d) Iteration process:**

| Joint | B | | C | |
|---|---|---|---|---|
| **Rotation Contribution** | **M'BA** | **M'BC** | **M'CB** | **M'CD** |
| **Rotation Factor** | **-0.2** | **-0.3** | **-0.3** | **-0.2** |
| Iteration 1 Stated with end B taking M'AB=0 and Assuming M'CB=0 | -0.2(-120+0) =24 | -0.3(-120+0) =36 | -0.2(120+36+0) = -46.8 | -0.2(120+36+0) = -31.2 |
| Iteration 2 | -0.2(-120-46.8) =33.6 | -0.3(-120-46.8) =50.04 | -0.3(120+50.04) = -51.01 | -0.2(120+50.04) = -34.01 |
| Iteration 3 | -0.2(-120-51.01) =34.2 | -0.3(-120-51.01) =51.3 | -0.3(120+51.3) = -51.39 | -0.2 (120+51.3) = -34.26 |
| Iteration 4 | -0.2(-120-51.39) =34.28 | -0.3(-120-51.39) =51.42 | -0.3(120+51.42) = -51.43 | -0.2 (120+51.42) = -34.28 |

The fixed end moments, sum of fixed and moments, rotation factors along with rotation contribution values at the end of each cycle in appropriate places is shown in figure 8(b).

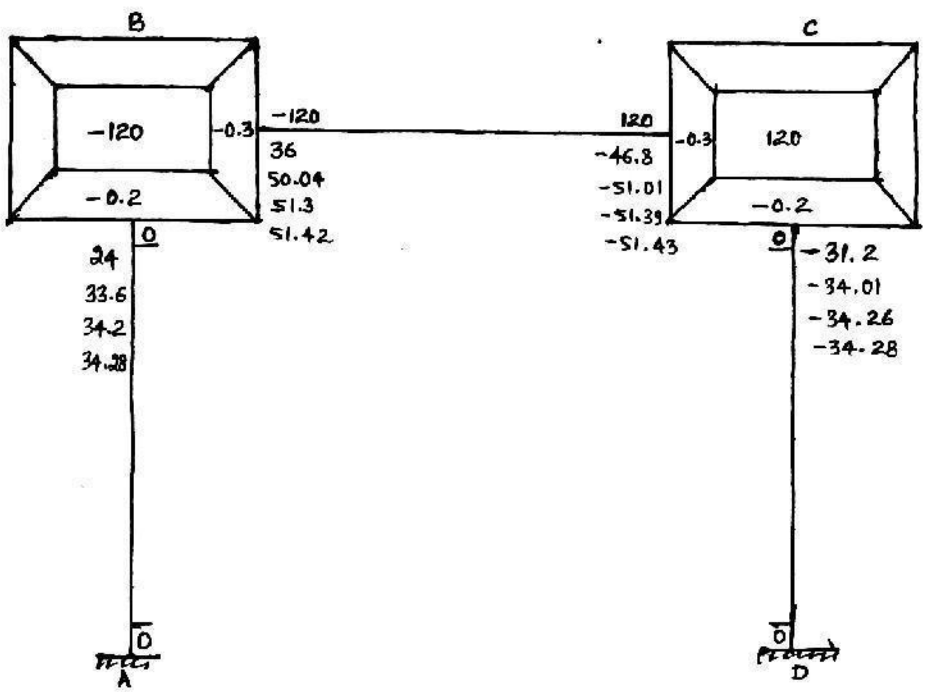
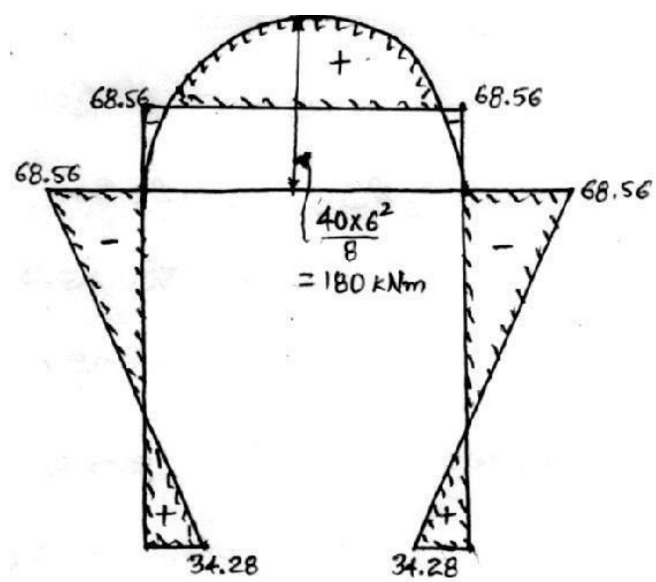



**Fig-8(b)**

**(e) Final moments:**

| Member (ij) | $M_{Fij}$ | $2M'_{ij}$(kNm) | $M'_{ji}$(kNm) | (kNm) Final moment = $M_{Fij}+ 2M'_{ij}+ M'_{ji}$ |
|---|---|---|---|---|
| AB | 0 | 0 | 34.28 | 34.28 |
| BA | 0 | 2 x 34.28 | 0 | 68.56 |
| BC | -120 | 2 x 51.42 | -51.43 | -68.59 |
| CB | 120 | 2 x (-51.43) | 51.42 | 68.56 |
| CD | 0 | 2 x (-34.28) | 0 | -68.56 |
| DC | 0 | 0 | -34.28 | -34.28 |

BMD is shown below in figure-8 (c)



# UNIT II: ANALYSIS OF FRAMES AND ARCHES

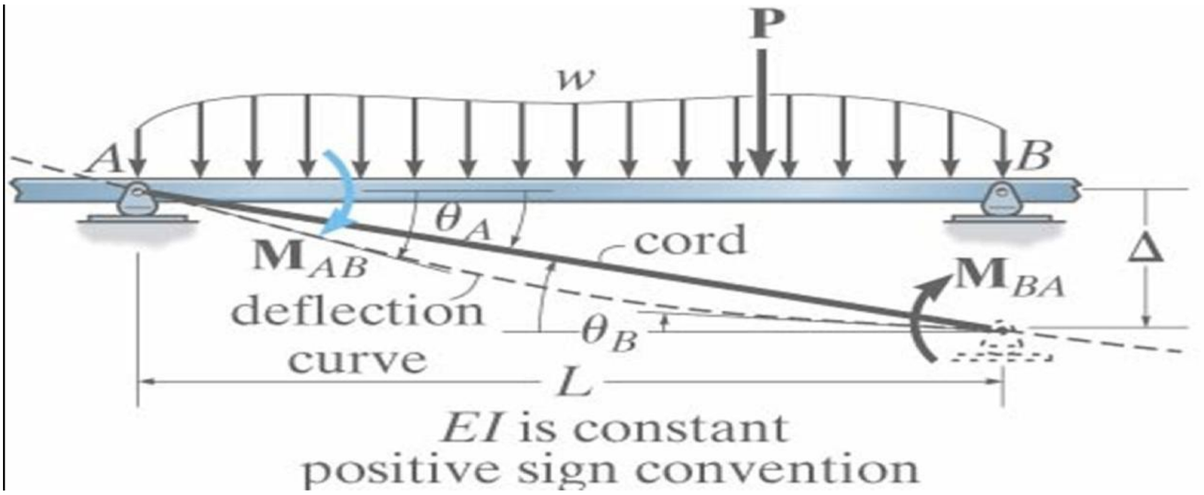
## THE SLOPE DEFLECTION METHOD

In the slope-deflection method, the relationship is established between moments at the ends of the members and the corresponding rotations and displacements.

The slope-deflection method can be used to analyze statically determinate and indeterminate beams and frames. In this method it is assumed that all deformations are due to bending only. In other words deformations due to axial forces are neglected. In the force method of analysis compatibility equations are written in terms of unknown reactions. It must be noted that all the unknown reactions appear in each of the compatibility equations making it difficult to solve resulting equations. The slope-deflection equations are not that lengthy in comparison. The basic idea of the slope deflection method is to write the equilibrium equations for each node in terms of the deflections and rotations. Solve for the generalized displacements. Using moment-displacement relations, moments are then known. The structure is thus reduced to a determinate structure. The slope-deflection method was originally developed by Heinrich Manderla and Otto Mohr for computing secondary stresses in trusses. The method as used today was presented by G.A.Maney in 1915 for analyzing rigid jointed structures.

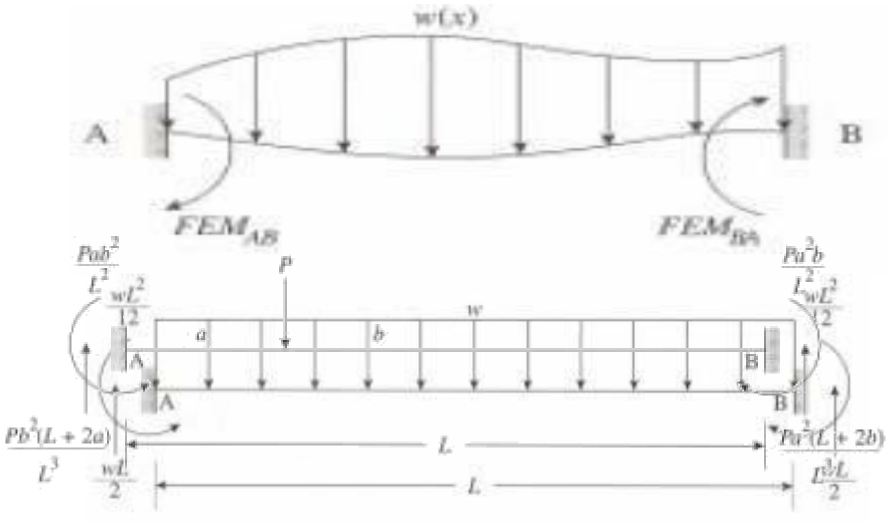
### FUNDAMENTAL SLOPE-DEFLECTION EQUATIONS:

The slope deflection method is so named as it relates the unknown slopes and deflections to the applied load on a structure. In order to develop general form of slope deflection equations, we will consider the typical span AB of a continuous beam which is subjected to arbitrary loading and has a constant EI. We wish to relate the beams internal end moments in terms of its three degrees of freedom, namely its angular displacements and linear displacement which could be caused by relative settlements between the supports. Since we will be developing a formula, moments and angular displacements will be considered positive, when they act clockwise on the span. The linear displacement will be considered positive since this displacement causes the chord of the span and the span's chord angle to rotate clockwise. The slope deflection equations can be obtained by using principle of superposition by considering separatelythemomentsdevelopedateachsupportsduetoeachofthedisplacements $\theta_A$ & $\theta_B$ & $\Delta$ and then the load.




Case A: fixed-end moments

$M_{AB} = FEM_{AB}$, $M_{BA} = FEM_{BA}$

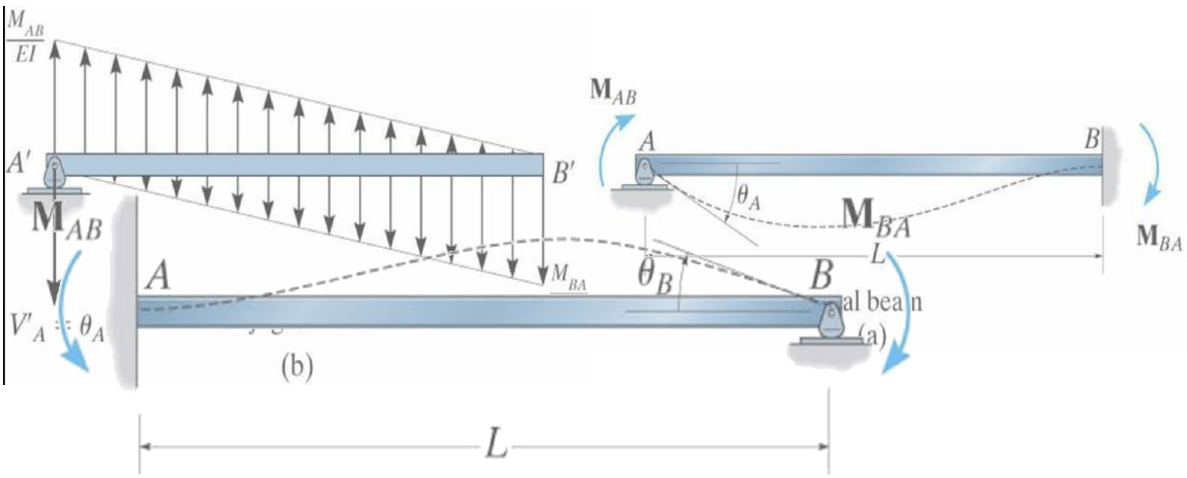



$FEM_{AB} = -\frac{wL^2}{12}$, $FEM_{BA} = \frac{wL^2}{12}$

$FEM_{AB} = -\frac{Pab^2}{L^2}$, $FEM_{BA} = \frac{Pa^2b}{L^2}$

Case B: rotation at A, (angular displacement at A)

Consider node A of the member as shown in figure to rotate while its far end B is fixed. To determine the moment needed to cause the displacement, we will use conjugate beam method. The end shear at A acts downwards on the beam since is clockwise.




$$\Sigma M'_A = 0, \quad \left[\frac{1}{2}\frac{M_{AB}}{EI}L\right]\frac{L}{3} - \left[\frac{1}{2}\frac{M_{BA}}{EI}L\right]\frac{2L}{3} = 0$$

$$\Sigma M'_B = 0, \quad \left[\frac{1}{2}\frac{M_{BA}}{EI}L\right]\frac{L}{3} - \left[\frac{1}{2}\frac{M_{BA}}{EI}L\right]\frac{2L}{3} + \theta_A L = 0$$

$$M_{AB} = \frac{4EI}{L}\theta_A, \; M_{BA} = \frac{2EI}{L}\theta_A$$

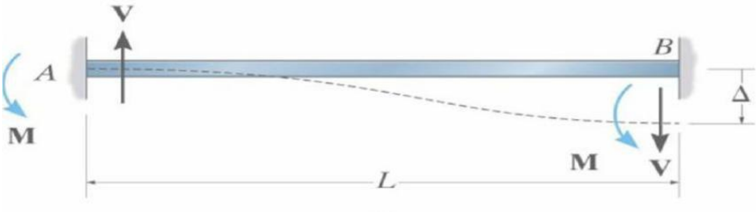
Case C: rotationatB, (angular displacement atB)

In a similar manner if the end B of the beam rotates to its final position, while end A is held fixed. We can relate the applied moment to the angular displacement and the reaction moment

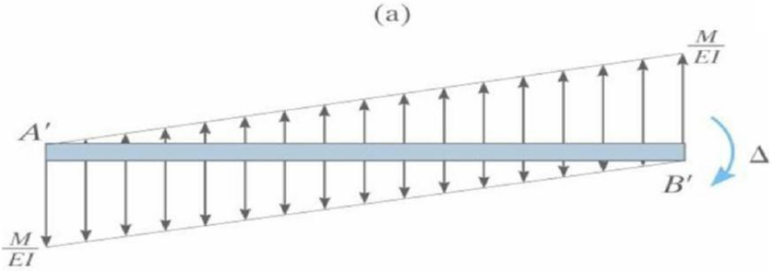
$$M_{AB} = \frac{2EI}{L}\theta_B, \; M_{BA} = \frac{4EI}{L}\theta_B$$

Case D: displacement of end B related to end A

If the far node B of the member is displaced relative to A so that so that the chord of the member rotates clockwise (positive displacement) .The moment M can be related to displacement by using conjugate beam method. The conjugate beam is free at both the ends as the real beam is fixed supported. Due to displacement of the real beam at B, the momentat the end B of the conjugate beam must have a magnitude of .Summing moments about B we have,



real beam
(a)



conjugate beam
(b)

$$\Sigma M_{B'} = 0, \quad \left[\frac{1}{2}\frac{M}{EI}(L)\right]\frac{2L}{3} - \left[\frac{1}{2}\frac{M}{EI}(L)\right]\frac{L}{3} - \Delta = 0$$

$$M_{AB} = M_{BA} = M = -\frac{6EI}{L^2}\Delta$$

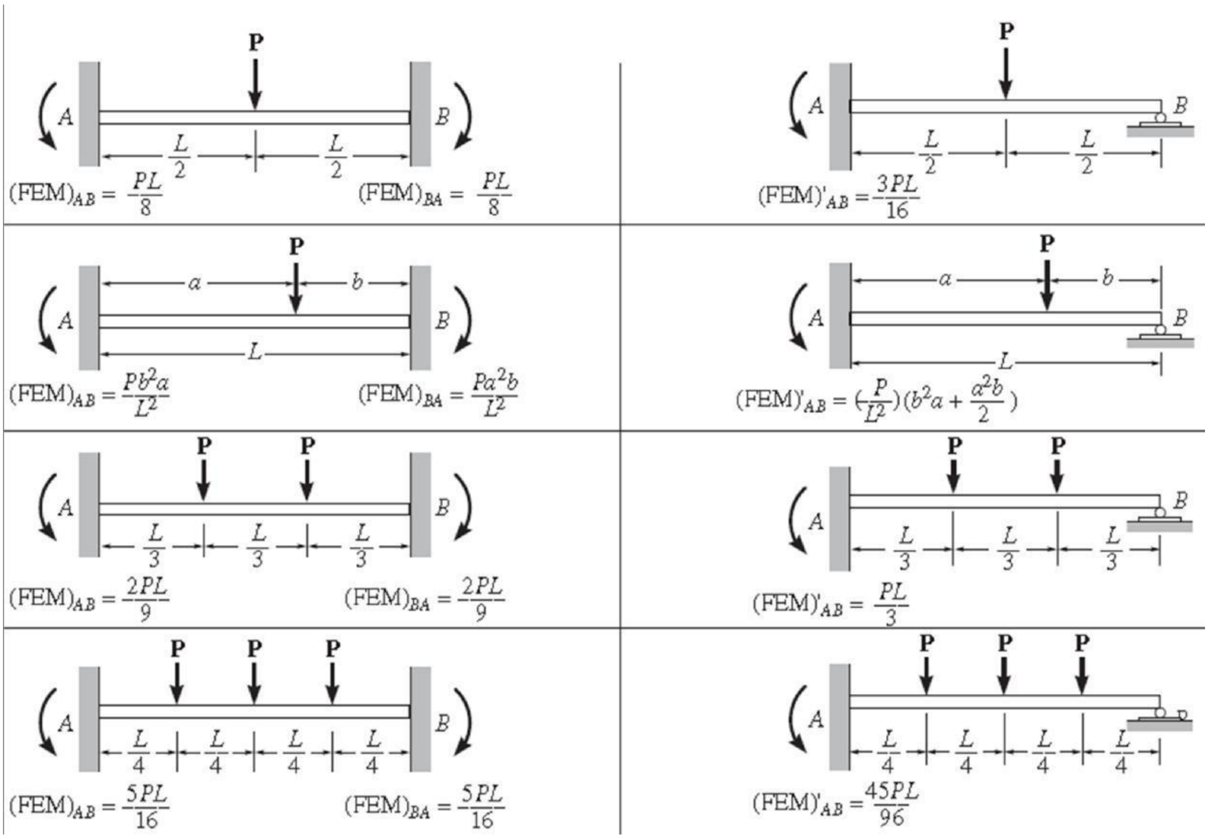
By our sign convention the induced moment is negative, since for equilibrium it acts counter clockwise on the member.

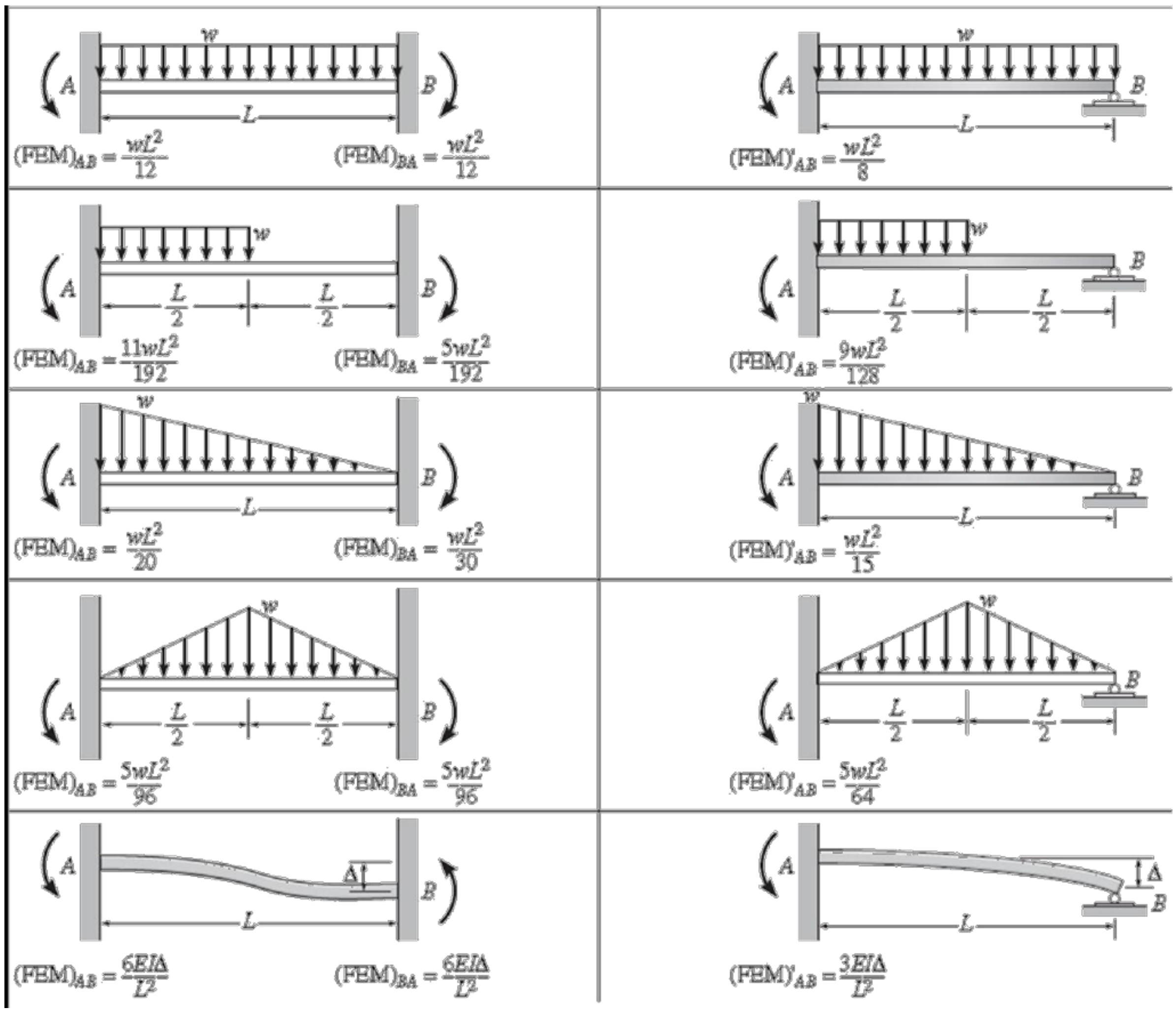
If the end moments due to the loadings and each displacements are added together, then the resultant moments at the ends can be writtenas,

$$M_{AB} = \frac{2EI}{L}[2\theta_A + \theta_B - \frac{3\Delta}{L}] + FEM_{AB}$$

$$M_{BA} = \frac{2EI}{L}[2\theta_B + \theta_A - \frac{3\Delta}{L}] + FEM_{BA}$$

## FIXED END MOMENT TABLE





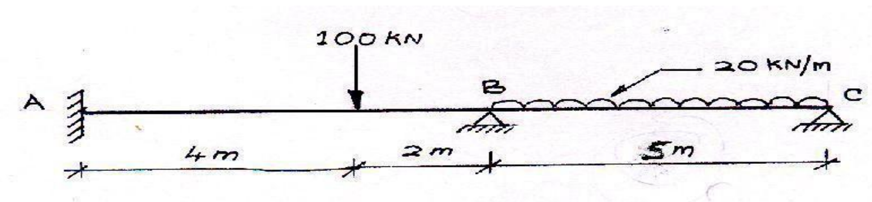
w
A
B
L
$(FEM)_{AB} = \frac{wL^2}{12}$
$(FEM)_{BA} = \frac{wL^2}{12}$
$(FEM)'_{AB} = \frac{wL^2}{8}$
$\frac{L}{2}$
$(FEM)_{AB} = \frac{11wL^2}{192}$
$(FEM)_{BA} = \frac{5wL^2}{192}$
$(FEM)'_{AB} = \frac{9wL^2}{128}$
$(FEM)_{AB} = \frac{wL^2}{20}$
$(FEM)_{BA} = \frac{wL^2}{30}$
$(FEM)'_{AB} = \frac{wL^2}{15}$
$(FEM)_{AB} = \frac{5wL^2}{96}$
$(FEM)_{BA} = \frac{5wL^2}{96}$
$(FEM)'_{AB} = \frac{5wL^2}{64}$
Δ
$(FEM)_{AB} = \frac{6EI\Delta}{L^2}$
$(FEM)_{BA} = \frac{6EI\Delta}{L^2}$
$(FEM)'_{AB} = \frac{3EI\Delta}{L^2}$

## GENERAL PROCEDURE OF SLOPE-DEFLECTION METHOD

- Find the fixed end moments of each span (both ends left &right).
- Apply the slope deflection equation on each span & identify the unknowns.
- Write down the joint equilibrium equations.
- Solve the equilibrium equations to get the unknown rotation & deflections.
- Determine the end moments and then treat each span as simply supported beam subjected to given load & end moments so we can work out the reactions & draw the bending moment & shear force diagram.

**Numerical Examples**

1. Q. Analyze two span continuous beam ABC by slope deflection method.Then draw Bending moment & Shear force diagram. Take EIconstant.




Fixed end moments are

$$M_{AB=}\ -\frac{Wab^2}{L^2} = -\frac{100\ \times\ 4\ \times 2^2}{6^2} = \ -44.44\ KNm$$

$$M_{BA=}\ \frac{Wa^2b}{L^2} = \frac{100\ \times\ 4^2 \times\ 2}{6^2} = \ 88.89KNm$$

$$M_{BC=}\ -\frac{wL^2}{12} = -\frac{20\ \times\ 5^2}{12} = \ -41.67KNm$$

$$M_{CB=}\ \frac{wL^2}{12} = \frac{20\ \times\ 5^2}{12} = 41.67\ KNm$$

Since A is fixed $\Theta_A\ = 0\ \&\ \Theta_B \&\ \Theta_c\ \neq 0$

Slope deflection equations are

$$M_{AB} = M_{FAB} + \frac{2EI}{L}[2\theta_A + \theta_B] = -44.44 + \frac{2EI}{6}\theta_B = -44.44 + \frac{EI}{3}\theta_B \quad \cdots\cdots(1)$$

$$M_{BA} = M_{FBA} + \frac{2EI}{L}[2\theta_B + \theta_A] = 88.89 + \frac{4EI}{6}\theta_B = 88.89 + \frac{2EI}{3}\theta_B \quad \cdots\cdots(2)$$

$$M_{BC} = M_{FBC} + \frac{2EI}{L}[2\theta_B + \theta_C] = -41.67 + \frac{4EI}{5}\theta_B + \frac{2EI}{5}\theta_C \quad \cdots\cdots(3)$$

$$M_{CB} = M_{FCB} + \frac{2EI}{L}[2\theta_C + \theta_B] = 41.67 + \frac{4EI}{5}\theta_C + \frac{2EI}{5}\theta_B \quad \cdots\cdots(4)$$

In all the above 4 equations there are only2unknowns $\theta_B \& \theta_C$ and accordinglythe boundary

conditions are

$$M_{BA} + M_{BC} = 0$$

$M_{CB} = 0$ as end C is simply supported.

Solving the equations (5) & (6), we get

$$\theta_B = -\frac{20.83}{EI}$$

$$\theta_C = -\frac{41.67}{EI}$$

Substituting the values in the slope deflections we have,

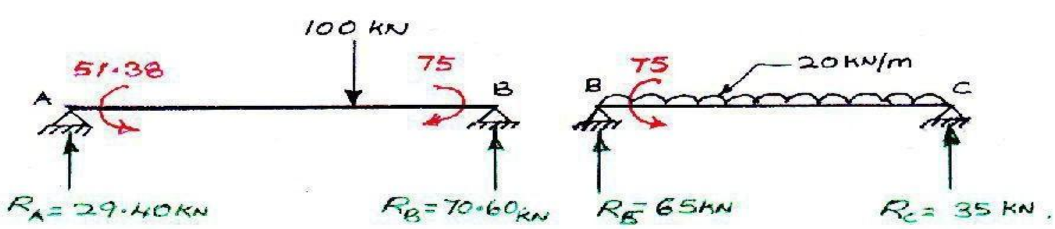
$$M_{AB} = -44.44 + \frac{EI}{3}\left(-\frac{20.83}{EI}\right) = -51.38 \text{ KNm}$$

$$M_{BA} = 88.89 + \frac{2EI}{3}\left(-\frac{20.83}{EI}\right) = 75 \text{ KNm}$$

$$M_{BC} = -41.67 + \frac{4EI}{5}\left(-\frac{20.83}{EI}\right) + \frac{2EI}{5}\left(-\frac{41.67}{EI}\right) = -75 \text{ KNm}$$

$$M_{CB} = 41.67 + \frac{4EI}{5}\left(-\frac{41.67}{EI}\right) + \frac{2EI}{5}\left(-\frac{20.83}{EI}\right) = 0$$

Reactions: Consider the free body diagram of the beam



Find reactions using equations of equilibrium.

Span AB: $M_A = 0$ , $R_B \times 6 = 100 \times 4 + 75 - 51.38$

$R_B = 70.60$ KN

$V = 0$, $R_A + R_B = 100$KN

$R_A = 100 - 70.60 = 29.40$ KN

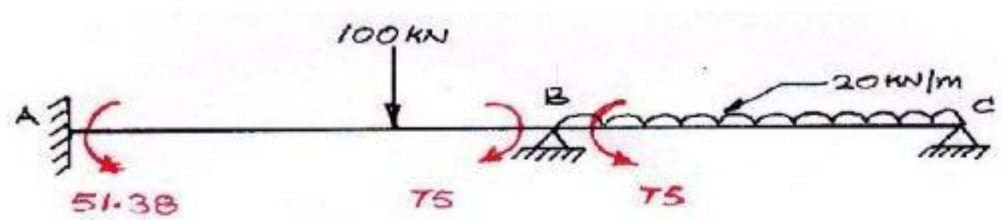
Span BC**:** $M_C = 0$, $R_B \times 5 = 20 \times 5 \times \pm 75$

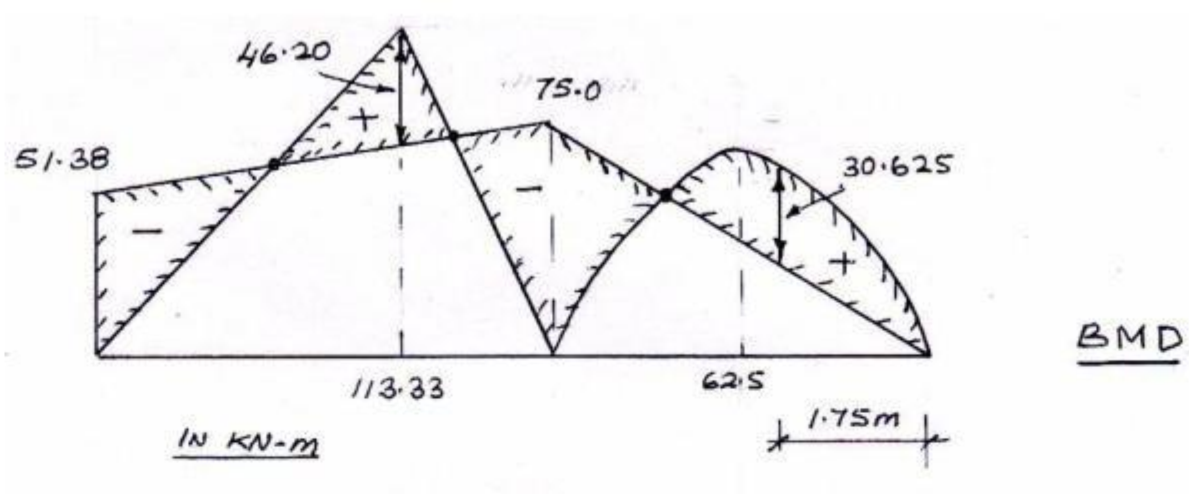
$R_B = 65$ KN

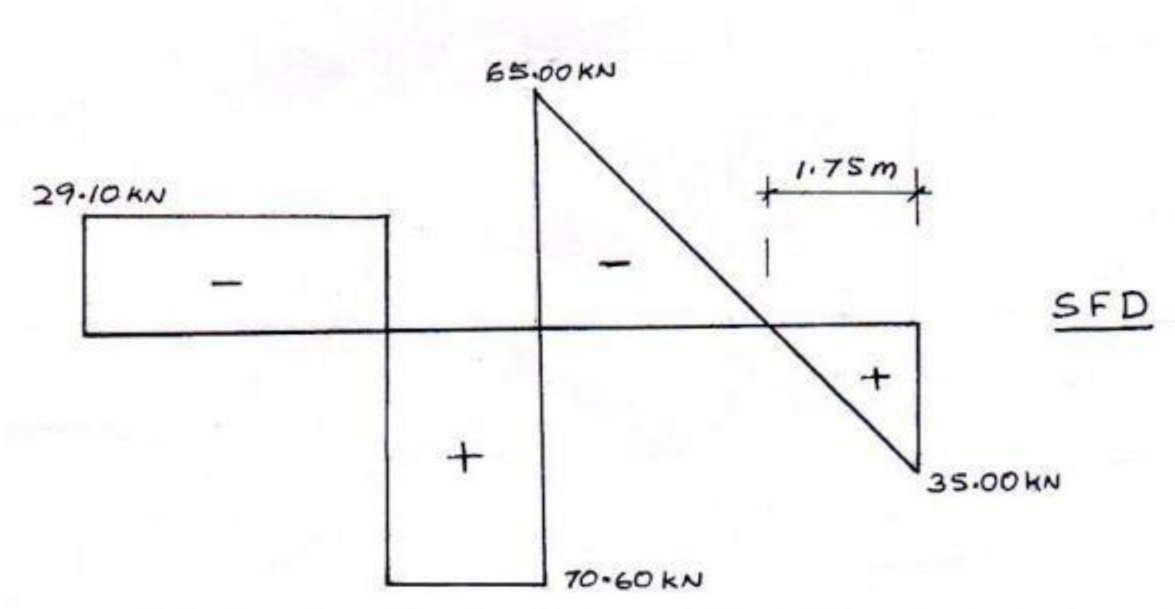
$V = 0$ $R_B + R_C = 20 \times 5 = 100$KN

$R_C = 100 - 65 = 35$ KN

Using these data BM and SF diagram can be drawn






**Max BM:**

**Span AB:** Max BM in span AB occurs under point load and can be found geometrically,

$$M_{max}=113.33-51.38-\left(\frac{75-51.38}{6}\right)\times 4 = 46.20 KNm$$

**Span BC:** Max BM in span BC occurs where shear force is zero or changes its sign. Henceconsider SF equation w.r.t C

$$S_x = 35-20x = 0$$

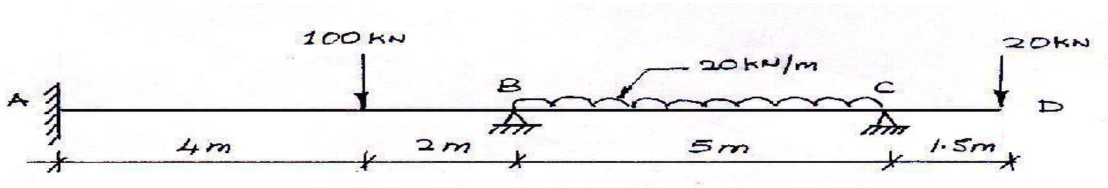
$$x = \frac{35}{20}=1.75m$$

—

Max BM occurs at 1.75m
from

$$\therefore M_{max} = 35 \times 1.75 - 20\times \frac{1.75^2}{2} = 30.625\ KNm$$

2. Q. Analyze continuous beam ABCD by slope deflection method and thendraw bending moment diagram. Take EIconstant.




$$\theta_A = 0 \;\&\; \theta_B \;\&\; \theta_c \neq 0$$

$$M_{AB=} -\frac{Wab^2}{L^2} = -\frac{100 \times 4 \times 2^2}{6^2} = -44.44 \text{ KNm}$$

$$M_{BA=} \frac{Wa^2b}{L^2} = \frac{100 \times 4^2 \times 2}{6^2} = 88.89\text{KNm}$$

$$M_{BC=} -\frac{wL^2}{12} = -\frac{20 \times 5^2}{12} = -41.67\text{KNm}$$

$$M_{CB=} \frac{wL^2}{12} = \frac{20 \times 5^2}{12} = 41.67 \text{ KNm}$$

$$M_{CD} = -20 \times 1.5 = -30 \text{ KNm}$$

Slope deflection equations are

$$M_{AB} = M_{FAB} + \frac{2EI}{L}[2\theta_A + \theta_B] = -44.44 + \frac{2EI}{6}\theta_B = -44.44 + \frac{EI}{3}\theta_B \quad \cdots\cdots(1)$$

$$M_{BA} = M_{FBA} + \frac{2EI}{L}[2\theta_B + \theta_A] = 88.89 + \frac{4EI}{6}\theta_B = 88.89 + \frac{2EI}{3}\theta_B \quad \cdots\cdots(2)$$

$$M_{BC} = M_{FBC} + \frac{2EI}{L}[2\theta_B + \theta_C] = -41.67 + \frac{4EI}{5}\theta_B + \frac{2EI}{5}\theta_C \quad \cdots\cdots(3)$$

$$M_{CB} = M_{FCB} + \frac{2EI}{L}[2\theta_C + \theta_B] = 41.67 + \frac{4EI}{5}\theta_C + \frac{2EI}{5}\theta_B \quad \cdots\cdots(4)$$

$$M_{CD} = -20 \times 1.5 = -30 \text{ KNm}$$

In all the above equations there are only 2 unknowns and accordingly the boundary conditions are

$$M_{BA} + M_{BC} = 0$$
$$M_{CB} + M_{CD} = 0$$

$$M_{BA} + M_{BC} = 88.89 + \frac{2EI}{3}\theta_B - 41.67 + \frac{4EI}{5}\theta_B + \frac{2EI}{5}\theta_C = 47.22 + \frac{22}{15}EI\theta_B + \frac{2}{5}EI\theta_C = 0 \quad \cdots\cdots \quad (5)$$

$$M_{CB} + M_{CD} = 41.67 + \frac{4EI}{5}\theta_C + \frac{2EI}{5}\theta_B - 30 = 11.67 + \frac{2EI}{5}\theta_B + \frac{4EI}{5}\theta_C = 0 \quad \cdots\cdots \quad (6)$$

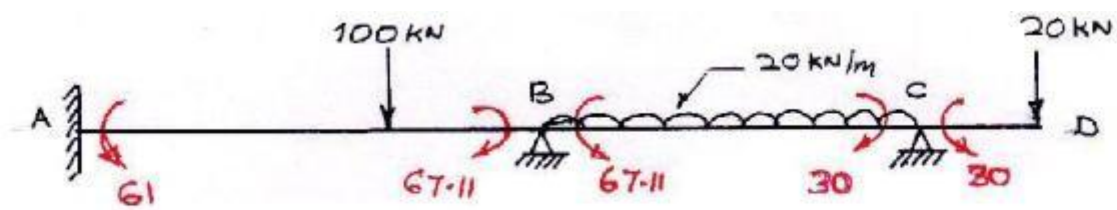
Solving equations (5) & (6),

$$\theta_B = -\frac{32.67}{EI}$$
$$\theta_C = \frac{1.75}{EI}$$

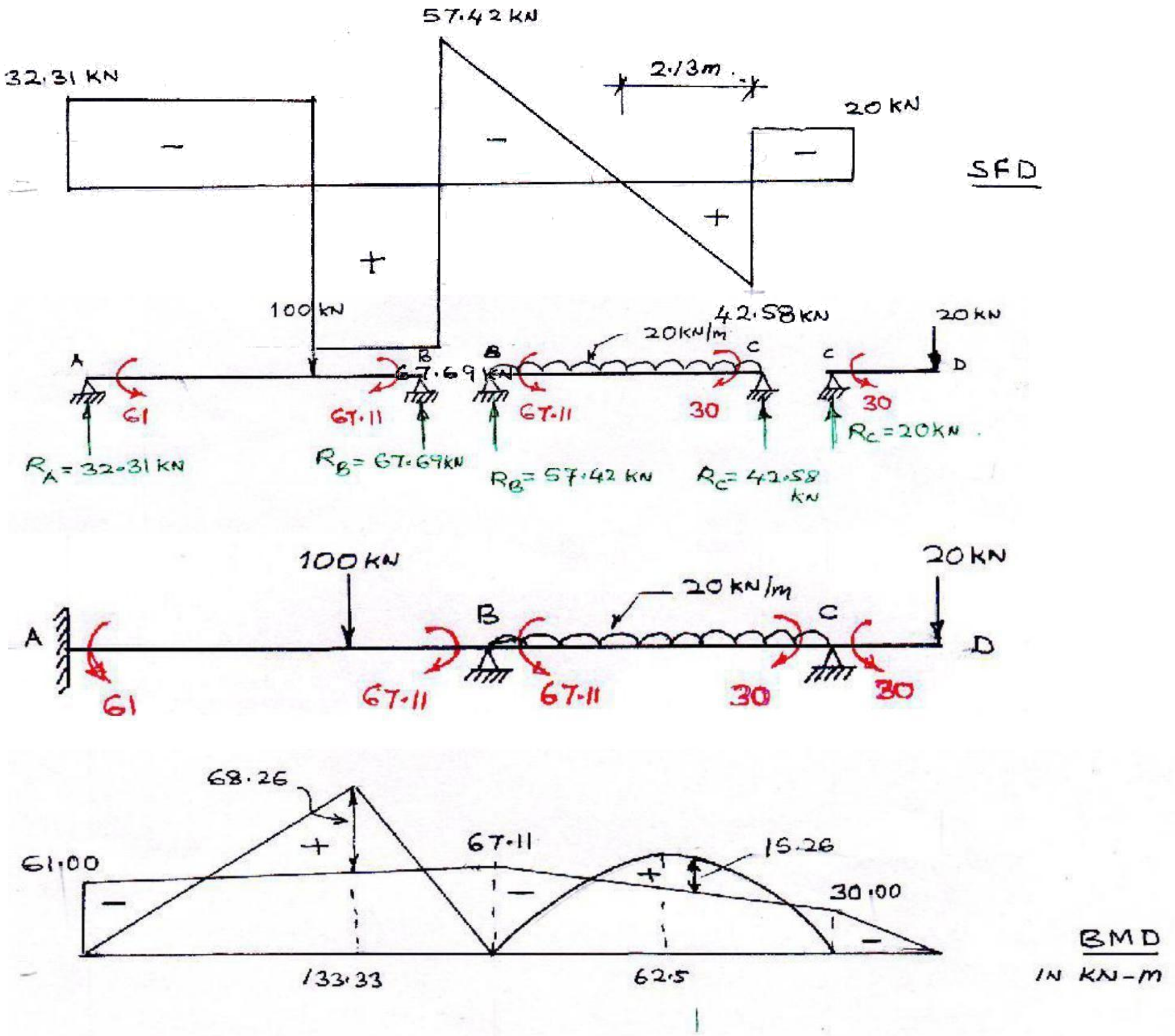
Substituting the values in the slope deflections we have,

$$M_{AB} = -44.44 + \frac{EI}{3} \times \left(-\frac{32.67}{EI}\right) = -61 \text{ KNm}$$

$$M_{BA} = 88.89 + \frac{2EI}{3} \times \left(-\frac{32.67}{EI}\right) = 67.11\text{KNm}$$
$$M_{BC} = -41.67 + \frac{4EI}{5}\left(-\frac{32.67}{EI}\right) + \frac{2EI}{5}\left(\frac{1.75}{EI}\right) = -67.11\text{KNm}$$
$$M_{CB} = 41.67 + \frac{4EI}{5}\left(\frac{1.75}{EI}\right) + \frac{2EI}{5}\left(-\frac{32.67}{EI}\right) = 30\text{KNm}$$
$$M_{CD} = -30\text{KNm}$$




**Reactions:** Consider free body diagram of beam AB, BC and CD as shown




Span AB:

$R_B \times 6 = 100 \times 4 + 67.11 - 61$

$R_B = 67.69$ KN

$R_A = 100 - R_B = 32.31$ KN

Span BC:

$R_C \times 5 = 20 \times \frac{5}{2} \times 5 + 30 - 67.11$

$R_C = 42.58$ KN

$R_B = 20 \times 5 - R_C = 57.42$ KN

Maximum Bending Moments:

Span AB: Occurs under point load

$M_{max}= 133.33 - 61 - \frac{67.11-61}{6} \times 4 = 68.26KNm$

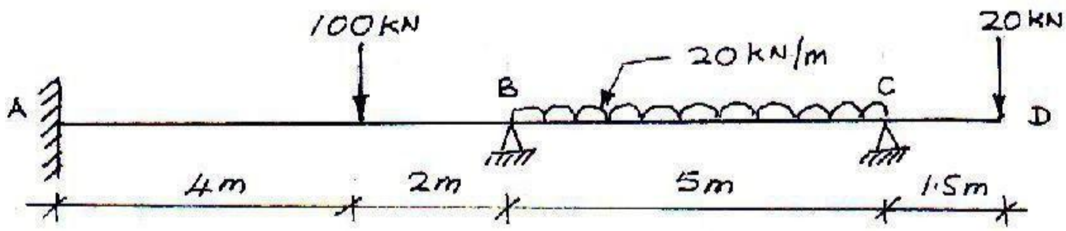
Span BC: Where SF=0, consider SF equation with C as reference

$S_x = 42.58\text{-}20x = 0$

$x = \frac{42.58}{20} = 2.13m$

$M_{max}= 42.58 \times 2.13 - 20 \times \frac{2.13^2}{2} - 30 = 15.26KNm$

3. Q. Analyse the continuous beam ABCD shown in figure by slope deflectionmethod. The support B sinks by 15mm.Take E =200 $10^5$ KN/$m^2$ and I =12010 $^{-6}$$m^4$




A. $\Theta_A = 0 \;\&\; \Theta_B \& \Theta_c \neq 0 \;\Delta = 15mm$

$$M_{AB=} -\frac{Wab^2}{L^2} = -\frac{100 \times 4 \times 2^2}{6^2} = -44.44\,KNm$$

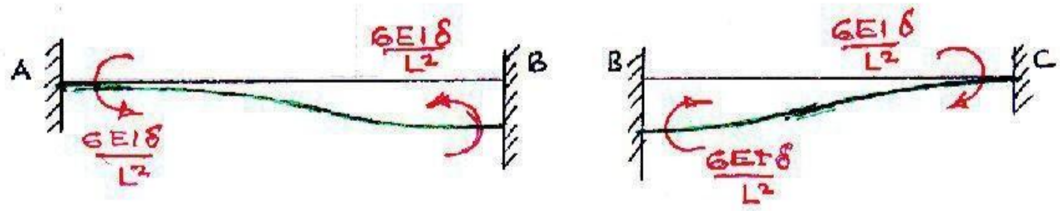
$$M_{BA=} \frac{Wa^2b}{L^2} = \frac{100 \times 4^2 \times 2}{6^2} = 88.89KNm$$

$$M_{BC=} -\frac{wL^2}{12} = -\frac{20 \times 5^2}{12} = -41.67KNm$$

$$M_{CB} = \frac{wL^2}{12} = \frac{20 \times 5^2}{12} = 41.67\,KNm$$

$$M_{CD} = -20 \times 1.5 = -30\,KNm$$

FEM due to yield of support B




For span AB:

$$M_{AB} = M_{BA} = -\frac{6EI}{L^2}\Delta = -\frac{6\times200\times10^5\times120\times10^{-6}}{6^2}\frac{15}{1000} = -6KNm$$

For span BC:

$$M_{BC} = M_{CB} = \frac{6EI}{L^2}\Delta = \frac{6\times200\times10^5\times120\times10^{-6}}{5^2}\frac{15}{1000} = 8.64KNm$$

Slope deflection equations are

$$M_{AB} = M_{FAB} + \frac{2EI}{6}\left[2\theta_A + \theta_B - 3\frac{\Delta}{6}\right] = -44.44 + \frac{EI}{3}\theta_B - 6 = -50.44 + \frac{EI}{3}\theta_B \cdots\cdots (1)$$

$$M_{BA} = M_{FBA} + \frac{2EI}{6}\left[2\theta_B + \theta_A - 3\frac{\Delta}{6}\right] = 88.89 + \frac{2EI}{3}\theta_B - 6 = 82.89 + \frac{2EI}{3}\theta_B \quad \cdots\cdots (2)$$

$$M_{BC} = M_{FBC} + \frac{2EI}{5}\left[2\theta_B + \theta_C + 3\frac{\Delta}{5}\right] = -41.67 + \frac{4EI}{5}\theta_B + \frac{2EI}{5}\theta_C + 8.64$$

$$= -33.03 + \frac{4EI}{5}\theta_B + \frac{2EI}{5}\theta_C \qquad \cdots\cdots (3)$$

$$M_{CB} = M_{FCB} + \frac{2EI}{5}\left[2\theta_C + \theta_B + 3\frac{\Delta}{5}\right] = 41.67 + \frac{4EI}{5}\theta_C + \frac{2EI}{5}\theta_B + 8.64$$

$$= 50.31 + \frac{4EI}{5}\theta_C + \frac{2EI}{5}\theta_B \qquad \cdots\cdots (4)$$

$$M_{CD} = -20 \times 1.5 = -30\ KNm$$

In all the above equations there are only2unknowns and accordinglythe boundary

conditions are

$$M_{BA} + M_{BC} = 0$$
$$M_{CB} + M_{CD} = 0$$

$$M_{BA} + M_{BC} = 82.89 + \frac{2EI}{3}\theta_B - 33.03 + \frac{4EI}{5}\theta_B + \frac{2EI}{5}\theta_C = 49.86 + \frac{22}{15}EI\theta_B + \frac{2}{5}EI\theta_C = 0 \quad \cdots\cdots \quad (5)$$

$$M_{CB} + M_{CD} = 50.31 + \frac{4EI}{5}\theta_C + \frac{2EI}{5}\theta_B - 30 = 20.31 + \frac{2EI}{5}\theta_B + \frac{4EI}{5}\theta_C = 0 \quad \cdots\cdots \quad (6)$$

Solving equations (5) & (6),

$$\theta_B = -\frac{31.35}{EI}$$

$$\theta_C = -\frac{9.71}{EI}$$

Substituting the values in the slope deflections we have,

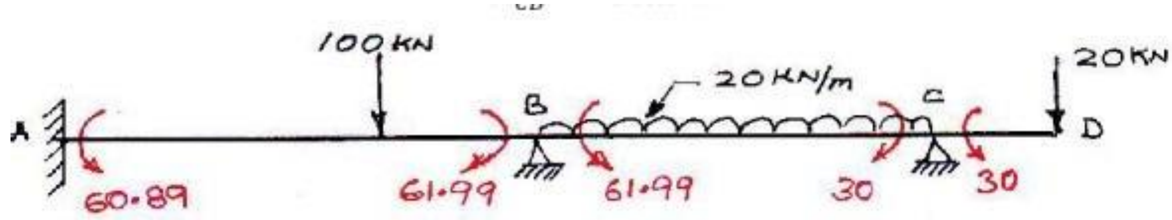
$$M_{AB} = -50.44 + \frac{EI}{3} \times \left(-\frac{31.35}{EI}\right) = -60.89 \text{ KNm}$$

$$M_{BA} = 82.89 + \frac{2EI}{3} \times \left(-\frac{31.35}{EI}\right) = 61.99 KNm$$

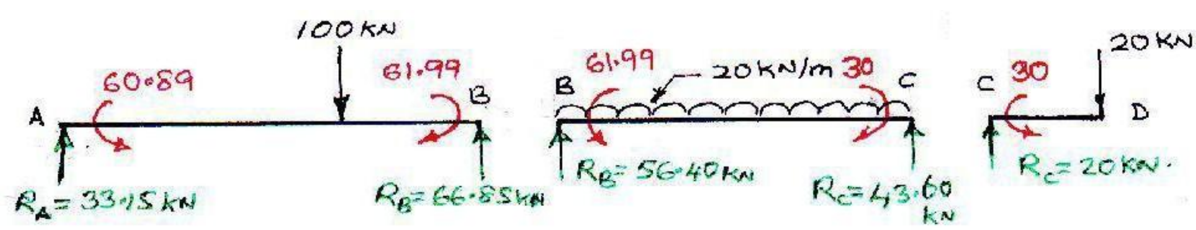
$$M_{BC} = -33.03 + \frac{4EI}{5}\left(-\frac{31.35}{EI}\right) + \frac{2EI}{5}\left(\frac{-9.71}{EI}\right) = -61.99 KNm$$

$$M_{CB} = 50.31 + \frac{4EI}{5}\left(\frac{-9.71}{EI}\right) + \frac{2EI}{5}\left(-\frac{31.35}{EI}\right) = 30 KNm$$

$$M_{CD} = -30 KNm$$




Consider the free body diagram of continuous beam for finding reactions




REACTIONS

Span AB:

$R_B \times 6 = 100 \times 4 + 61.99 - 60.89$

$R_B = 66.85KN$

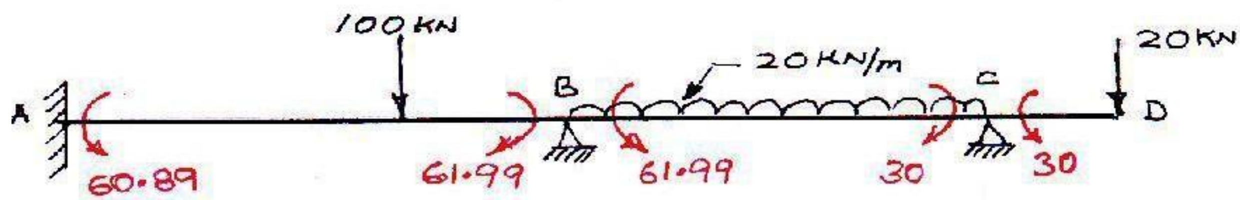
$R_A = 100 - R_B = 33.15\ KN$

Span BC:

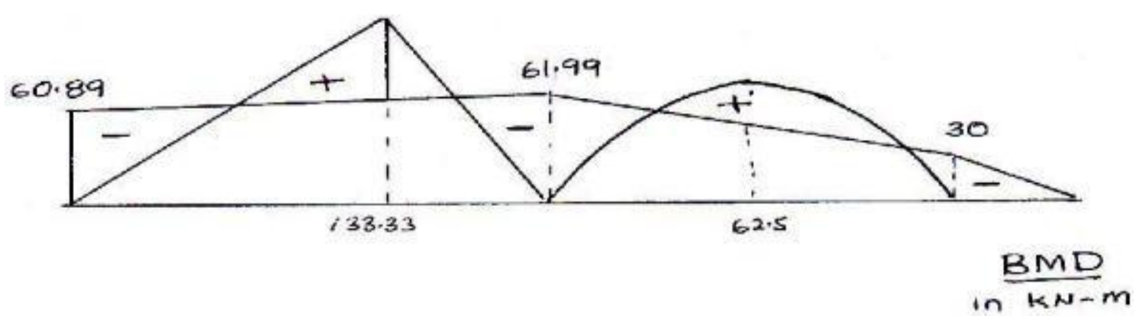
$R_B\ \times 5 = 20\ \times \frac{5}{2} \times 5 + 61.99 - 30$

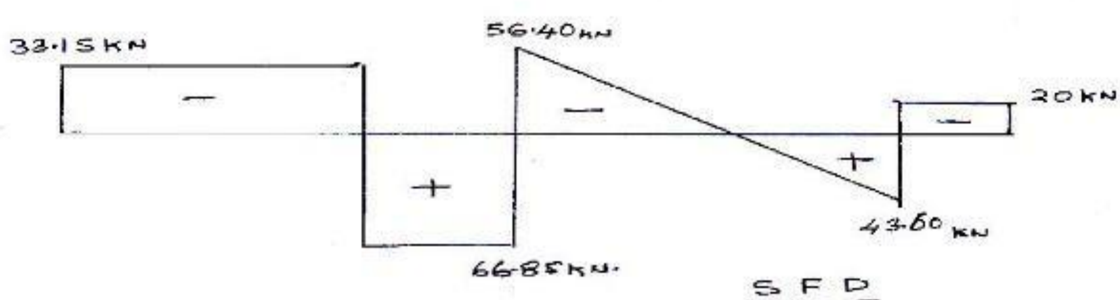
$R_B = 56.40\ KN$

$R_C = 20 \times 5 - R_B = 43.60KN$



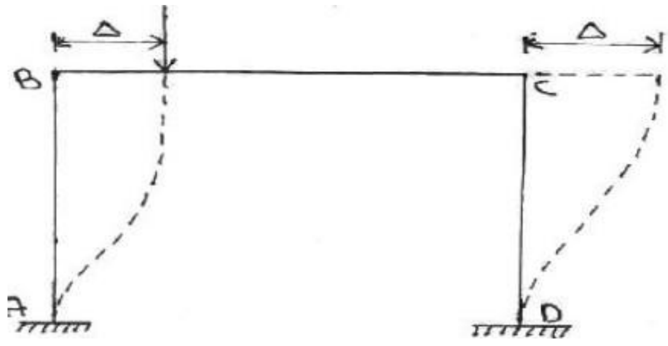




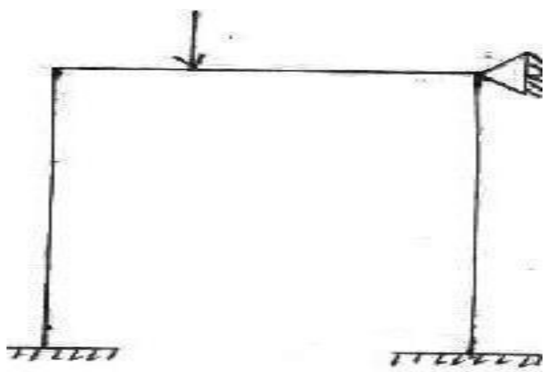



## ANALYSIS OF FRAMES (WITHOUT & WITH SWAY)

The side movement of the end of a column in a frame is called sway. Sway can be prevented by unyielding supports provided at the beam level as well as geometric or load symmetry about vertical axis.

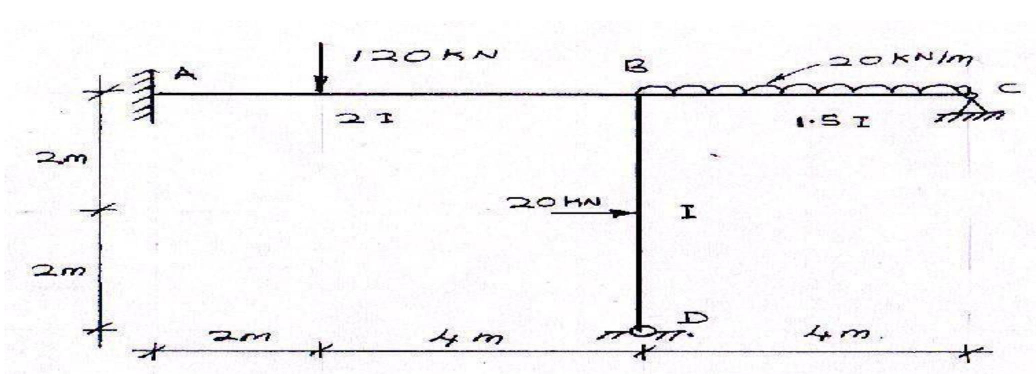



Frame with sway



Sway prevented by unyielding support

4. Q. Analyse the simple frame shown in figure. End A is fixed and ends B & Care hinged. Draw the bending momentdiagram.



$$M_{BA} + M_{BC} + M_{BD} = 0$$
$$M_{CB} = 0$$
$$M_{DB} = 0$$

$$M_{BA} + M_{BC} + M_{BD} = 53.33 + \frac{4EI}{3}\theta_B - 26.67 + \frac{3EI}{2}\theta_B + \frac{3EI}{4}\theta_C + 10 + EI\theta_B + \frac{EI}{2}\theta_D =$$
$$36.66 + \frac{23}{6}EI\theta_B + \frac{3}{4}EI\theta_C + \frac{EI}{2}\theta_D = 0 \qquad \cdots\cdots (7)$$
$$M_{CB} = 26.67 + \frac{3EI}{2}\theta_C + \frac{3EI}{4}\theta_B = 0 \qquad \cdots\cdots (8)$$
$$M_{DB} == -10 + EI\theta_D + \frac{EI}{2}\theta_B = 0 \qquad \cdots\cdots (9)$$

$$M_{BC=} -\frac{\quad}{12} - \frac{\quad}{12} = -26.67\,KNm$$
$$M_{CB} = \frac{wL^2}{12} = \frac{20 \times 4^2}{12} = 26.67\,KNm$$
$$M_{DB} = -\frac{wL}{8} = -\frac{20 \times 4}{8} = -10\,KNm$$
$$M_{BD} = \frac{wL}{8} = \frac{20 \times 4}{8} = 10\,KNm$$

Slope deflection equations are

$$M_{AB} = M_{FAB} + \frac{2EI}{L}[2\theta_A + \theta_B] = -106.67 + \frac{2E(2I)}{6}\theta_B$$
$$= -106.67 + \frac{2EI}{3}\theta_B \qquad \cdots\cdots (1)$$

$$M_{BA} = M_{FBA} + \frac{2EI}{L}[2\theta_B + \theta_A] = 53.33 + \frac{2E(2I)}{6}2\theta_B = 53.33 + \frac{4EI}{3}\theta_B \qquad \cdots\cdots (2)$$
$$M_{BC} = M_{FBC} + \frac{2EI}{L}[2\theta_B + \theta_C] = -26.67 + \frac{3EI}{2}\theta_B + \frac{3EI}{4}\theta_C \qquad \cdots\cdots (3)$$
$$M_{CB} = M_{FCB} + \frac{2EI}{L}[2\theta_C + \theta_B] = 26.67 + \frac{3EI}{2}\theta_C + \frac{3EI}{4}\theta_B \qquad \cdots\cdots (4)$$
$$M_{BD} = M_{FBD} + \frac{2EI}{L}[2\theta_B + \theta_D] = 10 + \frac{2EI}{4}2\theta_B + \frac{2EI}{4}\theta_D$$
$$= 10 + EI\theta_B + \frac{EI}{2}\theta_D \qquad \cdots\cdots (5)$$
$$M_{DB} = M_{FDB} + \frac{2EI}{L}[2\theta_D + \theta_B] = -10 + \frac{2EI}{4}2\theta_D + \frac{2EI}{4}\theta_B$$
$$= -10 + EI\theta_D + \frac{EI}{2}\theta_B \qquad \cdots\cdots (6)$$

In all the above equations there are only 3 unknowns and accordingly the boundary conditions are

Solving equations (7) & (8) & (9),

$$\theta_B = -\frac{8.83}{EI}$$
$$\theta_C = -\frac{13.36}{EI}$$
$$\theta_D = \frac{14.414}{EI}$$

Substituting the values in the slope deflections we have,

$$M_{AB} = -106.67 + \frac{2}{3}(-8.83) = -112.56KNm$$

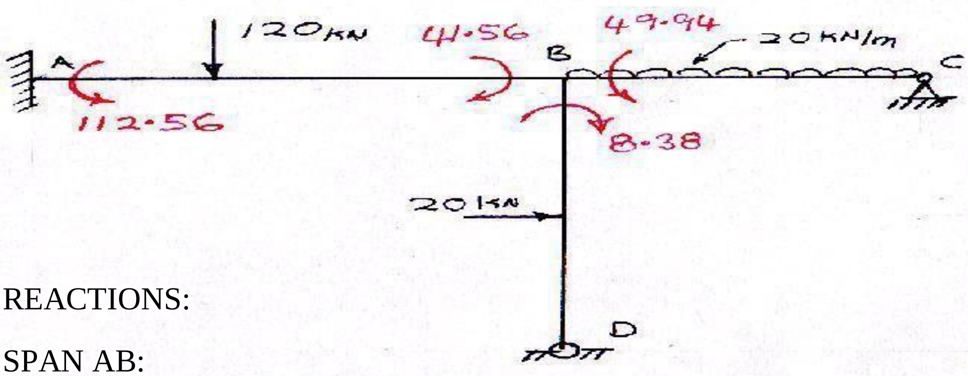
$$M_{BA} = 53.33 + \frac{4}{3}(-8.83) = 41.56KNm$$

$$M_{BC} = -26.67 + \frac{3}{2}(-8.83) + \frac{3}{4}(-13.36) = -49.94KNm$$

$$M_{CB} = 26.67 + \frac{3}{2}(-13.36) + \frac{3}{4}(-8.83) = 0$$

$$M_{BD} = 10 - 8.83 + \frac{1}{2}(14.414) = 8.38KNm$$

$$M_{DB} = -10 + \frac{1}{2}(-8.83) + (14.414) = 0$$




REACTIONS:

SPAN AB:

$$R_B = \frac{41.56 - 112.56 + 120 \times 2}{6} = 28.17KN$$

$$R_A = 120 - R_B = 91.83KN$$

SPAN BC:

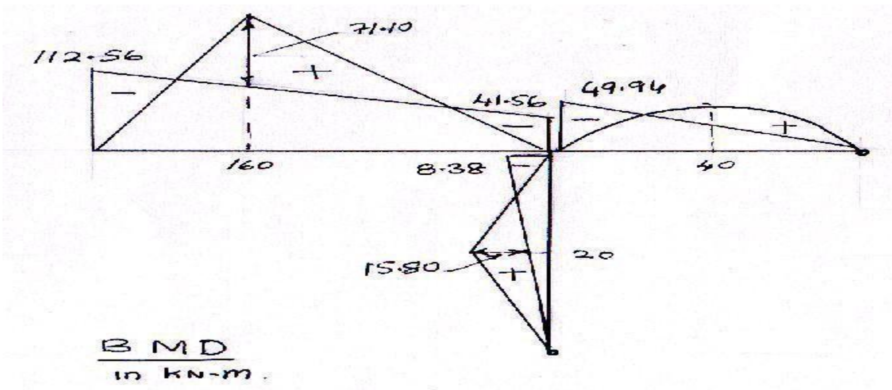
$$R_B = \frac{49.94 + 20 \times 4 \times 2}{4} = 52.485KN$$

$$R_C = 20 \times 4 - R_B = 27.515KN$$

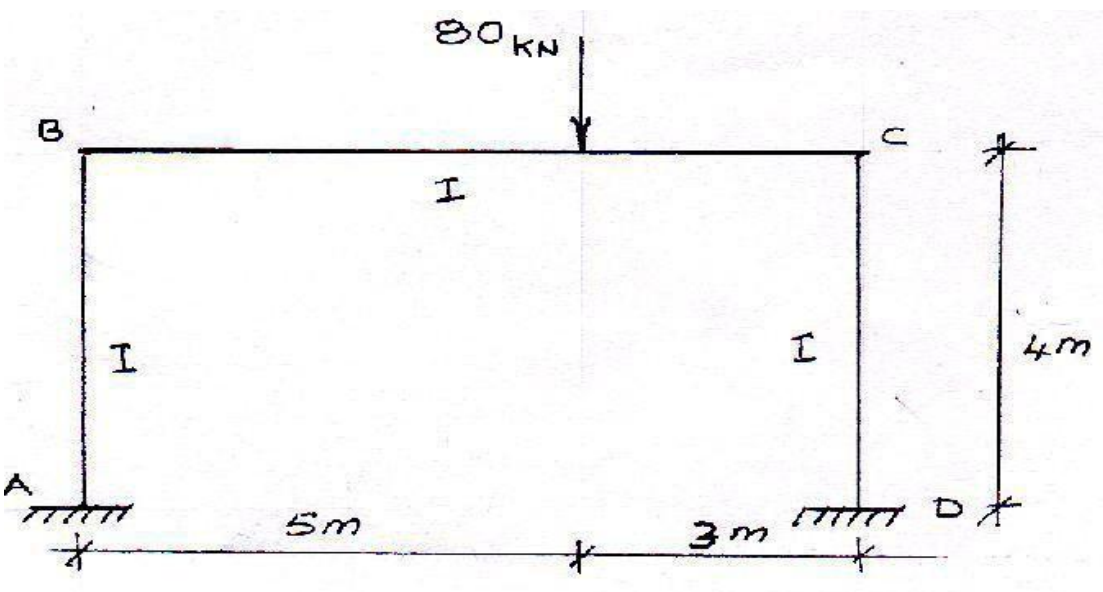
Column BD:

$$H_D = \frac{20 \times 2 - 8.38}{4} = 7.92KN$$

$$H_B = 12.78KN$$




Analyse the portal frame and then draw the bending momentdiagram




A. This is a symmetrical frame and unsymmetrically loaded, thus it is an unsymmetrical problem and there is a sway ,assume sway toright

$\theta_A = 0, \theta_D = 0, \theta_B \neq 0, \theta_C \neq 0$

FEMS:

$$M_{BC=} -\frac{wab^2}{L^2} = -\frac{80 \times 5 \times 3^2}{8^2} = -56.25KNm$$
$$M_{CB=} \frac{wa^2b}{L^2} = \frac{80 \times 3 \times 5^2}{8^2} = 93.75\ KNm$$

Slope deflection equations are

$$M_{AB} = M_{FAB} + \frac{2EI}{L}\left[2\theta_A + \theta_B - 3\frac{\Delta}{L}\right] = 0 + \frac{2EI}{4}\left(0 + \theta_B - 3\frac{\Delta}{L}\right)$$
$$= \frac{EI}{2}\theta_B - \frac{3EI\Delta}{8} \qquad \cdots\cdots(1)$$

$$M_{BA} = M_{FBA} + \frac{2EI}{L}\left[2\theta_B + \theta_A - 3\frac{\Delta}{L}\right]$$
$$= 0 + \frac{2EI}{4}\left(2\theta_B + 0 - 3\frac{\Delta}{4}\right) = EI\theta_B - \frac{3EI}{8}\Delta \qquad \cdots\cdots(2)$$
$$M_{BC} = M_{FBC} + \frac{2EI}{L}[2\theta_B + \theta_C] = -56.25 + \frac{2EI}{8}(2\theta_B + \theta_C)$$
$$= -56.25 + \frac{EI}{2}\theta_B + \frac{EI}{4}\theta_C \qquad \cdots\cdots(3)$$
$$M_{CB} = M_{FCB} + \frac{2EI}{L}[2\theta_C + \theta_B] = 93.75 + \frac{2EI}{8}[2\theta_C + \theta_B]$$
$$= 93.75 + \frac{EI}{2}\theta_C + \frac{EI}{4}\theta_B \qquad \cdots\cdots(4)$$

$$M_{CD} = M_{FCD} + \frac{2EI}{L}\left[2\theta_C + \theta_D - 3\frac{\Delta}{L}\right] = 0 + \frac{2EI}{4}\left(2\theta_C + 0 - 3\frac{\Delta}{L}\right)$$
$$= EI\theta_C - \frac{3EI\Delta}{8} \qquad \cdots\cdots(5)$$

$$M_{DC} = M_{FDC} + \frac{2EI}{L}\left[2\theta_D + \theta_C - 3\frac{\Delta}{L}\right]$$
$$= 0 + \frac{2EI}{4}\left(0 + \theta_C - 3\frac{\Delta}{4}\right) = \frac{EI}{2}\theta_C - \frac{3EI}{8}\Delta \qquad \cdots\cdots(6)$$

In the above equation there are three unknowns, $\theta_B, \theta_C$& $\Delta$,accordingly the boundary conditions are,joint conditions , $M_{BA} + M_{BC} = 0, M_{CB} + M_{CD} = 0$
shear condition, $H_A + H_D + \sum P_H = 0,\ \ \frac{M_{AB} + M_{BA}}{4} + \frac{M_{CD} + M_{DC}}{4} = 0$

$$\therefore M_{AB} + M_{BA} + M_{CD} + M_{DC} = 0$$

Now, $M_{BA} + M_{BC} = EI\theta_B - \frac{3EI}{8}\Delta - 56.25 + \frac{EI}{2}\theta_B + \frac{EI}{4}\theta_C$ =-56.25+$\frac{3EI}{2}\theta_B$+$\frac{EI}{4}\theta_C - \frac{3EI}{8}\Delta = 0$ ⋯⋯ (7)

$M_{CB} + M_{CD} = 93.75 + \frac{EI}{2}\theta_C + \frac{EI}{4}\theta_B + EI\theta_C - \frac{3EI\Delta}{8} = 93.75 + \frac{EI}{4}\theta_B + \frac{3EI}{2}\theta_C - \frac{3EI\Delta}{8} = 0$ ⋯⋯ (8)

$M_{AB} + M_{BA} + M_{CD} + M_{DC} = \frac{EI}{2}\theta_B - \frac{3EI\Delta}{8} + EI\theta_B - \frac{3EI}{8}\Delta + EI\theta_C - \frac{3EI\Delta}{8} + \frac{EI}{2}\theta_C - \frac{3EI}{8}\Delta = \frac{3EI}{2}\theta_B + \frac{3EI}{2}\theta_C - \frac{3EI}{2}\Delta = 0$

$\therefore EI\Delta = EI\theta_B + EI\theta_C$ ⋯⋯ (9)

Substitute in (7) & (8), equation (9),

$-56.25 + \frac{3EI}{2}\theta_B + \frac{EI}{4}\theta_C - \frac{3EI}{8}(\theta_B + \theta_C) = 0$

$-56.25 + \frac{9EI}{8}\theta_B - \frac{EI}{8}\theta_C = 0$ ⋯⋯ (10)

$93.75 + \frac{EI}{4}\theta_B + \frac{3EI}{2}\theta_C - \frac{3EI}{8}(\theta_B + \theta_C) = 0$

$93.75 - \frac{EI}{8}\theta_B + \frac{9EI}{8}\theta_C = 0$ ⋯⋯ (11)

Solving equations (10) & (11), we get

$$\theta_B = \frac{41.25}{EI}$$

By equation (10),

$$EI\theta_C = 8\left[-56.25 + \frac{9EI}{8}\theta_B\right] = 8\left[-56.25 + \frac{9}{8}(41.25)\right] = -78.75$$

$$\theta_C = \frac{-78.75}{EI}$$

$EI\Delta = EI\theta_B + EI\theta_C = 41.25 - 78.75 = -37.5$

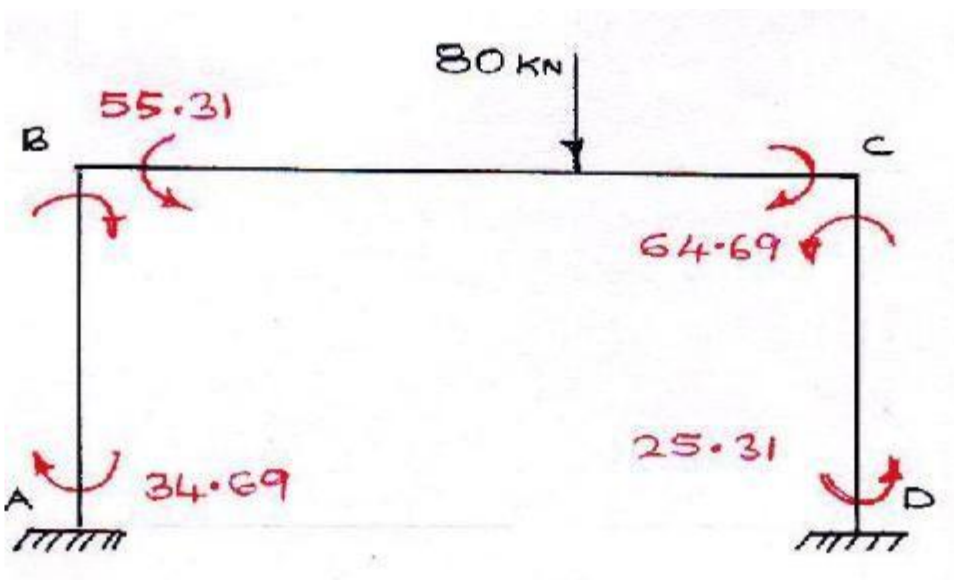
$$\Delta = \frac{-37.5}{EI}$$

Substituting these values in slope deflection equation, we have,

$$M_{AB} = \frac{1}{2}(41.25) - \frac{3}{8}(-37.5) = 34.69KNm$$

$$M_{BA} = 41.25 - \frac{3}{8}(-37.5) = 55.31KNm$$
$$M_{BC} = -56.25 + \frac{1}{2}(41.25) + \frac{1}{4}(-78.75) = -55.31KNm$$
$$M_{CB} = 93.75 + \frac{1}{2}(-78.75) + \frac{1}{4}(41.25) = 64.69KNm$$

$$M_{CD} = -78.75 - \frac{3}{8}(-37.5) = -64.69KNm$$
$$M_{DC} = \frac{1}{2}(-78.75) - \frac{3}{8}(-37.5) = -25.31KNm$$




**Reactions:** consider the free body diagram of beam and columns

Column AB:

$$H_A = \frac{34.69 + 55.31}{4} = 22.5KN$$

Span BC:

$$R_B = \frac{55.31 - 64.69 + 80 \times 3}{8} = 28.83KN$$
$$R_C = 80 - R_B = 51.17KN$$

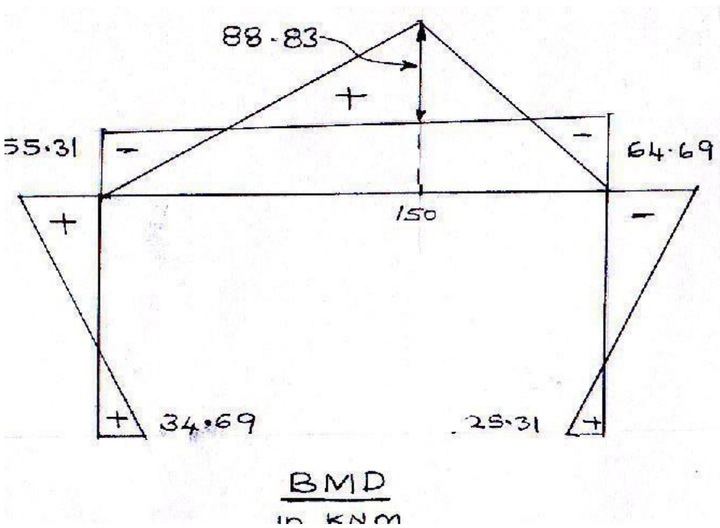
Column CD:

$$H_D = \frac{64.69 + 25.31}{4} = 22.5KN$$

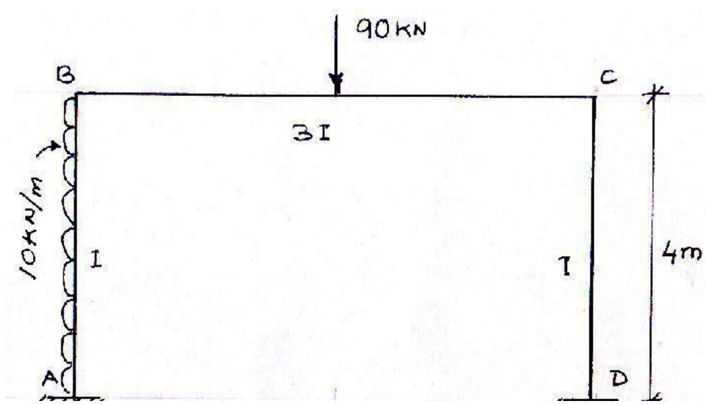
Check:

$\Sigma H = 0$, $H_A + H_D = 0$, $22.5 - 22.5 = 0$

Hence okay




6. Q. Frame ABCD is subjected to a horizontal force of 20 KN at joint C as shown in figure. Analyse and draw bending moment diagram.



The frame is symmetrical but loading is unsymmetrical. Hence there is a sway, assume sway towards right. In this problem

FEMS

$$M_{FAB=}\ -\frac{wL^2}{12}=-\frac{10\ \times 4^2}{12}=\ -13.33KNm$$

$$M_{FBA=}\ \frac{wL^2}{12}=\frac{10\ \times 4^2}{12}=13.33\ KNm$$

$$M_{FBC=}\ -\frac{wL}{8}=-\frac{90\times 10}{8}=-112.5KNm$$

$$M_{FCB=}\ \frac{wL}{8}=\frac{90\times10}{8}=112.5KNm$$

Slope deflection equations:

$$M_{AB}\ = M_{FAB}+\frac{2EI}{L}\left[2\theta_A+\ \theta_B-3\frac{\Delta}{L}\right]=\ -13.33+\frac{2EI}{4}\left(0+\theta_B-3\frac{\Delta}{4}\right)$$
$$=-13.33+\frac{EI}{2}\theta_B-\frac{3EI\Delta}{8} \qquad \cdots\cdots(1)$$

$$M_{BA}\ = M_{FBA}+\frac{2EI}{L}\left[2\theta_B+\ \theta_A-3\frac{\Delta}{L}\right]$$
$$=13.33$$
$$+\frac{2EI}{4}\left(2\theta_B+0-3\frac{\Delta}{4}\right)=13.33+EI\theta_B-\frac{3EI}{8}\Delta \qquad \cdots\cdots(2)$$

$$M_{BC}\ = M_{FBC}+\frac{2EI}{L}[2\theta_B+\ \theta_C]=\ -112.5+\frac{2E3I}{10}(2\theta_B+\theta_C)$$
$$=-112.5+\frac{6EI}{5}\theta_B+\frac{3EI}{5}\theta_C \qquad \cdots\cdots(3)$$

$$M_{CB}\ = M_{FCB}+\frac{2EI}{L}[2\theta_C+\ \theta_B]=\ 112.5+\frac{2E3I}{10}[2\theta_C+\ \theta_B]$$
$$=112.5+\frac{6EI}{5}\theta_C+\frac{3EI}{5}\theta_B \qquad \cdots\cdots(4)$$

$$M_{CD}\ = M_{FCD}+\frac{2EI}{L}\left[2\theta_C+\ \theta_D-3\frac{\Delta}{L}\right]=\ 0+\frac{2EI}{4}\left(2\theta_C+0-3\frac{\Delta}{4}\right)$$
$$=EI\theta_C-\frac{3EI\Delta}{8} \qquad \cdots\cdots(5)$$

$$M_{DC}\ = M_{FDC}+\frac{2EI}{L}\left[2\theta_D+\ \theta_C-3\frac{\Delta}{L}\right]$$
$$=0+\frac{2EI}{4}\left(0+\theta_C-3\frac{\Delta}{4}\right)=\frac{EI}{2}\theta_C-\frac{3EI}{8}\Delta \qquad \cdots\cdots(6)$$

In the above equation there are three unknowns, $\theta_B, \theta_C$ & $\Delta$, accordingly the boundary conditions are,

Joint conditions , $M_{BA} + M_{BC} = 0, M_{CB} + M_{CD} = 0$

Shear condition, $H_A + H_D + \sum P_H = 0, \quad H_A + H_D + 40 = 0$

$$H_A \times 4 = M_{AB} + M_{BA} - 10 \times 4 \times \frac{4}{2}$$

$$H_A = \frac{M_{AB} + M_{BA} - 80}{4}$$

$$H_D \times 4 = M_{CD} + M_{DC}$$

$$H_D = \frac{M_{CD} + M_{DC}}{4}$$

$$\therefore \frac{M_{AB} + M_{BA} - 80}{4} + \frac{M_{CD} + M_{DC}}{4} + 40 = 0$$

$$M_{AB} + M_{BA} + M_{CD} + M_{DC} + 80 = 0$$

Now, $M_{BA} + M_{BC} = 0$

$$13.33 + EI\theta_B - \frac{3EI}{8}\Delta - 112.5 + \frac{6EI}{5}\theta_B + \frac{3EI}{5}\theta_C = 0$$

$2.2EI\theta_B + 0.6EI\theta_C - 0.375\, EI\Delta - 99.17 = 0$ ……(7)

$M_{CB} + M_{CD} = 0$

$$112.5 + \frac{6EI}{5}\theta_C + \frac{3EI}{5}\theta_B + EI\theta_C - \frac{3EI\Delta}{8} = 0$$

$112.5 + 2.2\, EI\theta_C + 0.6EI\theta_B - 0.375\, EI\Delta = 0$ ……(8)

$$M_{AB} + M_{BA} + M_{CD} + M_{DC} + 80 = 0$$

$$-13.33 + \frac{EI}{2}\theta_B - \frac{3EI\Delta}{8} + 13.33 + EI\theta_B - \frac{3EI}{8}\Delta + EI\theta_C - \frac{3EI\Delta}{8} + \frac{EI}{2}\theta_C - \frac{3EI}{8}\Delta + 80 = 0$$

$1.5\, EI\theta_B + 1.5\, EI\theta_C - 1.5\, EI\Delta + 80 = 0$ ……(9)

Solving equations (7), (8)& (9)

$$\theta_B = \frac{72.65}{EI}$$

$$\theta_C = -\frac{59.64}{EI}$$

$$\Delta = \frac{66.34}{EI}$$

Substituting these values in slope deflection equation, we have,

$$M_{AB} = -13.33 + \frac{1}{2}(72.65) - \frac{3}{8}(66.34) = -1.88KNm$$

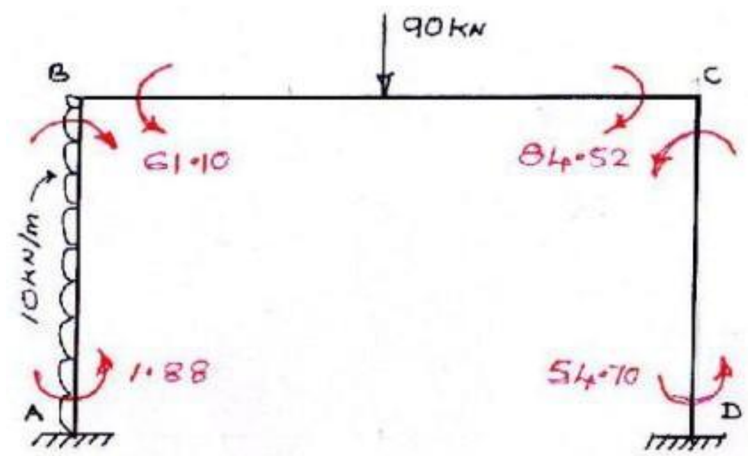
$$M_{BA} = 72.65 - \frac{3}{8}(66.34) = 61.10KNm$$

$$M_{BC} = -112.5 + \frac{6}{5}(72.65) + \frac{3}{5}(-59.64) = -61.10KNm$$

$$M_{CB} = 112.5 + \frac{6}{5}(-59.64) + \frac{3}{5}(72.65) = 84.52KNm$$

$$M_{CD} = -59.64 - \frac{3}{8}(66.34) = -84.52KNm$$

$$M_{DC} = \frac{1}{2}(-59.64) - \frac{3}{8}(66.34) = -54.70KNm$$




Reactions: Consider the free body diagram of various members

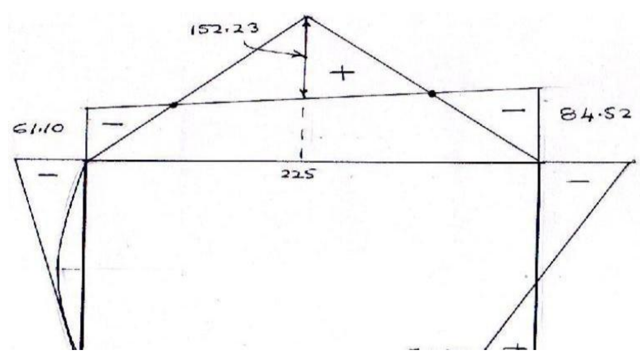
Member AB:

$$H_A = \frac{61.10 - 1.88 - 10 \times 4 \times 2}{4} = -5.195 \text{ KN}$$

Span BC:

$$R_C = \frac{84.52 - 61.10 + 90 \times 5}{10} = 47.34 \text{ KN}$$

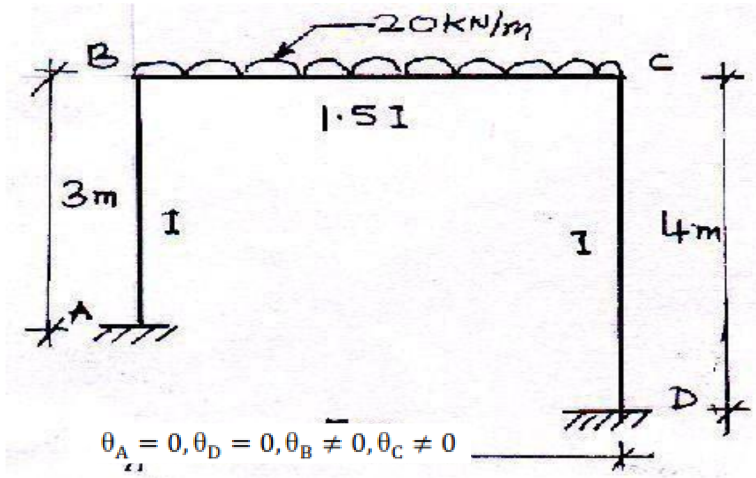
$$R_B = 90 - R_C = 38.34\text{KN}$$




Column CD:

$$H_D = \frac{84.52 + 54.7}{4} = 34.81\text{KN}$$

Analyse the portal frame and draw theB.M.D.




A. It is an unsymmetrical problem, hence there is a sway be towards right.

FEMS:

$$M_{FAB=}\ M_{FBA=}\ 0 =\ M_{FCD=}\ M_{FDC}$$

$$M_{FBC=}\ -\frac{wL^2}{12} = -\frac{20\times 5^2}{12} = -41.67\text{KNm}$$

$$M_{FCB=}\ \frac{wL^2}{12} = \frac{20\times 5^2}{12} = 41.67\text{KNm}$$

Slope deflection equations:

$$M_{AB} = M_{FAB} + \frac{2EI}{L}\left[2\theta_A + \theta_B - 3\frac{\Delta}{L}\right] = \frac{2EI}{3}\left(0 + \theta_B - 3\frac{\Delta}{3}\right)$$
$$= \frac{2EI}{3}\theta_B - \frac{2EI\Delta}{3} \qquad \cdots\cdots(1)$$

$$M_{BA} = M_{FBA} + \frac{2EI}{L}\left[2\theta_B + \theta_A - 3\frac{\Delta}{L}\right]$$
$$= \frac{2EI}{3}\left(2\theta_B + 0 - 3\frac{\Delta}{3}\right) = \frac{4EI}{3}\theta_B - \frac{2EI}{3}\Delta \qquad \cdots\cdots(2)$$

$$M_{BC} = M_{FBC} + \frac{2EI}{L}[2\theta_B + \theta_C] = -41.67 + \frac{2E(1.5)I}{5}(2\theta_B + \theta_C)$$
$$= -41.67 + \frac{6EI}{5}\theta_B + \frac{3EI}{5}\theta_C \qquad \cdots\cdots(3)$$

$$M_{CB} = M_{FCB} + \frac{2EI}{L}[2\theta_C + \theta_B] = 41.67 + \frac{2E(1.5)I}{5}[2\theta_C + \theta_B]$$

$$= 41.67 + \frac{6EI}{5}\theta_C + \frac{3EI}{5}\theta_B \qquad \cdots\cdots(4)$$

$$M_{CD} = M_{FCD} + \frac{2EI}{L}\left[2\theta_C + \theta_D - 3\frac{\Delta}{L}\right] = 0 + \frac{2EI}{4}\left(2\theta_C + 0 - 3\frac{\Delta}{4}\right)$$

$$= EI\theta_C - \frac{3EI\Delta}{8} \qquad \cdots\cdots(5)$$

$$M_{DC} = M_{FDC} + \frac{2EI}{L}\left[2\theta_D + \theta_C - 3\frac{\Delta}{L}\right]$$

$$= 0 + \frac{2EI}{4}\left(0 + \theta_C - 3\frac{\Delta}{4}\right) = \frac{EI}{2}\theta_C - \frac{3EI}{8}\Delta \qquad \cdots\cdots(6)$$

In the above equation there are three unknowns, $\theta_B, \theta_C \& \Delta$, accordingly the boundary conditions are,

Joint conditions , $M_{BA} + M_{BC} = 0, M_{CB} + M_{CD} = 0$

Shear condition, $H_A + H_D = 0, \quad \therefore \frac{M_{AB} + M_{BA}}{3} + \frac{M_{CD} + M_{DC}}{4} + 40 = 0$

$$4(M_{AB} + M_{BA}) + 3(M_{CD} + M_{DC}) = 0$$

Now, $\qquad M_{BA} + M_{BC} = 0$

$$\frac{4EI}{3}\theta_B - \frac{2EI}{3}\Delta - 41.67 + \frac{6EI}{5}\theta_B + \frac{3EI}{5}\theta_C = 0$$

$$2.53EI\theta_B + 0.6EI\theta_C - \frac{2EI}{3}\Delta - 41.67 = 0 \qquad \cdots\cdots(7)$$

$$M_{CB} + M_{CD} = 0$$

$$41.67 + \frac{6EI}{5}\theta_C + \frac{3EI}{5}\theta_B + EI\theta_C - \frac{3EI\Delta}{8} = 0$$

$41.67 + 2.2\,EI\theta_C + 0.6EI\theta_B - 0.375\,EI\Delta = 0$ ……(8)

$$4(M_{AB} + M_{BA}) + 3(M_{CD} + M_{DC}) = 0$$

$$4\left(\frac{2EI}{3}\theta_B - \frac{2EI\Delta}{3} + \frac{4EI}{3}\theta_B - \frac{2EI}{3}\Delta\right) + 3(EI\theta_C - \frac{3EI\Delta}{8} + \frac{EI}{2}\theta_C - \frac{3EI}{8}\Delta) = 0$$

$8\,EI\theta_B + 4.5\,EI\theta_C - 7.53\,EI\Delta = 0$ ……(9)

Solving equations (7), (8)& (9)

$$\theta_B = \frac{25.46}{EI}$$

$$\theta_C = \frac{-23.17}{EI}$$

$\Delta = \frac{12.8}{EI}$

Substituting these values in slope deflection equation, we have,

$$M_{AB} = \frac{2}{3}(25.46) - \frac{2}{3}(12.8) = 8.44\text{KNm}$$

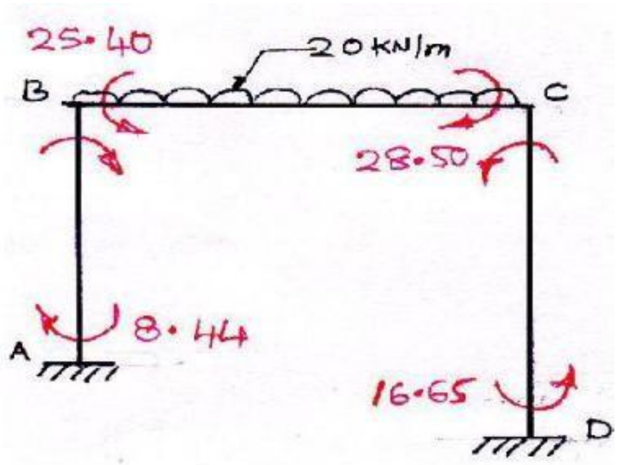
$$M_{BA} = \frac{4}{3}(25.46) - \frac{2}{3}(12.8) = 25.4\text{KNm}$$

$$M_{BC} = -41.67 + \frac{6}{5}(25.46) + \frac{3}{5}(-23.17) = -25.4\text{KNm}$$

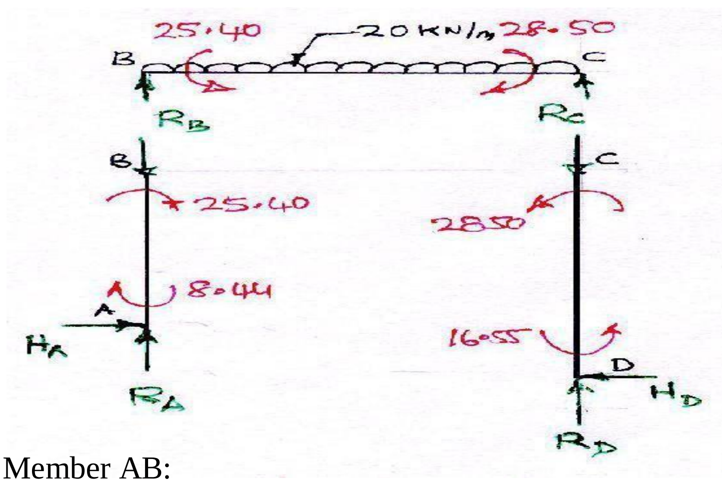
$$M_{CB} = 41.67 + \frac{6}{5}(-23.17) + \frac{3}{5}(20.46) = 28.5\text{KNm}$$

$$M_{CD} = -23.17 - \frac{3}{8}(12.8) = -28.5\text{KNm}$$

$$M_{DC} = \frac{1}{2}(-23.17) - \frac{3}{8}(12.8) = -16.65\text{KNm}$$




**Reactions:** Consider the free body diagram




Member AB:

$$H_A = \frac{25.4 + 8.44}{3} = 11.28 \text{ KN}$$

Span BC:

$$R_C = \frac{28.5 - 25.4 + 20 \times 5 \times \frac{5}{2}}{5} = 51.64 \text{ KN}$$

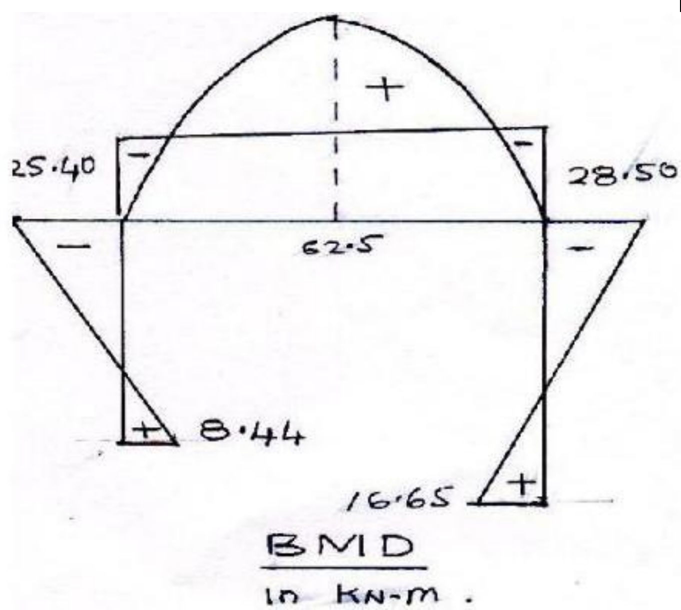
$$R_B = 20 \times 5 - 51.64 = 48.36\text{KN}$$

Column CD:

$$H_D = \frac{28.5 + 16.65}{4} = 11.28\text{KN}$$

**Check:**

$\Sigma H = 0, H_A + H_D = 0$ Satisfied, hence okay



# PART B : TWO HINGED ARCHES

## INTRODUCTION

Mainly three types of arches are used in practice: three-hinged, two-hinged and hingeless arches. In the early part of the nineteenth century, three-hinged arches were commonly used for the long span structures as the analysis of such arches could be done with confidence. However, with the development in structural analysis, for long span structures starting from late nineteenth century engineers adopted two-hinged and hingeless arches. Two-hinged arch is the statically indeterminate structure to degree one. Usually, the horizontal reaction is treated as the redundant and is evaluated by the method of least work. In this lesson, the analysis of two-hinged arches is discussed and few problems are solved to illustrate the procedure for calculating the internal forces.

## ANALYSIS OF TWO-HINGED ARCH

A typical two-hinged arch is shown in Fig. 33.1a. In the case of two-hinged arch, we have four unknown reactions, but there are only three equations of equilibrium available. Hence, the degree of statical indeterminacy is one for two- hingedarch.

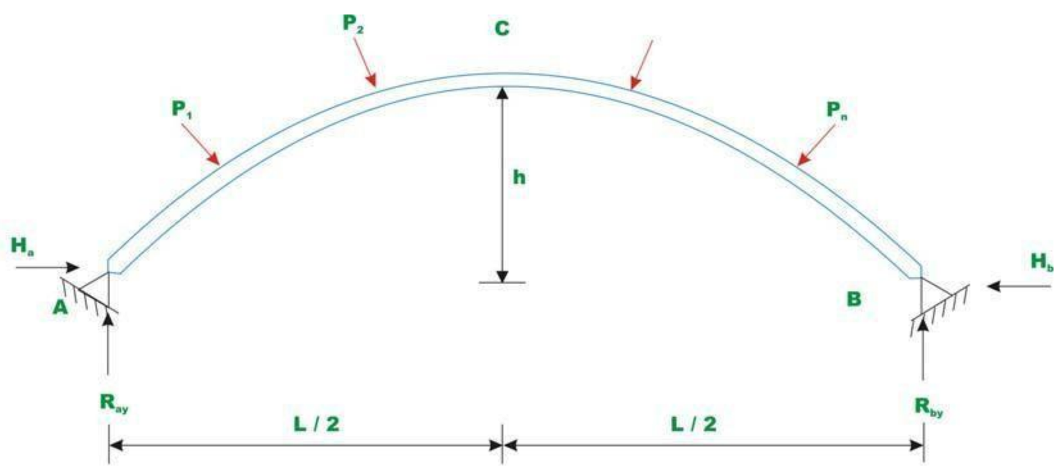
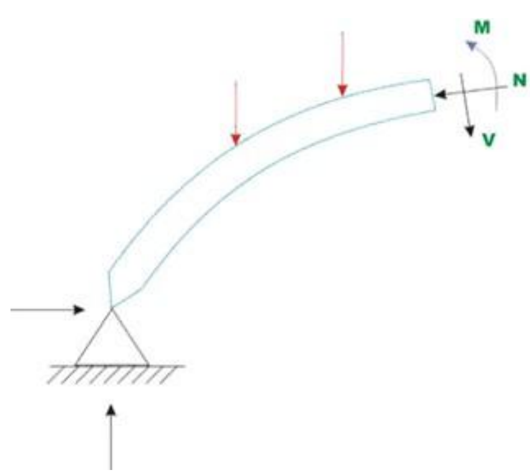



Fig. 33.1a Two - hinged arch.



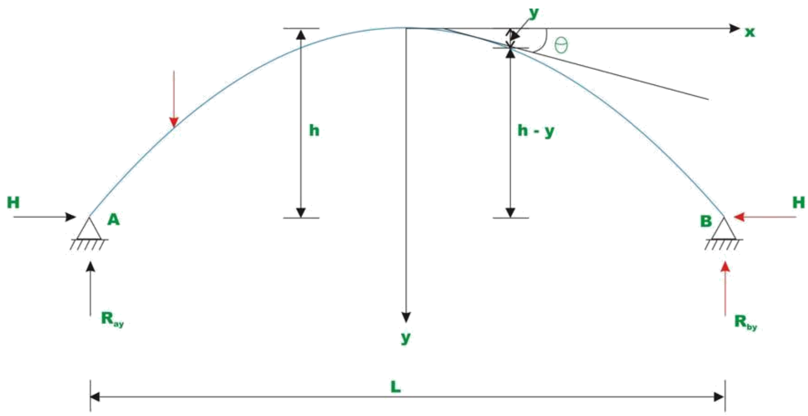



**Fig. 33.2a**

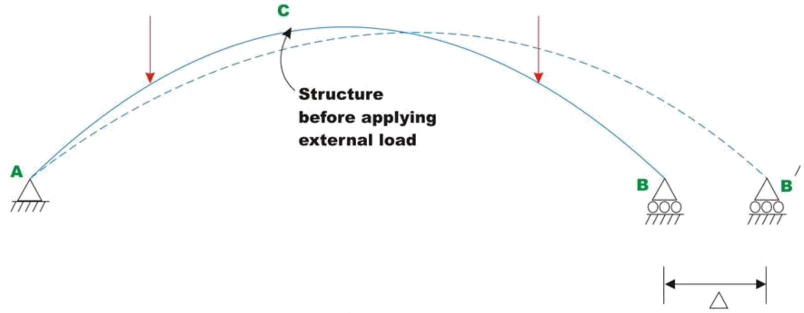



**Fig. 33.2b.**

The fourth equation is written considering deformation of the arch. The unknown redundant reaction $H_b$is calculated by noting that the horizontal displacement of hinge $B$ is zero. In general the horizontal reaction in the two hinged arch is evaluated by straightforward application of the theorem of least work (see module 1, lesson 4), which states that the partial derivative of the strain energy of a statically indeterminate structure with respect to statically indeterminate action should vanish. Hence to obtain, horizontal reaction, one must develop an expression for strain energy. Typically, any section of the arch (vide Fig 33.1b) is subjected to shear force$V$, bending moment $M$ and the axial compression $N$ . The strain energy due to bending $U_b$is calculated from the following expression.

$$U_b = \int_0^s \frac{M^2}{2EI} ds \qquad (33.1)$$

The above expression is similar to the one used in the case of straight beams. However, in this case, the integration needs to be evaluated along the curved arch length. In the above equation, $s$ is the length of the centerline of the arch, $I$ is the moment of inertia of the arch cross section, $E$ is the Young's modulus of the arch material. The strain energy due to shear is small as compared to the strain energy due to bending and is usually neglected in the analysis. In the case of flat arches, the strain energy due to axial compression can be appreciable and is givenby,

$$U_a = \int_0^s \frac{N^2}{2AE} ds \qquad (33.2)$$

The total strain energy of the arch is given by

$$U = \int_0^s \frac{M^2}{2EI} ds + \int_0^s \frac{N^2}{2AE} ds \qquad (33.3)$$

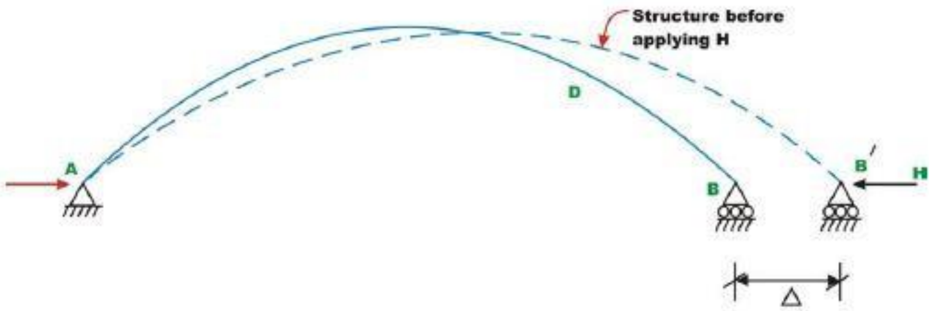



Fig. 33.2c.

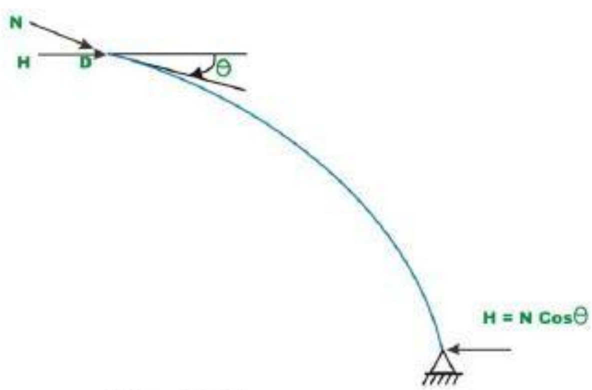



Fig. 33.2d.

From Fig. 33.2b and Fig 33.2c, the bending moment at any cross section of the arch (say $D$), may be written as

$$M = M_0 - H(h - y) \tag{33.5}$$

The axial compressive force at any cross section (say $D$) may be written as

$$N = N_0 + H\cos\theta \tag{33.6}$$

Where $\theta$ is the angle made by the tangent at $D$ with horizontal (vide Fig 33.2d). Substituting the value of $M$ and $N$ in the equation (33.4),

$$\frac{\partial U}{\partial H} = 0 = -\int_0^s \frac{M_0 - H(h-y)}{EI}(h-y)ds + \int_0^s \frac{N_0 + H\cos\theta}{EA}\cos\theta\, ds \tag{33.7a}$$

Let, $\tilde{y} = h - y$

$$-\int_0^s \frac{M_0 - H\tilde{y}}{EI}\tilde{y}\, ds + \int_0^s \frac{N_0 + H\cos\theta}{EA}\cos\theta\, ds = 0 \tag{33.7b}$$

Solving for $H$, yields

$$-\int_0^s \frac{M_0}{EI}\tilde{y}\, ds + \int_0^s \frac{H\tilde{y}^2}{EI} ds + \int_0^s \frac{N_0}{EA}\cos\theta\, ds + \int_0^s \frac{H\cos^2\theta}{EA} ds = 0$$

$$H = \frac{\int_0^s \frac{M_0}{EI}\tilde{y}\, ds - \int_0^s \frac{N_0}{EA}\cos\theta\, ds}{\int_0^s \frac{\tilde{y}^2}{EI} ds + \int_0^s \frac{\cos^2\theta}{EA} ds} \tag{33.8}$$

Using the above equation, the horizontal reaction $H$ for any two-hinged symmetrical arch may be calculated. The above equation is valid for any general type of loading. Usually the above equation is further simplified. The second term in the numerator is small compared with the first terms and is neglected in the analysis. Only in case of very accurate analysis second term s considered. Also for flat arched, $\cos\theta \cong 1$ as $\theta$ is small. The equation (33.8) is now written as,

$$H = \frac{\int_0^s \frac{M_0}{EI}\tilde{y}\, ds}{\int_0^s \frac{\tilde{y}^2}{EI} ds + \int_0^s \frac{ds}{EA}} \tag{33.9}$$

As axial rigidity is very high, the second term in the denominator may also be neglected. Finally the horizontal reaction is calculated by the equation

$$H = \frac{\int_0^s \frac{M_0}{EI} \tilde{y}\, ds}{\int_0^s \frac{\tilde{y}^2}{EI} ds} \qquad (33.10)$$

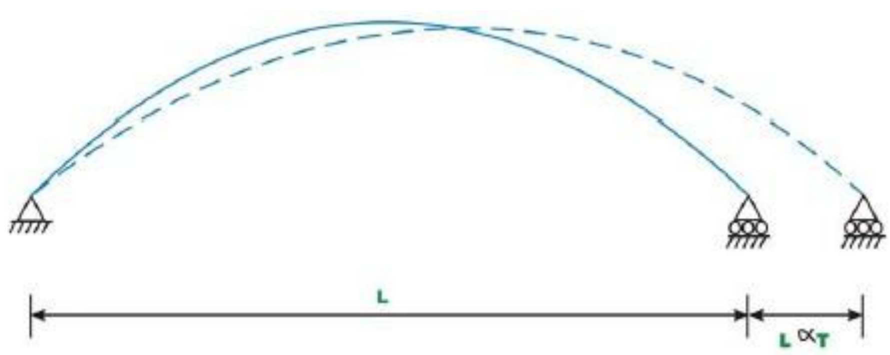
For an arch with uniform cross section $EI$ is constant and hence,

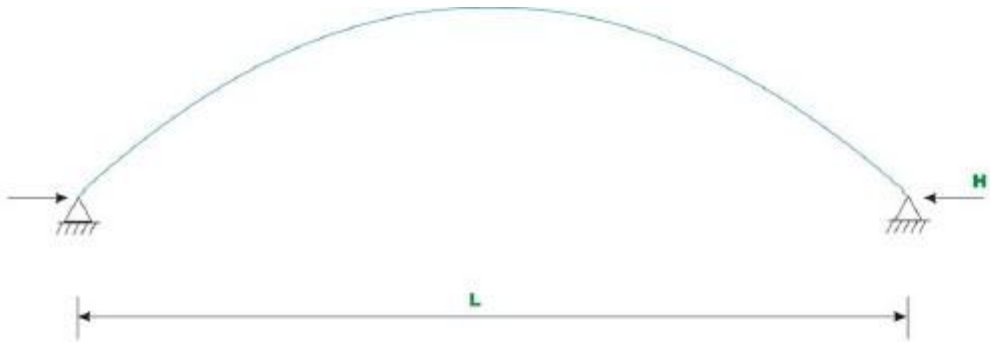
$$H = \frac{\int_0^s M_0 \tilde{y}\, ds}{\int_0^s \tilde{y}^2\, ds} \qquad (33.11)$$

In the above equation, $M_0$ is the bending moment at any cross section of the arch when one of the hinges is replaced by a roller support. $\tilde{y}$ is the height of the arch as shown in the figure. If the moment of inertia of the arch rib is not constant, then equation (33.10) must be used to calculate the horizontal reaction $H$.

## TEMPERATURE EFFECT

Consider an unloaded two-hinged arch of span $L$. When the arch undergoes a uniform temperature change of $T$, then its span would increase by $C°TL\alpha$ if it were allowed to expand freely (vide Fig 33.3a). α is the co-efficient of thermal expansion of the arch material. Since the arch is restrained from the horizontal movement, a horizontal force is induced at the support as the temperature is increase






Now applying the Castigliano's first theorem,

$$\frac{\partial U}{\partial H} = \alpha LT = \int_0^s \frac{H\tilde{y}^2}{EI} ds + \int_0^s \frac{H\cos^2\theta}{EA} ds$$

Solving for *H*,

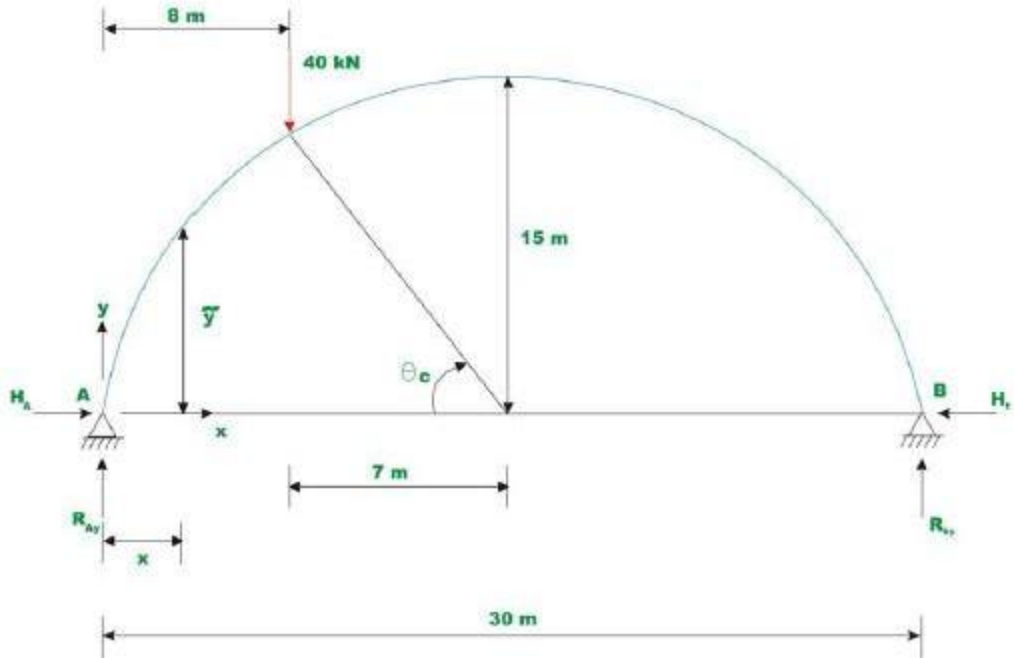
$$H = \frac{\alpha LT}{\int_0^s \frac{\tilde{y}^2}{EI} ds + \int_0^s \frac{\cos^2\theta}{EA} ds}$$

The second term in the denominator may be neglected, as the axial rigidity is quite high. Neglecting the axial rigidity, the above equation can be written as

$$H = \frac{\alpha LT}{\int^s \tilde{y}^2 \, ds}$$

**Example**

A semicircular two hinged arch of constant cross section is subjected to a concentrated load as shown in Fig. Calculate reactions of the arch and draw bending moment diagram.

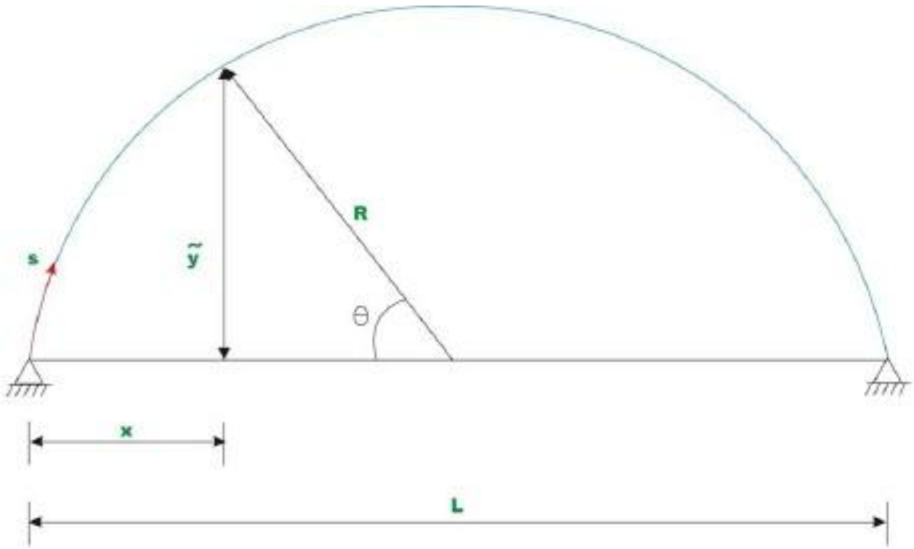



**Solution:**

Taking moment of all forces about hinge B leads to,

$$R_{ay} = \frac{40 \times 22}{30} = 29.33 \text{ kN } (\uparrow)$$

$$\sum Fy = 0 \qquad \Rightarrow R_{by} = 10.67 \text{ kN } (\uparrow)$$




From figure,

$$\tilde{y} = R\sin\theta$$

$$x = R(1-\cos\theta)$$

$$ds = R\,d\theta \qquad (2)$$

$$\tan\theta_c = \frac{13.267}{7} \qquad \Rightarrow \theta_c = 62.18° = \pi/2.895\,rad$$

Now, the horizontal reaction $H$ may be calculated by the following expression,

$$H = \frac{\int_0^s M_0\tilde{y}\,ds}{\int_0^s \tilde{y}^2\,ds} \qquad (3)$$

Now $M_0$ the bending moment at any cross section of the arch when one of the hinges is replaced by a roller support is given by,

$$M_0 = R_{ay}x = R_{ay}R(1-\cos\theta) \qquad 0 \le \theta \le \theta_c$$

and,

$$\begin{aligned} M_0 &= R_{ay}R(1-\cos\theta) - 40(x-8) \\ &= R_{ay}R(1-\cos\theta) - 40\{R(1-\cos\theta)-8\} \qquad \theta_c \le \theta \le \pi \end{aligned} \qquad (4)$$

Integrating the numerator in equation (3),

$$\int_0^s M_0\bar{y}ds = \int_0^{\theta_c} R_{ay}R^3(1-\cos\theta)\sin\theta\, d\theta + \int_{\theta_c}^{\pi}[R_{ay}R(1-\cos\theta) - 40\{R(1-\cos\theta)-8\}]R\sin\theta\, Rd\theta$$

$$= R_{ay}R^3\int_0^{\pi/2.895}(1-\cos\theta)\sin\theta\, d\theta + R^2\int_{\pi/2.895}^{\pi}[R_{ay}R(1-\cos\theta)\sin\theta - 40\{R(1-\cos\theta)\sin\theta - 8\sin\theta\}]d\theta$$

$$= R_{ay}R^3[-\cos\theta]_0^{\pi/2.895} + R^2\left[[R_{ay}R(-\cos\theta)]_{\pi/2.895}^{\pi} - [40R(-\cos\theta)]_{\pi/2.895}^{\pi} + [40\times 8(-\cos\theta)]_{\pi/2.895}^{\pi}\right]$$

$$= 0.533R_{ay}R^3 + R^2[[1.4667R_{ay}R] - [40R(1.4667)] + [40\times 8(1.4667)]]$$

$$= 52761.00 + 225(645.275 - 410.676) = 105545.775 \qquad (5)$$

The value of denominator in equation (3), after integration is,

$$\begin{aligned} \int_0^s \bar{y}^2 ds &= \int_0^{\pi}(R\sin\theta)^2 Rd\theta \\ &= R^3\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta = R^3\left(\frac{\pi}{2}\right) = 5301.46 \end{aligned} \qquad (6)$$

Hence, the horizontal thrust at the support is,

$$H = \frac{105545.775}{5301.46} = 19.90 \text{ kN} \qquad (7)$$

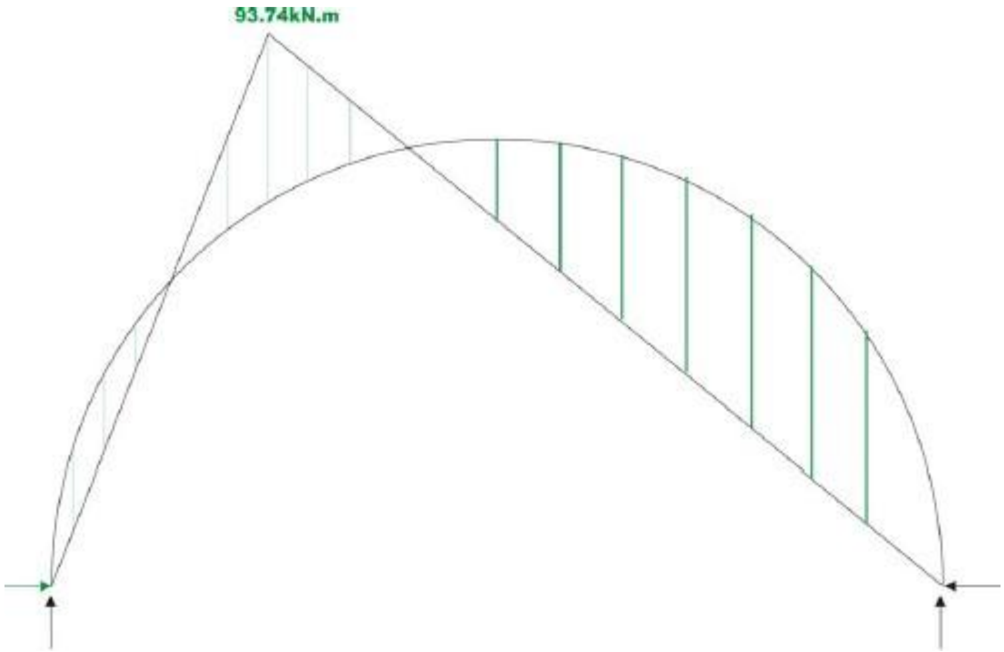
**Bending moment diagram**

Bending moment $M$ at any cross section of the arch is given by,

$$\begin{aligned} M &= M_0 - H\bar{y} \\ &= R_{ay}R(1-\cos\theta) - HR\sin\theta \qquad 0 \le \theta \le \theta_c \\ &= 439.95(1-\cos\theta) - 298.5\sin\theta \end{aligned} \qquad (8)$$

$$M = 439.95(1-\cos\theta) - 298.5\sin\theta - 40(15(1-\cos\theta)-8) \qquad \theta_c \le \theta \le \pi \qquad (9)$$

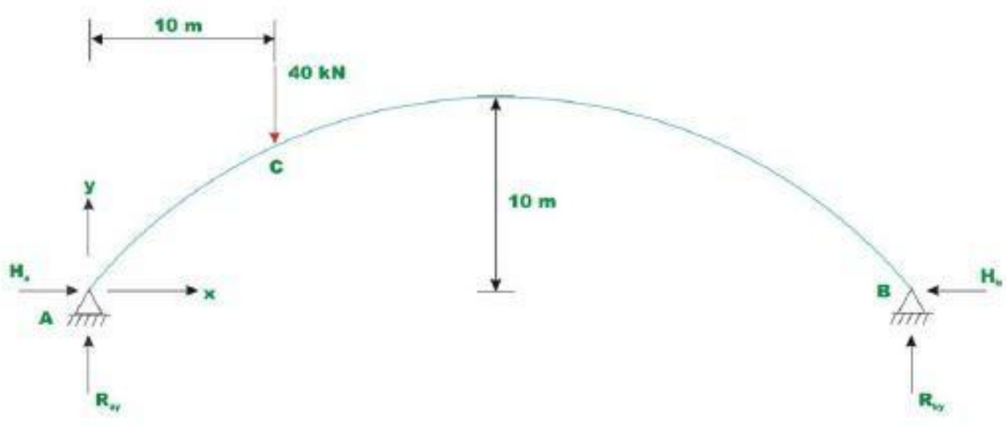
Using equations (8) and (9), bending moment at any angle θ can be computed. The bending moment diagram is shown in Fig.




Example

A two hinged parabolic arch of constant cross section has a span of 60m and a rise of 10m. It is subjected to loading as shown in Fig.. Calculate reactions of the arch if the temperature of the arch is raised by. Assume co-efficient of thermal expansion as

$\alpha = 12 \times 10^{-6} / °C.$



Taking A as the origin, the equation of two hinged parabolic arch may be written as,

$$y = \frac{2}{3}x - \frac{10}{30^2}x^2 \qquad (1)$$

The given problem is solved in two steps. In the first step calculate the horizontal reaction due to 40kN load applied at C. In the next step calculate the horizontal reaction due to rise in temperature. Adding both, one gets the horizontal reaction at the hinges due to 40kN combined external loading and temperature change. The horizontal reaction due to load may be calculated by the following equation,

$$H_1 = \frac{\int_0^s M_0 y \, ds}{\int_0^s \tilde{y}^2 \, ds} \qquad (2a)$$

For temperature loading, horizontal reaction is given by,

$$H_2 = \frac{\alpha L T}{\int_0^s \frac{y^2}{EI} ds} \qquad (2b)$$

Where $L$ is the span of the arch.

For 40 kN load,

$$\int_0^s M_0 y \, ds = \int_0^{10} R_{ay} \, x \, y \, dx + \int_{10}^{60} \left[ R_{ay} x - 40(x-10) \right] y \, dx \qquad (3)$$

Please note that in the above equation, the integrations are carried out along the x- axis instead of the curved arch axis. The error introduced by this change in the variables in the case of flat arches is negligible. Using equation (1), the above equation (3) can be easily evaluated.

The vertical reaction A is calculated by taking moment of all forces about B. Hence,

$$R_{ay} = \frac{1}{60}\left[40\times 50\right] = 33.33 \quad \text{kN}$$

$$R_{by} = 6.67 \quad \text{kN}.$$

Now consider the equation (3),

$$\int_0^l M_0 y\, dx = \int_0^{10} (33.33)\, x\, (\frac{2}{3}x - \frac{10}{30^2}x^2)\, dx + \int_{10}^{60} [(33.33)x - 40(x-10)](\frac{2}{3}x - \frac{10}{30^2}x^2)\, dx$$

$$= 6480.76 + 69404.99 = 74885.75 \qquad (4)$$

$$\int_0^l y^2\, dx = \int_0^{60} \left[\frac{2}{3}x - \frac{10}{30^2}x^2\right]^2 dx$$

$$= 3200 \qquad (5)$$

Hence, the horizontal reaction due to applied mechanical loads alone is given by,

$$H_1 = \frac{\int_0^l M_0 y \quad dx}{\int_0^l y^2 \quad dx} = \frac{75885.75}{3200} = 23.71 \quad \text{kN} \qquad (6)$$

The horizontal reaction due to rise in temperature is calculated by equation (2b),

$$H_2 = \frac{12\times 10^{-6}\times 60\times 40}{3200/EI} = \frac{EI\times 12\times 10^{-6}\times 60\times 40}{3200}$$

Taking $E = 200 \quad \text{kN/mm}^2$ and $I = 0.0333 m^4$

$$H_2 = 59.94 \text{ kN.} \tag{7}$$

Hence the total horizontal thrust $H = H_1 + H_2 = 83.65$ kN.

When the arch shape is more complicated, the integrations $\int_0^s \frac{M_0 y}{EI} ds$ and $\int_0^s \frac{y^2}{EI} ds$ are accomplished numerically. For this purpose, divide the arch span in to $n$ equals divisions. Length of each division is represented by $(\Delta s)_i$ (vide Fig.33.5b). At the midpoint of each division calculate the ordinate $y_i$ by using the equation $y = \frac{2}{3}x - \frac{10}{30^2}x^2$. The above integrals are approximated as,

$$\int_0^s \frac{M_0 y}{EI} ds = \frac{1}{EI} \sum_{i=1}^{n} (M_0)_i y_i (\Delta s)_i \tag{8}$$

$$\int_0^s \frac{y^2}{EI} ds = \frac{1}{EI} \sum_{i=1}^{n} (y)_i^2 (\Delta s)_i \tag{9}$$

The complete computation for the above problem for the case of external loading is shown in the following table.

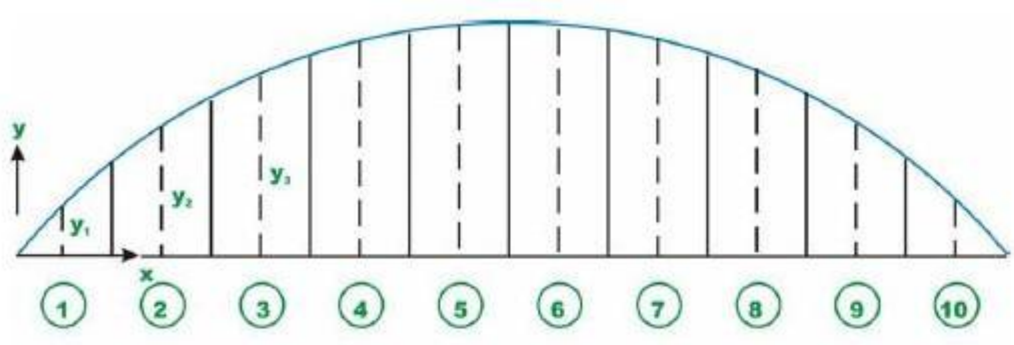



Table 1. Numerical integration of equations (8) and (9)

| Segment No | Horizontal distance *x* Measured from A (m) | Corresponding $y_i$ (m) | Moment at that Point $(M_0)_i$ (kNm) | $(M_0)_i y_i (\Delta s)_i$ | $(y)_i^2 (\Delta s)_i$ |
|---|---|---|---|---|---|
| 1 | 3 | 1.9 | 99.99 | 1139.886 | 21.66 |
| 2 | 9 | 5.1 | 299.97 | 9179.082 | 156.06 |
| 3 | 15 | 7.5 | 299.95 | 13497.75 | 337.5 |
| 4 | 21 | 9.1 | 259.93 | 14192.18 | 496.86 |
| 5 | 27 | 9.9 | 219.91 | 13062.65 | 588.06 |
| 6 | 33 | 9.9 | 179.89 | 10685.47 | 588.06 |
| 7 | 39 | 9.1 | 139.87 | 7636.902 | 496.86 |
| 8 | 45 | 7.5 | 99.85 | 4493.25 | 337.5 |
| 9 | 51 | 5.1 | 59.83 | 1830.798 | 156.06 |
| 10 | 57 | 1.9 | 19.81 | 225.834 | 21.66 |
| | | | $\sum$ | 75943.8 | 3300.3 |

$$H_1 = \frac{\sum (M_0)_i y_i (\Delta s)}{\sum (y)_i^2 (\Delta s)_i} = \frac{75943.8}{3200.3} = 23.73 \text{ kN} \qquad (10)$$

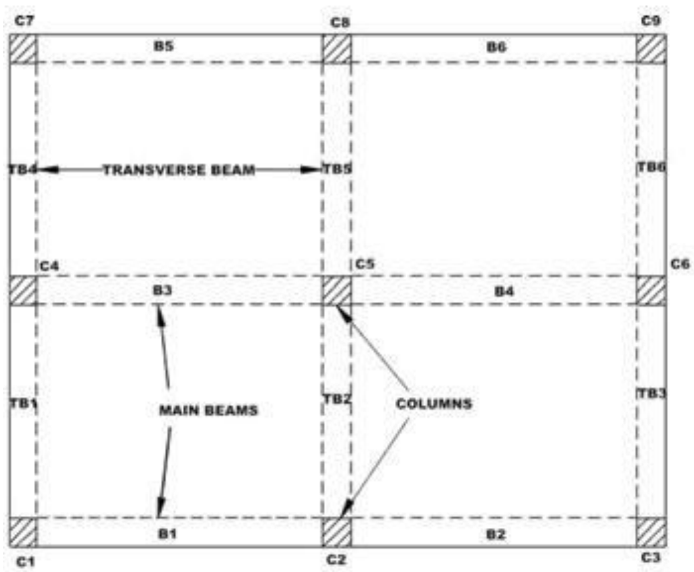
This compares well with the horizontal reaction computed from the exact integration.

# UNIT III: APPROXIMATE METHODS OFANALYSIS OF BUILDING FRAMES

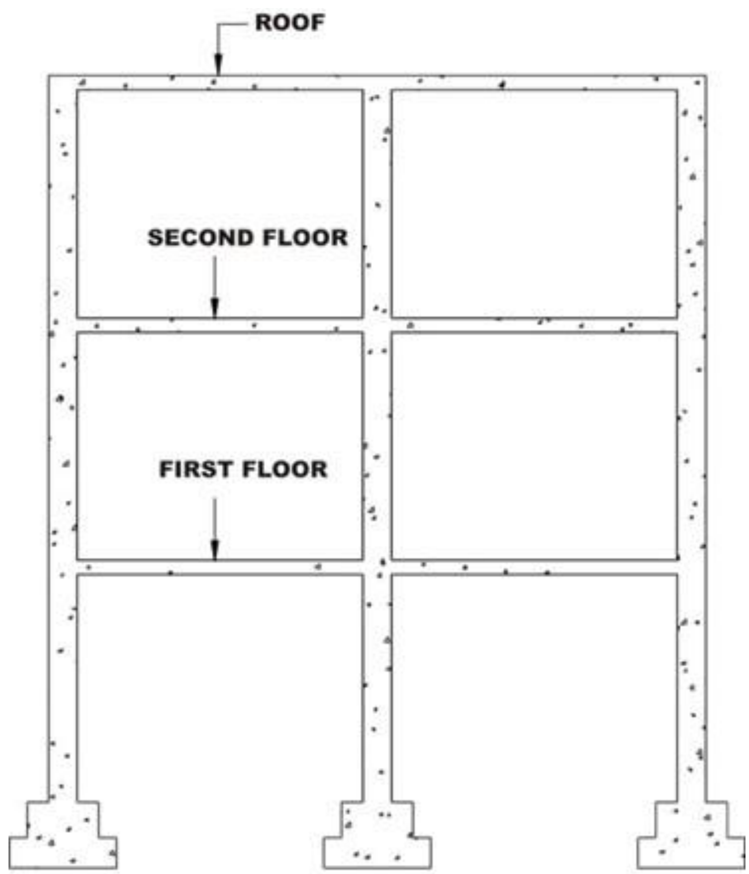
## INTRODUCTION

The building frames are the most common structural form, an analyst/engineer encounters in practice. Usually the building frames are designed such that the beam column joints are rigid. A typical example of building frame is the reinforced concrete multistory frames. A two-bay, three-storey building plan and sectional elevation are shown in Fig. In principle this is a three dimensional frame.

However, analysis may be carried out by considering planar frame in two perpendicular directions separately for both vertical and horizontal loads as shown in Fig. 36.2 and finally superimposing moments appropriately. In the case of building frames, the beam column joints are monolithic and can resist bending moment, shear force and axial force. Any exact method, such as slope-deflection method, moment distribution method or direct stiffness method may be used to analyse this rigid frame. However, in order to estimate the preliminary size of different members, approximate methods are used to obtain approximate design values of moments, shear and axial forces in various members. Before applying approximate methods, it is necessary to reduce the given indeterminate structure to a determinate structure by suitable assumptions. These will be discussed in this lesson. In next section, analysis of building frames to vertical loads is discussed and in section after that, analysis of building frame to horizontal loads will be discussed.




Plan




**Sectional Elevation Along $C_1$ - $C_3$**

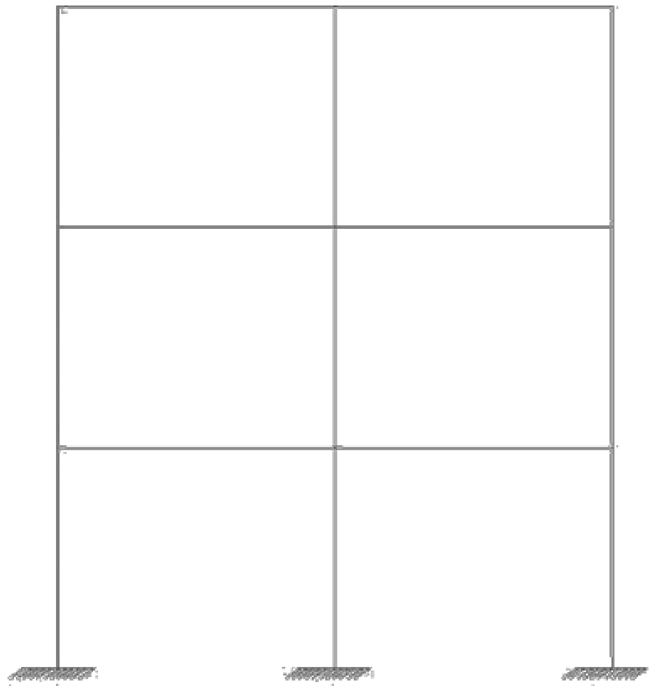


**Fig.36.2 Idealized frame for analysis**

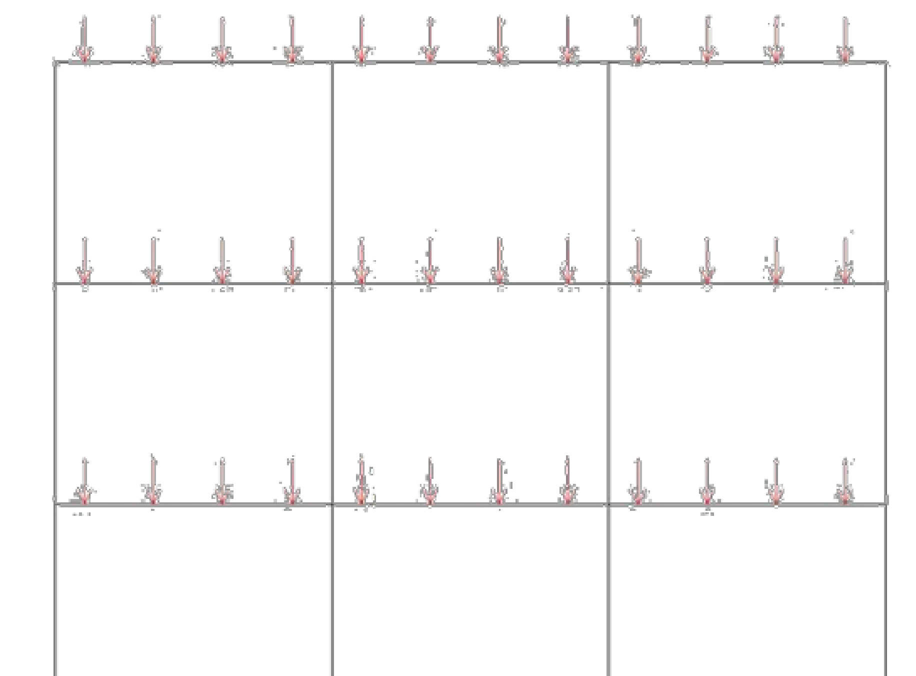


**Fig.36.3 Building frame subjected to vertical loads**

## SUBSTITUTE FRAME METHOD

Consider a building frame subjected to vertical loads as shown in Fig.36.3. Any typical beam, in this building frame is subjected to axial force, bending moment and shear force. Hence each beam is statically indeterminate to third degree and hence 3 assumptions are required to reduce this beam to determinate beam.

Before we discuss the required three assumptions consider a simply supported beam. In this case zero moment (or point of inflexion) occurs at the supports as shown in Fig.36.4a. Next consider a fixed-fixed beam, subjected to vertical loads as shown in Fig. 36.4b. In this case, the point of inflexion or point of zero moment occurs at $0.21L$ from both ends of the support.

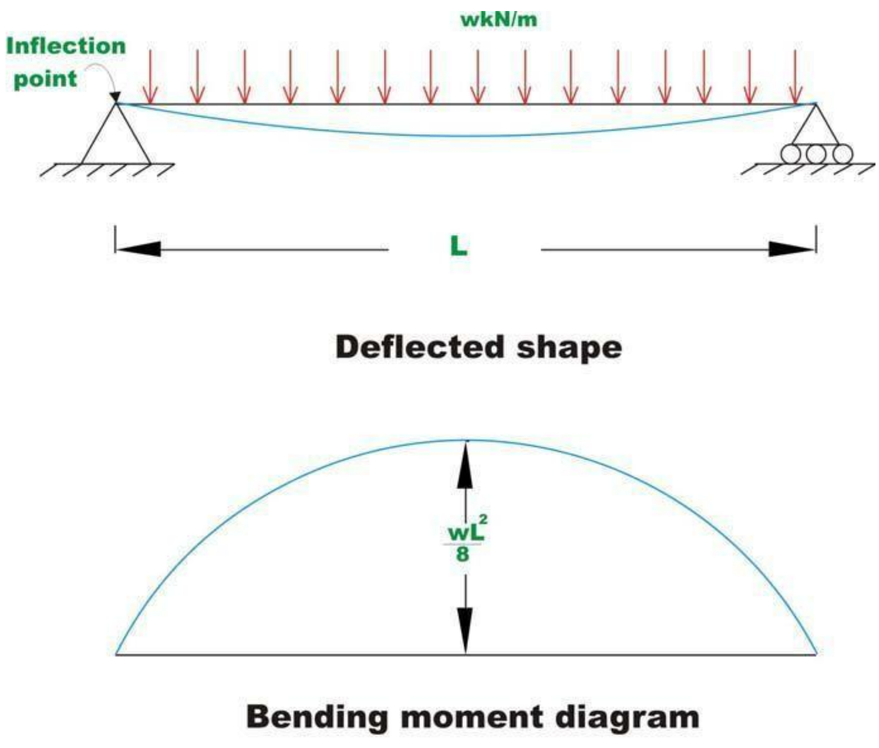
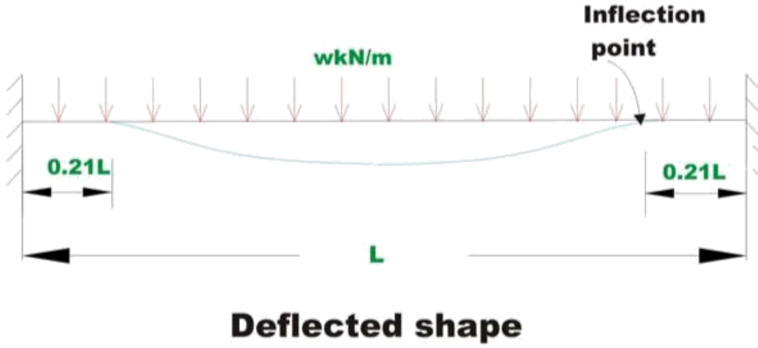



**Fig.36. 4a Simply Supported beam**



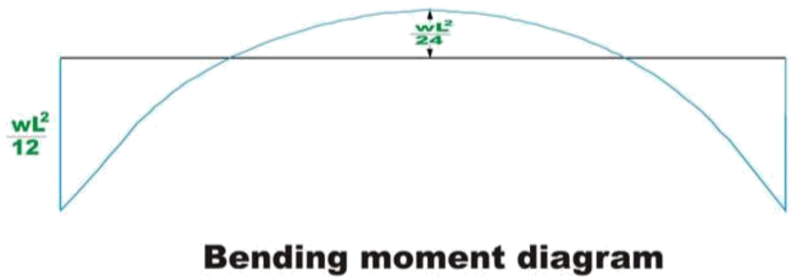




**Fig.36. 4b Fixed - Fixed beam**

Now consider a typical beam of a building frame as shown in Fig.36.4c. In this case, the support provided by the columns is neither fixed nor simply supported.

For the purpose of approximate analysis the inflexion point or point of zero moment is assumed to occur at $\left(\frac{0+0.21L}{2}\right) \approx 0.1L$ from the supports. In reality the point of zero moment varies depending on the actual rigidity provided by the columns. Thus the beam is approximated for the analysis as shown in Fig.

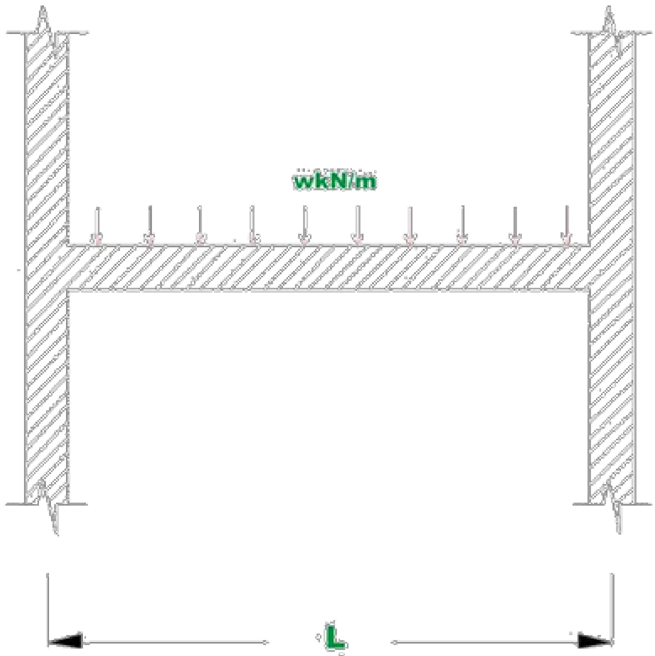
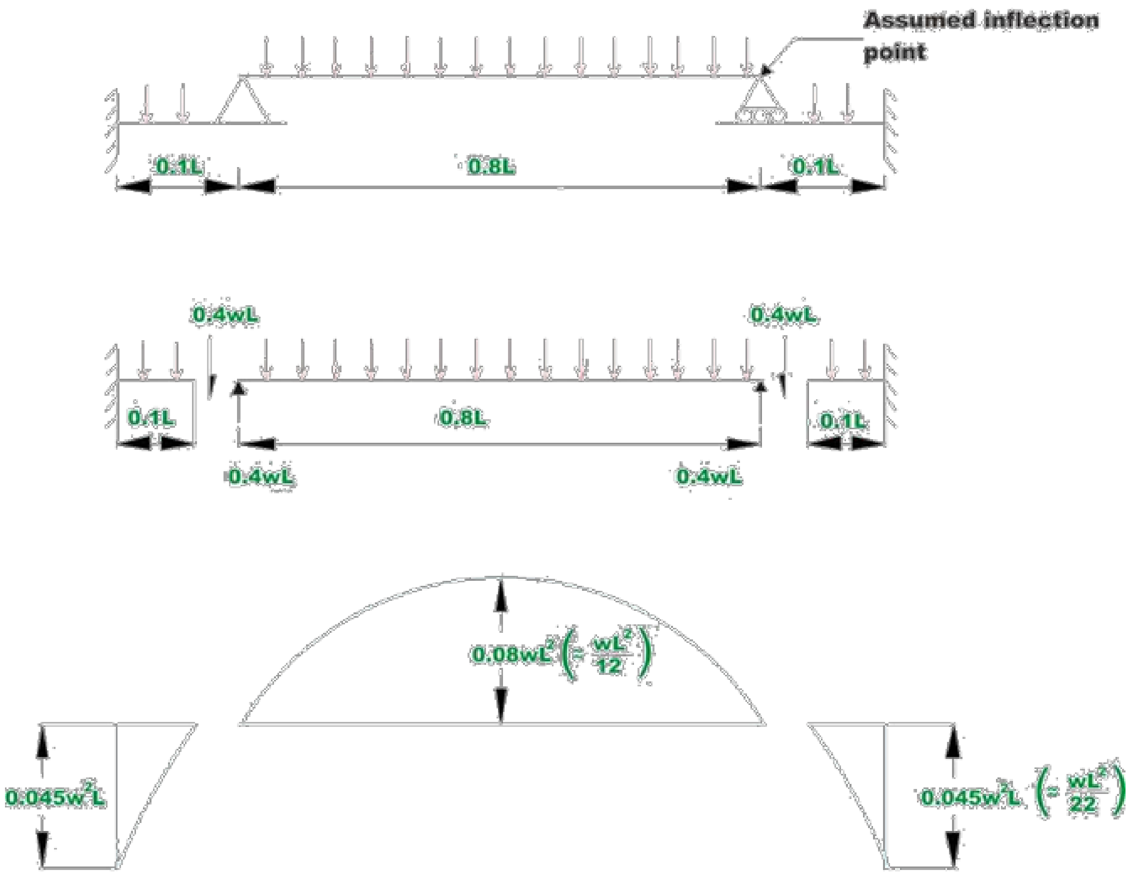



Fig.36.4c




Bending moment diagram

Fig.36.4d

For interior beams, the point of inflexion will be slightly more than $0.1L$ . An experienced engineer will use his past experience to place the points of inflexion appropriately. Now redundancy has reduced by two for each beam. The third assumption is that axial force in the beams is zero. With these three assumptions one could analyse this frame for vertical loads.

**Example 36.1**

Analyse the building frame shown in Fig. 36.5a for vertical loads using approximate methods.

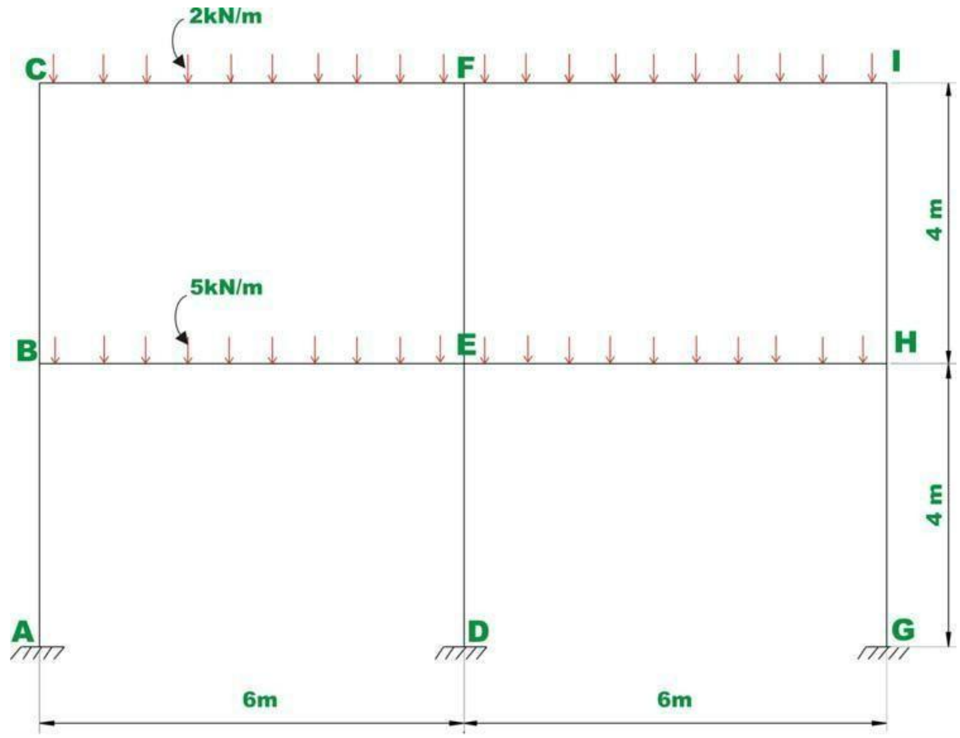



**Fig.36.5a**

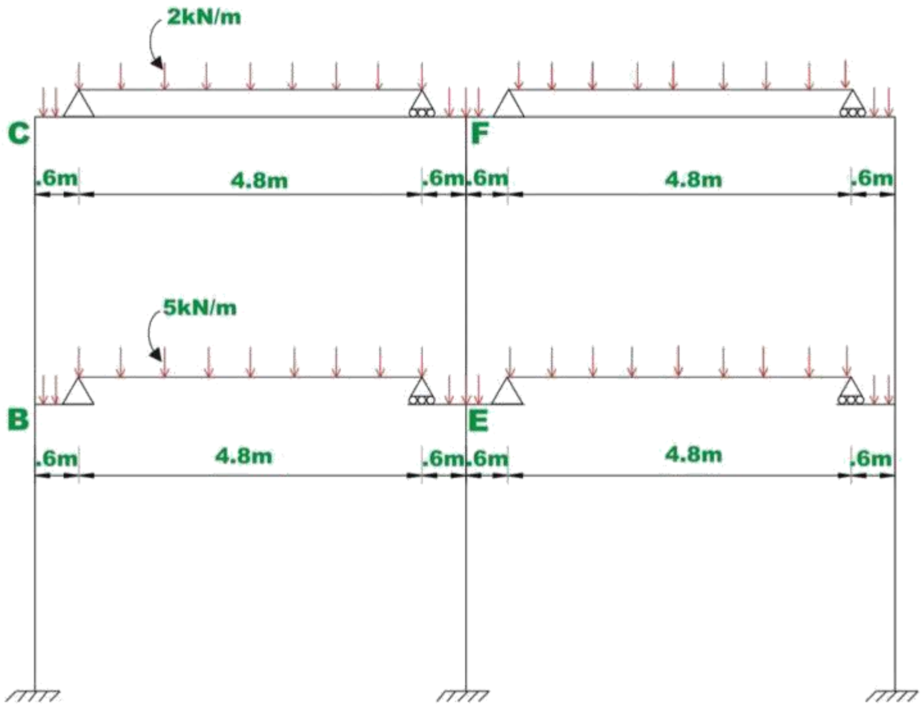
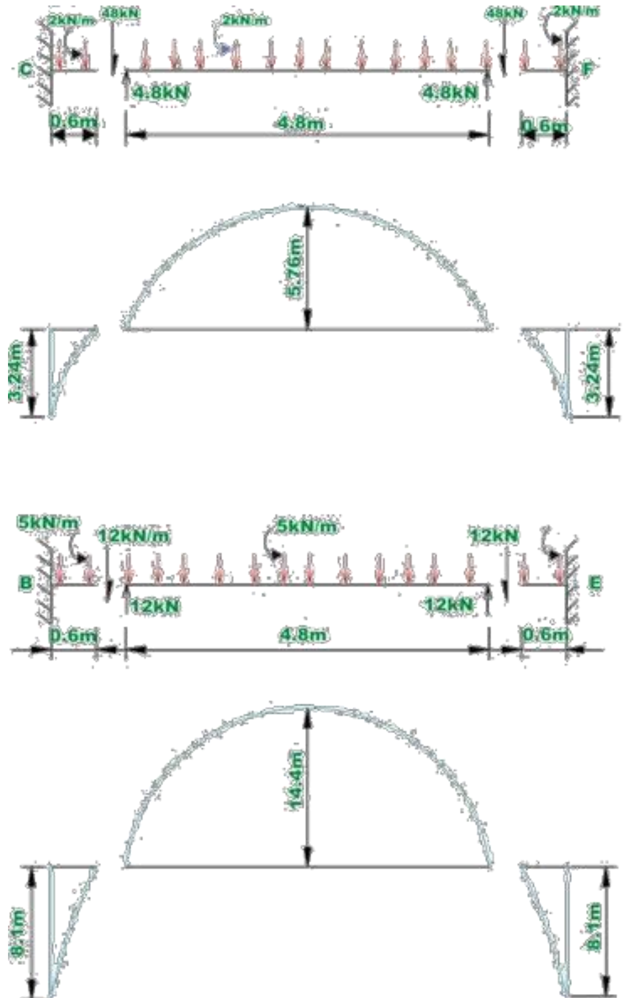



**Fig.36.5 b**

## Solution:

In this case the inflexion points are assumed to occur in the beam at $0.1L(=0.6m)$ from columns as shown in Fig. 36.5b. The calculation of beam moments is shown in Fig. 36.5c.




**Bending moment diagrams**

**Fig.36.5c**

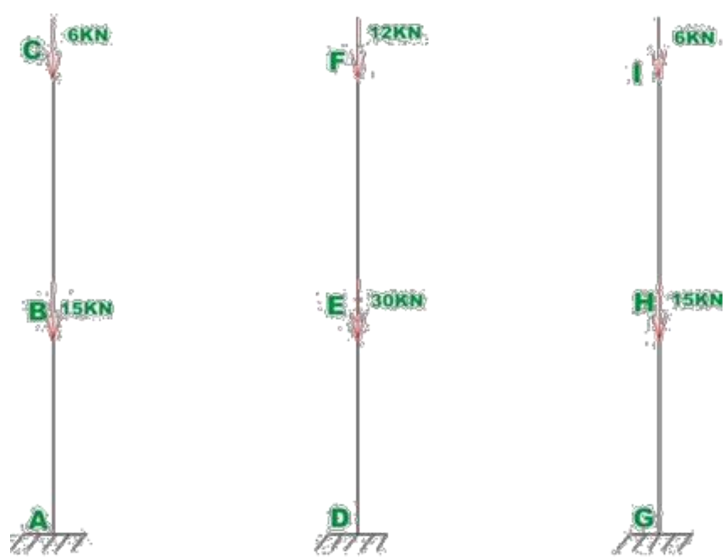



**Fig.36.5d Axial force in columns**

## ANALYSIS OF BUILDING FRAMES TO LATERAL (HORIZONTAL) LOADS

A building frame may be subjected to wind and earthquake loads during its life time. Thus, the building frames must be designed to withstand lateral loads. A two-storey two-bay multistory frame subjected to lateral loads is shown in Fig.36.6. The actual deflected shape (as obtained by exact methods) of the frame is also shown in the figure by dotted lines. The given frame is statically indeterminate to degree 12.

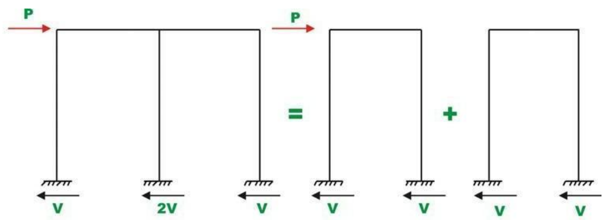



**Fig.36.6 Shear in columns**

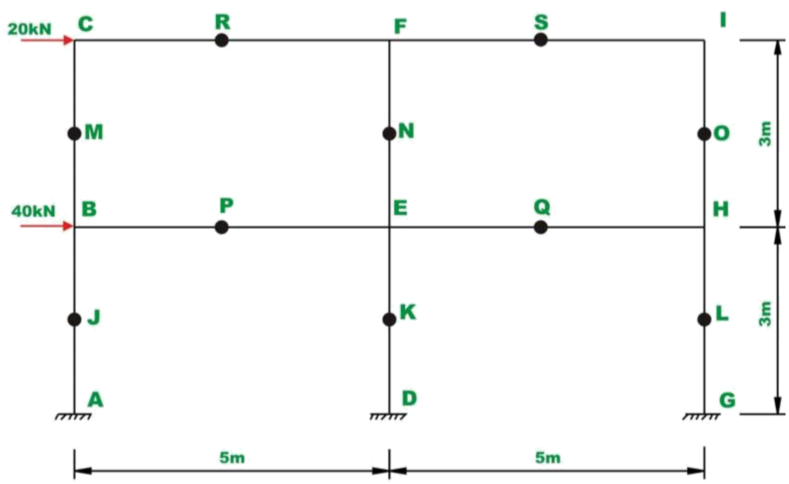



**Fig.36.7a Two storey building frame subjected to lateral load of Example 36.2**

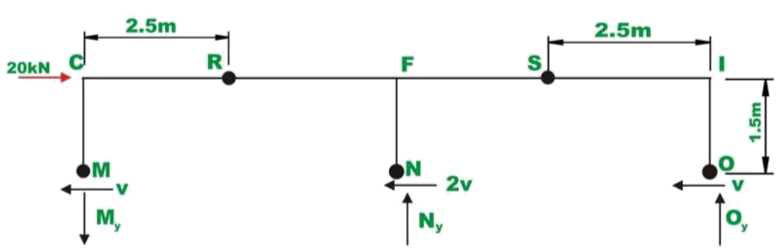



**Fig.36.7b**

Hence it is required to make 12 assumptions to reduce the frame in to a statically determinate structure. From the deformed shape of the frame, it is observed that inflexion point (point of zero moment) occur at mid height of each column and mid point of each beam. This leads to 10 assumptions. Depending upon how the remaining two assumptions are made, we have two different methods of analysis: *i*) Portal method and *ii*) cantilever method. They will be discussed in the subsequent sections.

## PORTAL METHOD

In this method following assumptions are made.

1) An inflexion point occurs at the mid height of eachcolumn.
2) An inflexion point occurs at the mid point of eachgirder.
3) The total horizontal shear at each storey is divided between the columns of that storey such that the interior column carries twice the shear of exterior column.

The last assumption is clear, if we assume that each bay is made up of a portal thus the interior column is composed of two columns (Fig. 36.6). Thus the interior column carries twice the shear of exterior column. This method is illustrated in example 36.2.

## Example

Analyse the frame shown in Fig. 36.7a and evaluate approximately the column end moments, beam end moments and reactions.

**Solution:**

The problem is solved by equations of statics with the help of assumptions made in the portal method. In this method we have hinges/inflexion points at mid height of columns and beams. Taking the section through column hinges *M*.*N*,*O*weget, (ref. Fig. 36.7b).

$$\sum F_X = 0 \qquad \Rightarrow \qquad V + 2V + V = 20$$

or $V = 5\,\text{kN}$

Taking moment of all forces left of hinge $R$ about $R$ gives,

$$V \times 1.5 - M_y \times 2.5 = 0$$
$$M_y = 3\ \text{kN}(\downarrow)$$

Column and beam moments are calculates as,

$$M_{CB} = 5 \times 1.5 = 7.5\,\text{kN.m} \ ; \ M_{IH} = +7.5\,\text{kN.m}$$

$$M_{CF} = -7.5\,\text{kN.m}$$

Taking moment of all forces left of hinge $S$ about $S$ gives,

$$5 \times 1.5 - O_y \times 2.5 = 0$$

$$O_y = 3\,\text{kN}(\uparrow)$$

$$N_y = 0$$

Taking a section through column hinges $J, K, L$ we get, (ref. Fig. 36.7c).

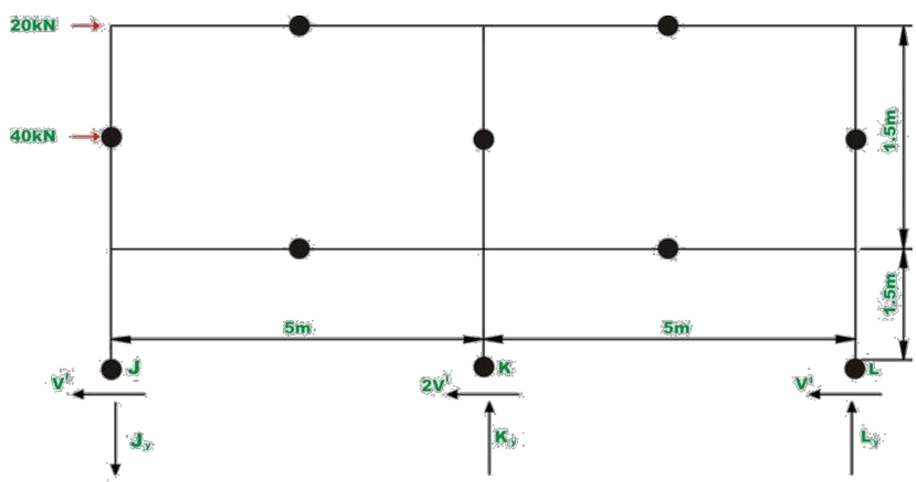



Fig.36.7c

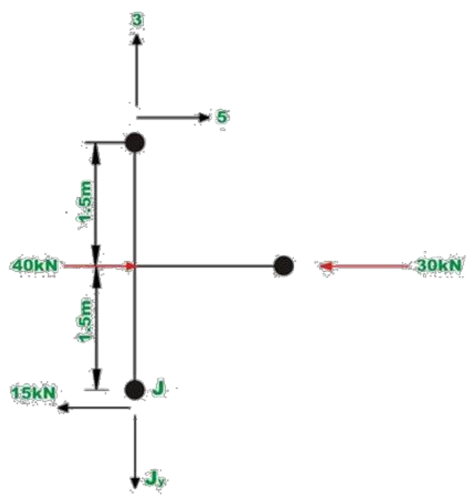



Fig.36.7d

$$\sum F_X = 0 \quad \Rightarrow \quad V'+2V'+V' = 60$$

or $V' = 15 \quad \text{kN}$

Taking moment of all forces about $P$ gives (vide Fig. 36.7d)

$$\sum M_p = 015 \times 1.5 + 5 \times 1.5 + 3 \times 2.5 - J_y \times 2.5 = 0$$

$$J_y = 15 \quad \text{kN}(\downarrow)$$

$$L_y = 15 \quad \text{kN}(\uparrow)$$

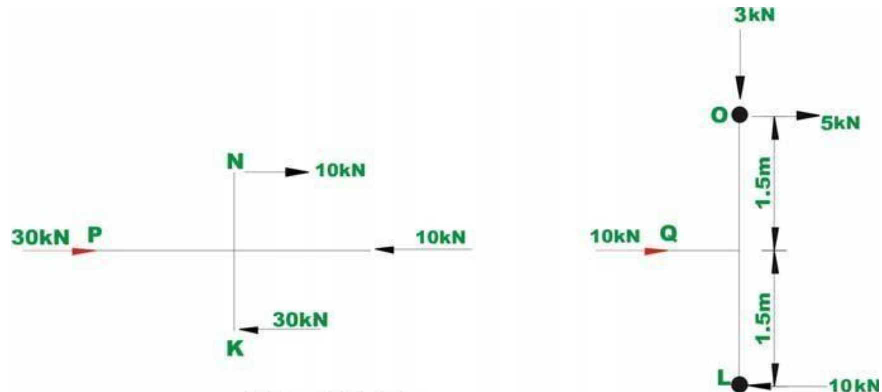



Fig.36.7e

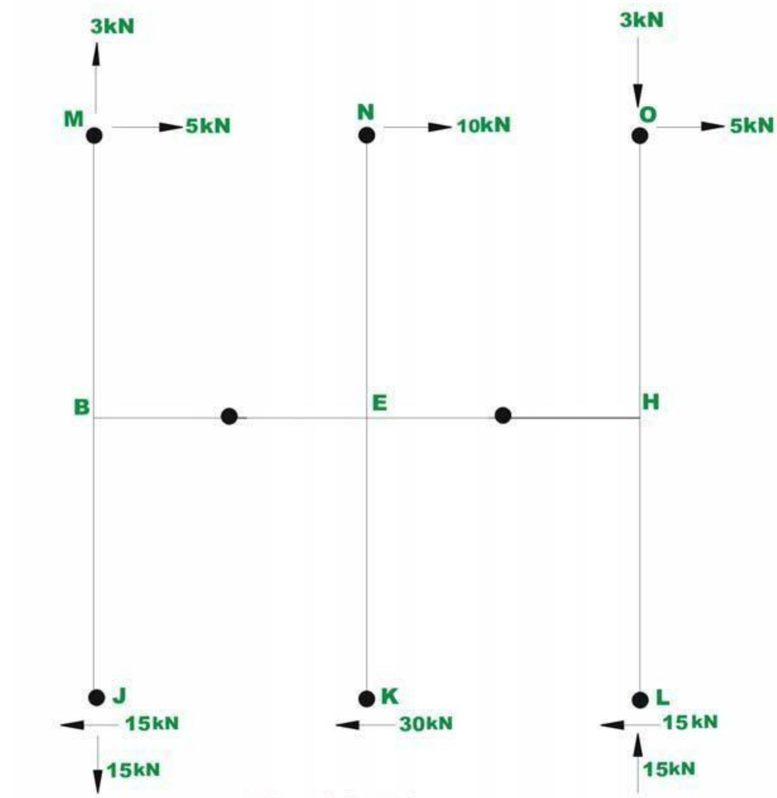



Fig.36.7f

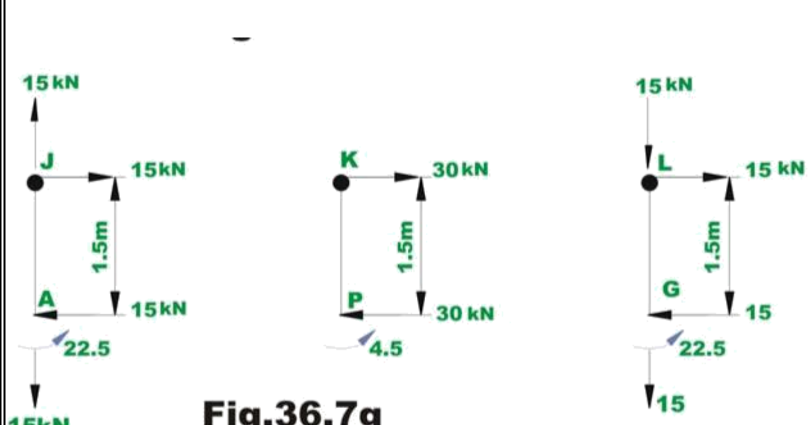



Fig.36.7g

Column and beam moments are calculated as, (ref. Fig. 36.7f)

$M_{BC}$=5x1.5=7.5 $kN.m$;$M_{BA}$=15x1.5=22.5 kN.m

$M_{BE}$=–30 kN.m

$M_{EF}$=10x1.5=15 kN.m ;$M_{ED}$=30x1.5=45 kN.m

$M_{EB}$=–30kN.m $M_{EH}$=–30kN.m

$M_{HI}$=5$x$1.5=7.5 kN.m ;$M_{HG}$=15$x$1.5=22.5 kN.m

$M_{HE}$=–30 kN.m

Reactions at the base of the column are shown in Fig. 36.7g**.**

CANTILEVER METHOD

The cantilever method is suitable if the frame is tall and slender. In the cantilever method following assumptions are made.

An inflexion point occurs at the mid point of eachgirder.An inflexion point occurs at mid height of eachcolumn.In a storey, the intensity of axial stress in a column is proportional to its horizontal distance from the center of gravity of all the columns in that storey. Consider a cantilever beam acted by a horizontal load P as shown in Fig. 36.8. In such a column the bending stress in the column cross section varies linearlyfrom its neutral axis. The last assumption in the cantilever method is based on this fact. The method is illustrated in example36.3.

Example 36.3

Estimate approximate column reactions, beam and column moments using cantilever method of the frame shown in Fig. 36.8a. The columns are assumed to have equal cross sectional areas.

Solution:

This problem is already solved by portal method. The center of gravity of all column passes through centre column.

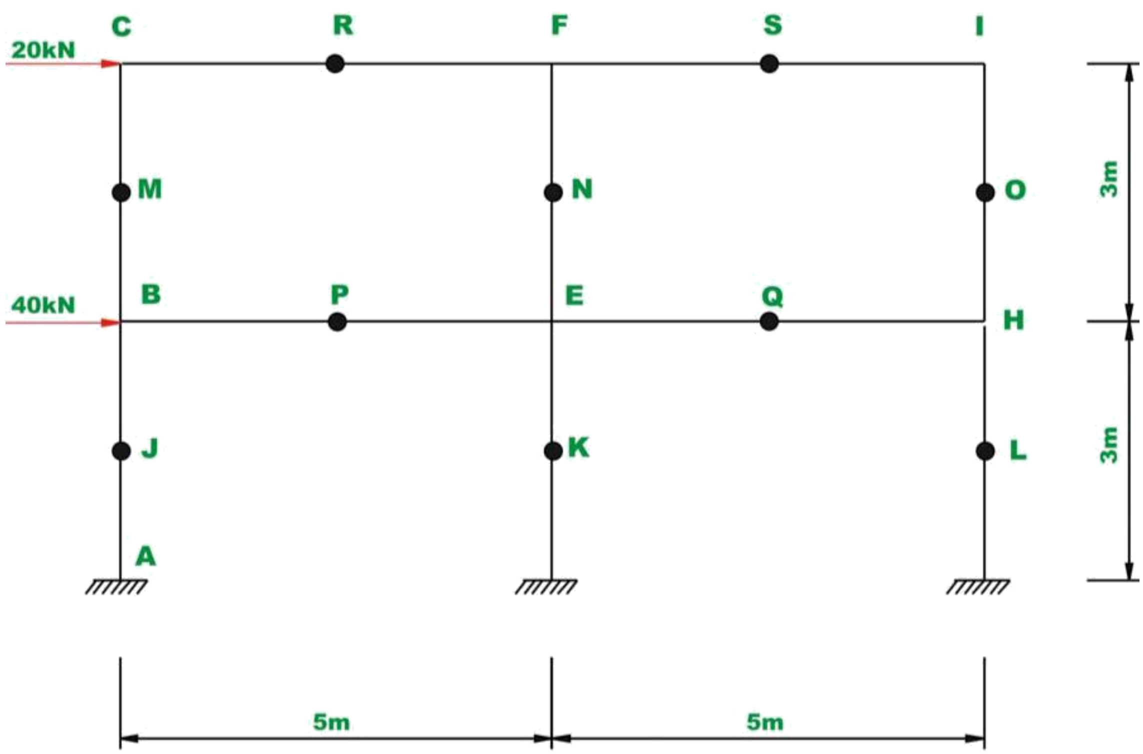



**Fig.36.8b**

Taking a section through first storey hinges gives us the free body diagram as shown in Fig. 36.8b. Now the column left of C.G. *i.e.CB*must be subjected to tension and one on the right is subjected to compression.

From the third assumption,

$$\frac{M_y}{5\times A} = -\frac{O_y}{5\times A} \qquad \Rightarrow \qquad M_y = -O_y$$

Taking moment about $O$ of all forces gives,

$$20\times 1.5 - M_y \times 10 = 0$$

$$M_y = 3\,\text{kN}(\downarrow) \quad ; \qquad O_y = 3\,\text{kN}(\uparrow)$$

Taking moment about $R$ of all forces left of $R$ ,

$$V_M \times 1.5 - 3\times 2.5 = 0$$

$$V_M = 5 \quad \text{kN}(\leftarrow)$$

Taking moment of all forces right of $S$ about $S$ ,

$$V_O \times 1.5 - 3\times 2.5 = 0 \quad \Rightarrow \qquad V_O = 5\,\text{kN.}$$

$$\sum F_X = 0 \qquad V_M + V_N + V_O - 20 = 0$$

$$V_N = 10\,\text{kN.}$$

Moments

$M_{CB}$=5·1.5=7.5 kN.m
$M_{CF}$=–7.5 kN.m
$M_{FE}$=15 kN.m
$M_{FC}$=–7.5 kN.m
$M_{FI}$=–7.5 kN.m
$M_{IH}$=7.5 kN.m
$M_{IF}$=–7.5 kN.m

Tae a section through hinges *J*,*K*,*L*(ref. Fig. 36.8c). Since the center of gravity passes through centre column the axial force in that column is zero.

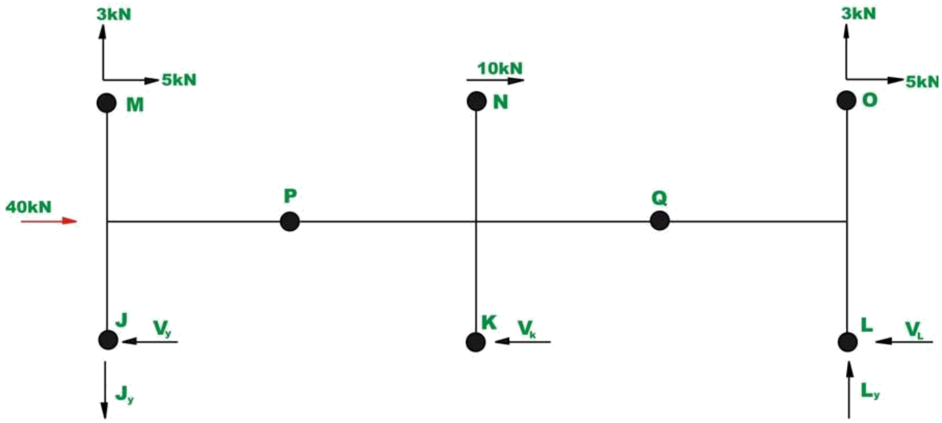



**Fig.36.8c**

Taking moment about hinge $L$ ,$J_y$ can be evaluated. Thus,

$$20\times 3+40\times 1.5+3\times 10-J_y\times 10=0$$

$$J_y=15\text{kN}(\downarrow) \quad ; \quad L_y=15\text{kN}(\uparrow)$$

Taking moment of all forces left of $P$ about $P$ gives,

$$5\times 1.5+3\times 2.5-15\times 2.5+V_j\times 1.5=0$$

$$V_J=15\text{kN}(\leftarrow)$$

Similarly taking moment of all forces right of $Q$ about $Q$ gives,

$$5\times 1.5+3\times 2.5-15\times 2.5+V_L\times 1.5=0$$

$$V_L=15\text{kN}(\leftarrow)$$

$$\sum F_X=0 \qquad V_J+V_K+V_L-60=0$$

$$V_K=30 \quad \text{kN.}$$

Moments

$M_{BC}=5\cdot1.5=7.5$ kN.m ; $M_{BA}=15\cdot1.5=22.5$ kN.m

$M_{BE}=-30$ kN.m

$M_{EF}=10\cdot1.5=15$ kN.m ; $M_{ED}=30\cdot1.5=45$ kN.m

$M_{EB}=-30$ kN.m $M_{EH}=-30$ kN.m

$M_{HI}=5\cdot1.5=7.5$ kN.m ; $M_{HG}=15\cdot1.5=22.5$ kN.m

$M_{HE}=-30$ kN.m

**PART B**

**Approximate Lateral Load Analysis by Portal Method**

Portal Frame

Portal frames, used in several Civil Engineering structures like buildings, factories, bridges have the primary purpose of transferring horizontal loads applied at their tops to their foundations. Structural requirements usually necessitate the use of statically indeterminate layout for portal frames, and approximate solutions are often used in their analyses.

## Assumptions for the Approximate Solution

In order to analyze a structure using the equations of statics only, the number of independent force components must be equal to the number of independent equations of statics.

If there are *n*more independent force components in the structure than there are independent equations of statics, the structure is statically indeterminate to the *nth* degree. Therefore to obtain an approximate solution of the structure based on statics only, it will be necessary to make *n*additional independent assumptions. A solution based on statics will not be possible by making fewer than *n* assumptions, while more than *n* assumptions will not in general be consistent.

Thus, the first step in the approximate analysis of structures is to find its degree of statical indeterminacy (dosi) and then to make appropriate number of assumptions.

For example, the dosi of portal frames shown in (i), (ii), (iii) and (iv) are 1, 3, 2 and 1 respectively. Based on the type of frame, the following assumptions can be made for portal structures with a *vertical axis of symmetry* that are *loaded horizontally at thetop*

1. The horizontal support reactions areequal

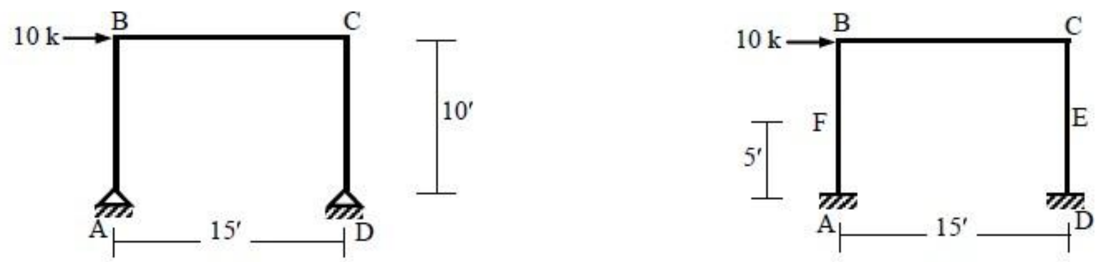
2. There is a point of inflection at the center of the unsupported height of each fixed basedcolumn

Some additional assumptions can be made in order to solve the structure approximately for different loading and support conditions.

3. Horizontal body forces not applied at the top of a column can be divided into two forces (i.e., applied at the top and bottom of the column) based on simple supports

4. For hinged and fixed supports, the horizontal reactions for fixed supports can be assumed to be four times the horizontal reactions for hinged supports Example

Draw the axial force, shear force and bending moment diagrams of the frames loaded as shown below.



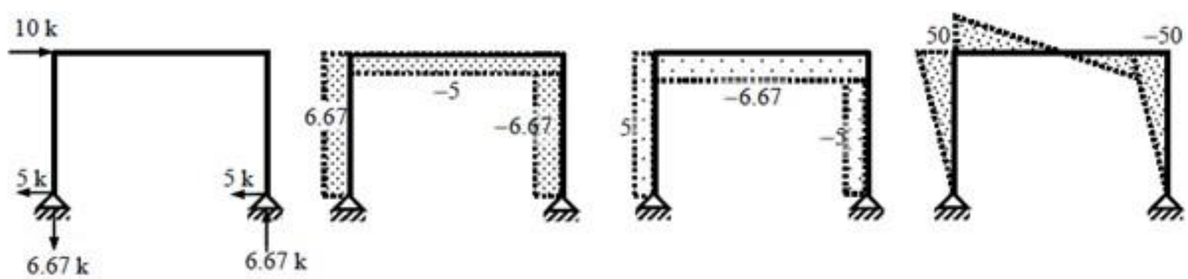
(i) (ii)

<u>Solution</u>

(i) For this frame, dosi = 3 × 3 + 4 – 3 × 4 = 1; i.e., Assumption 1 ⇒ $H_A = H_D = 10/2 = 5$ k

$\therefore \Sigma M_A = 0 \Rightarrow 10 \times 10 - V_D \times 15 = 0 \Rightarrow V_D = 6.67$ k

$\therefore \Sigma F_y = 0 \Rightarrow V_A + V_D = 0 \Rightarrow V_A = -6.67$ k



Reactions AFD (k) SFD (k) BMD

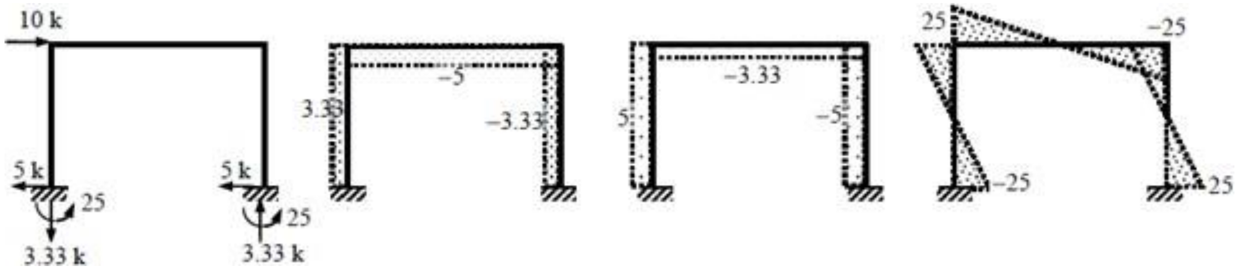
(ii) dosi = 3 × 3 + 6 – 3 × 4 = 3

Assumption 1 ⇒ $H_A = H_D = 10/2 = 5$ k, Assumption 2 ⇒ $BM_E = BM_F = 0$

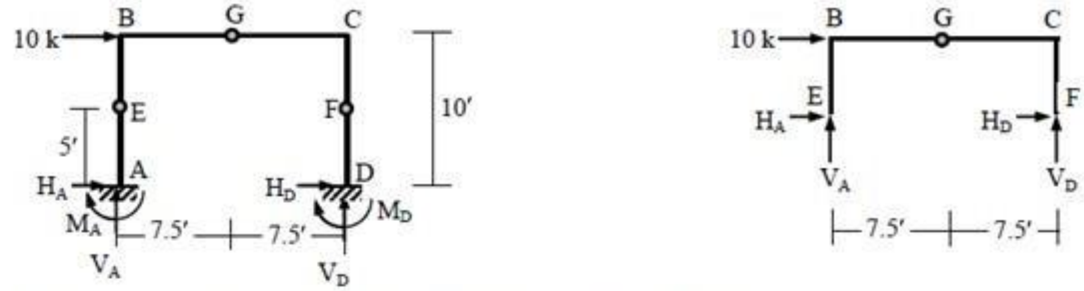
$\therefore BM_F = 0 \Rightarrow H_A \times 5 + M_A = 0 \Rightarrow M_A = -25$ k-ft; Similarly $BM_E = 0 \Rightarrow M_D = -25$

$\therefore \Sigma M_A = 0 \Rightarrow -25 - 25 + 10 \times 10 - V_D \times 15 = 0 \Rightarrow V_D = 3.33$ k

$\therefore \Sigma F_y = 0 \Rightarrow V_A + V_D = 0 \Rightarrow V_A = -3.33$ k




(iii)




$dosi = 3 \times 4 + 6 - 3 \times 5 - 1 = 2$; $\therefore$ Assumption 1 and 2 $\Rightarrow BM_E = BM_F = 0$

$\therefore BM_E = 0$ (bottom) $\Rightarrow -H_A \times 5 + M_A = 0 \Rightarrow M_A = 5H_A$; Similarly $BM_F = 0 \Rightarrow M_D = 5H_D$

Also $BM_E = 0$ (free body of EBCF) $\Rightarrow 10 \times 5 - V_D \times 15 = 0 \Rightarrow V_D = 3.33$ k

$\therefore \Sigma F_y = 0 \Rightarrow V_A + V_D = 0 \Rightarrow V_A = -V_D = -3.33$ k

$BM_G = 0$ (between E and G) $\Rightarrow V_A \times 7.5 - H_A \times 5 = 0 \Rightarrow H_A = -5$ k $\Rightarrow M_A = 5H_A = -25$

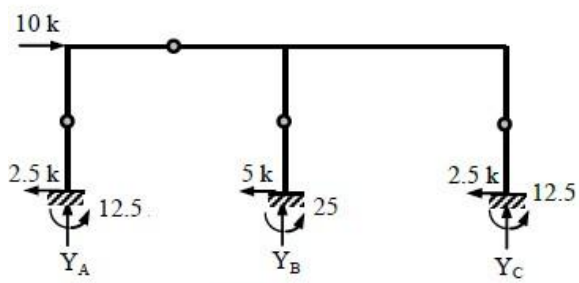
$\Sigma F_x = 0$ (entire structure) $\Rightarrow H_A + H_D + 10 = 0 \Rightarrow -5 + H_D + 10 = 0 \Rightarrow H_D = -5$ k $\Rightarrow M_D = 5H_D = -25$

(iv) $dosi = 3 \times 5 + 9 - 3 \times 6 = 6 \Rightarrow$ 6 Assumptions needed to solve the structure

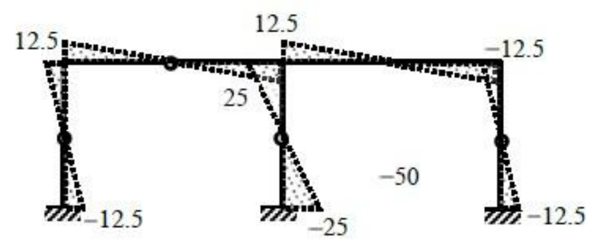
Assumption 1 and 2 $\Rightarrow H_A : H_B : H_C = 1 : 2 : 1 \Rightarrow H_A = 10/4 = 2.5$ k, $H_B = 5$ k, $H_C = 2.5$ k

$\therefore M_A = M_C = 2.5 \times 5 = 12.5$ k-ft, $M_B = 5 \times 5 = 25$

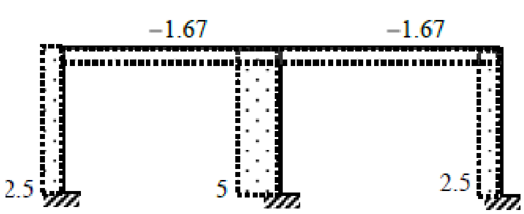
The other 4 assumptions are the assumed internal hinge locations at midpoints of columns and one beam



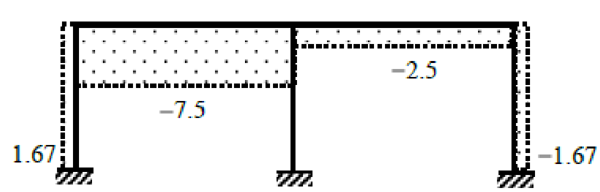

Reactions




BMD




SFD (k)




AFD (k)

## Analysis of Multi-storied Structures by Portal Method

Approximate methods of analyzing multi-storied structures are important because such structures are statically highly indeterminate. The number of assumptions that must be made to permit an analysis by statics alone is equal to the degree of statical indeterminacy of the structure.

Assumptions

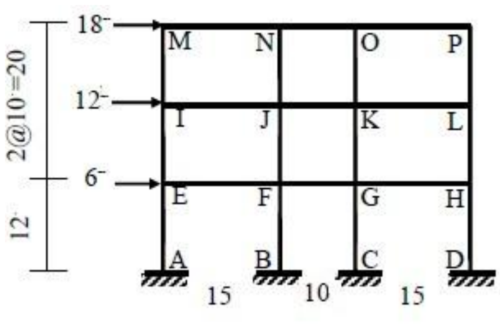
The assumptions used in the approximate analysis of portal frames can be extended for the lateral load analysis of multi-storied structures. The *Portal Method* thus formulated is based on three assumptions

1. The shear force in an interior column is twice the shear force in an exterior column.
2. There is a point of inflection at the center of eachcolumn.
3. There is a point of inflection at the center of eachbeam.

Assumption 1 is based on assuming the interior columns to be formed by columns of two adjacent bays or portals. Assumption 2 and 3 are based on observing the deflected shape of thestructure.

### Example

Use the Portal Method to draw the axial force, shear force and bending moment diagrams of the three-storied frame structure loaded as shown below.




Column shear forces are at the ratio of 1:2:2:1.

$\therefore$ Shear force in (V) columns IM, JN, KO, LP are $[18 \times 1/(1+2+2+1) =]\ 3$, $[18 \times 2/(1+2+2+1) =]\ 6$, 6, 3 respectively. Similarly,

$V_{EI} = 30 \times 1/(6) = 5$, $V_{FJ} = 10$, $V_{GK} = 10$, $V_{HL} = 5$; and

$V_{AE} = 36 \times 1/(6) = 6$, $V_{BF} = 12$, $V_{CG} = 12$, $V_{DH} = 6$

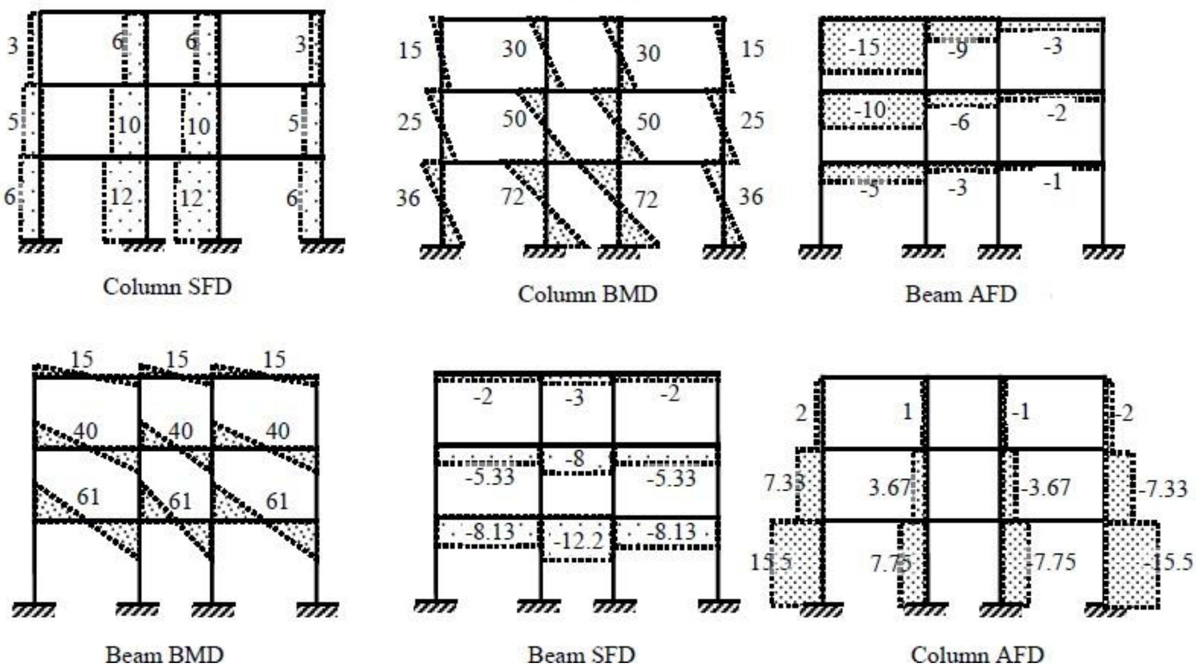
Bending moments are

$M_{IM} = 3 \times 10/2 = 15$, $M_{JN} = 30$, $M_{KO} = 30$, $M_{LP} = 15$

$M_{EI} = 5 \times 10/2 = 25$, $M_{FJ} = 50$, $M_{GK} = 50$, $M_{HL} = 25$

$M_{AE} = 6 \times 10/2 = 30$, $M_{FJ} = 60$, $M_{GK} = 60$, $M_{HL} = 30$

The rest of the calculations follow from the free-body diagrams




**Analysis of Multi-storied Structures by Cantilever Method**

Although the results using the *Portal Method* are reasonable in most cases, the method suffers due to the lack of consideration given to the variation of structural response due to the difference between sectional properties of various members. The *Cantilever Method* attempts to rectify this limitation by considering the cross- sectional areas of columns in distributing the axial forces in various columns of a story.

Assumptions

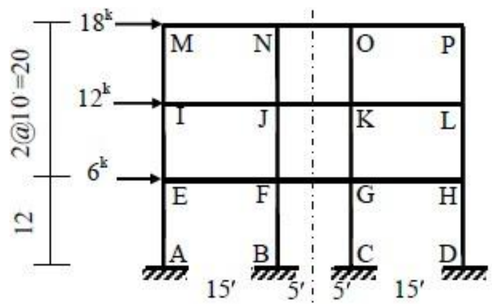
The *Cantilever Method* is based on three assumptions

1. The axial force in each column of a storey is proportional to its horizontal distance from the centroidal axis of all the columns of thestorey.
2. There is a point of inflection at the center of eachcolumn.
3. There is a point of inflection at the center of eachbeam.

Assumption 1 is based on assuming that the axial stresses can be obtained by a method analogous to that used for determining the distribution of normal stresses on a transverse section of a cantilever beam. Assumption 2 and 3 are based on observing the deflected shape of the structure.

**Example**

Use the Cantilever Method to draw the axial force, shear force and bending moment diagrams of the three -storied frame structure loaded as shown below.



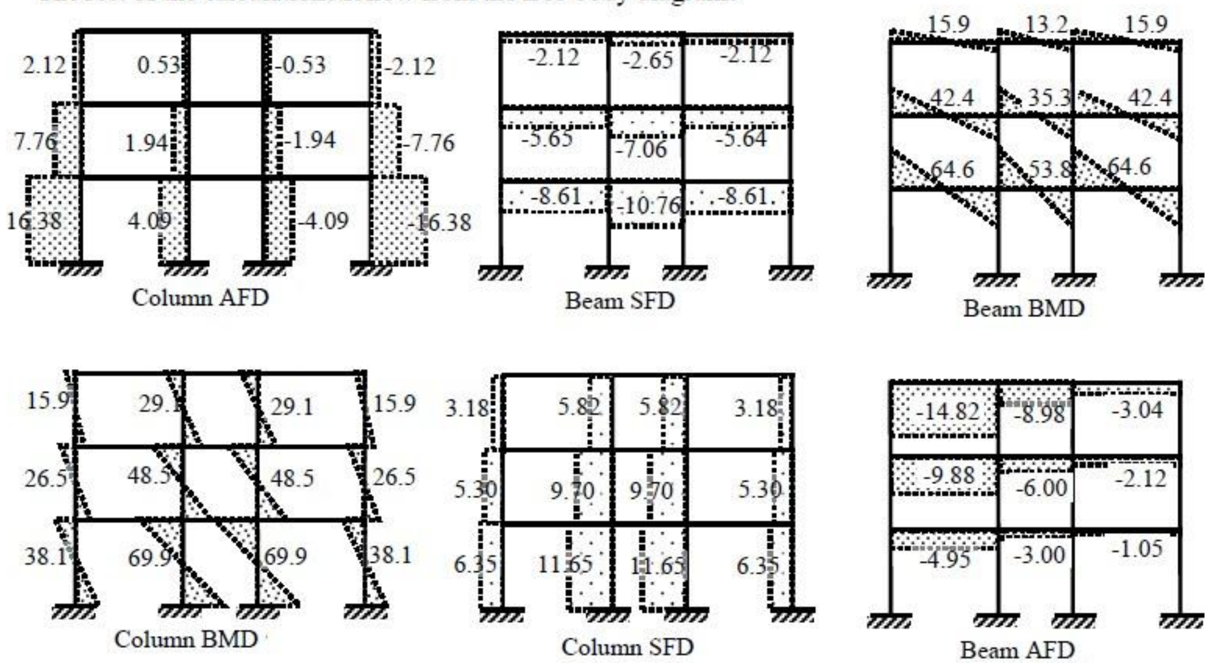

The dotted line is the column centerline (at all floors)

∴ Column axial forces are at the ratio of 20: 5: –5: –20.

∴ Axial force in (P) columns IM, JN, KO, LP are $[18 \times 5 \times 20/\{20^2 + 5^2 + (-5)^2 + (-20)^2\} = ]$ 2.12, $[18 \times 5 \times 5/(20^2 + 5^2 + (-5)^2 + (-20)^2\} = ]$ 0.53, –0.53, –2.12 respectively.

Similarly, $P_{EI} = 330\times20/(850) = 7.76$, $P_{FJ} = 1.94$, $P_{GK} = -1.94$, $P_{HL} = -7.76$; and $P_{AE} = 696\times20/(850) = 16.38$, $P_{BF} = 4.09$, $P_{CG} = -4.09$, $P_{DH} = 16.38$

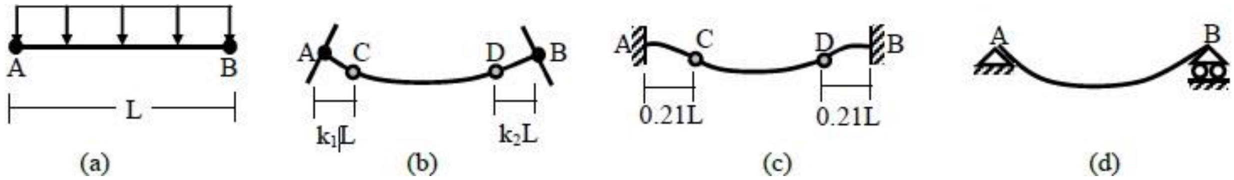
The rest of the calculations follow from the free-body diagrams




**Approximate Vertical Load Analysis**

Approximation based on the Location of Hinges

If a beam AB is subjected to a uniformly distributed vertical load of w per unit length [Fig. (a)], both the joints A and B will rotate as shown in Fig. (b), because although the joints A and B are partly restrained against rotation, the restraint is not complete. Had the joints A and B been completely fixed against rotation [Fig. (c)] the points of inflection would be located at a distance 0.21L from each end. If, on the other hand, the joints A and B are hinged [Fig. (d)], the points of zero moment would be at the end of the beam. For the actual case of partial fixity, the points of inflection can be assumed to be somewhere between 0.21 L and 0 from the end of the beam. *For approximate analysis, they are often assumed to be located at one-tenth (0.1 L) of the span length from each end joint.*




Depending on the support conditions (i.e., hinge ended, fixed ended or continuous), a beam in general can be statically indeterminate up to a degree of three. Therefore, to make it statically determinate, the following three assumptions are often made in the vertical load analysis of a beam

1. The axial force in the beam iszero
2. Points of inflection occur at the distance 0.1 L measured along the span from the left and rightsupport.

Bending Moment and Shear Force from Approximate Analysis

Based on the approximations mentioned (i.e., points of inflection at a distance 0.1 L from the ends), the maximum positive bending moment in the beam is calculated to be

$M(+) = w(0.8L)^2/8 = 0.08\ wL^2$, at the midspan of the beam The maximum negative bending moment is

$M() = wL^2/8\ 0.08\ wL^2 = 0.045\ wL^2$, at the joints A and B of the beam

The shear forces are maximum (positive or negative) at the joints A and B and are calculated to be

$VA = wL/^2$, and $VB = wL/2$

Moment and Shear Values using ACI Coefficients

Maximum allowable LL/DL = 3, maximum allowable adjacent span difference = 20%

1. PositiveMoments

(i) For EndSpans

(a) If discontinuous end is unrestrained, $M(+) = wL^2/11$

(b) If discontinuous end is restrained, $M(+) = wL^2/14$

(ii) For Interior Spans, $M(+) = wL^2/16$

2. NegativeMoments

(i) At the exterior face of first interiorsupports

(a) Two spans, $M(-) = wL^2/9$

(b) More than two spans, $M(-) = wL^2/10$

(ii) At the other faces of interior supports, $M(-) = wL^2/11$

(iii) Forspansnotexceeding10,ofwherecolumnsaremuchstifferthanbeams,

M(-) = wL2/12

(iv) At the interior faces of exteriorsupports

(a) If the support is a beam, $M(-) = wL^2/24$

(b) If the support is a column, $M(-) = wL^2/16$

3. ShearForces

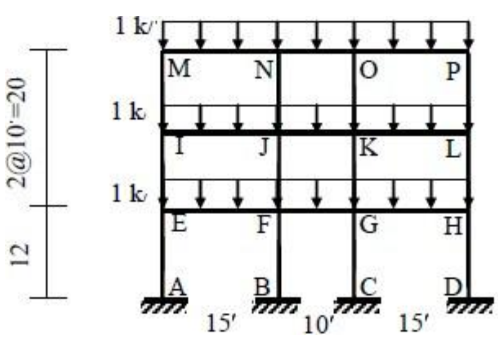
(i) In end members at first interior support, V =1.15wL/2

(ii) At all other supports, V =wL/2

[where L = clear span for M(+) and V, and average of two adjacent clear spans for M(-)]

Example

Analyze the three-storied frame structure loaded as shown below using the approximate location of hinges to draw the axial force, shear force and bending moment diagrams of the beams and columns.




The maximum positive and negative beam moments and shear forces are as follows.

For the 15 beam, $M_{(+)} = 0.08 \times 1 \times 15^2 = 18$

$M_{(-)} = 0.045 \times 1 \times 15^2 = 10.13$

$V_{(\pm)} = 1 \times 15/2 = 7.5$ k

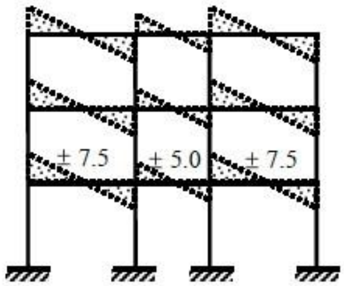
For the 10′ beam, $M_{(+)} = 0.08 \times 1 \times 10^2 = 8$

$M_{(-)} = 0.045 \times 1 \times 10^2 = 4.5$

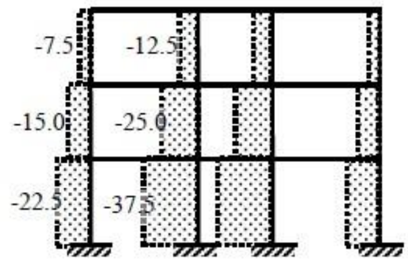
$V_{(\pm)} = 1 \times 10/2 = 5$ k

Axial Force P in all the beams = 0

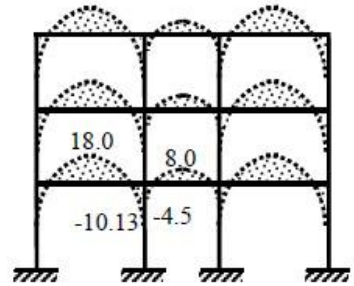
The rest of the calculations follow from the free-body diagrams



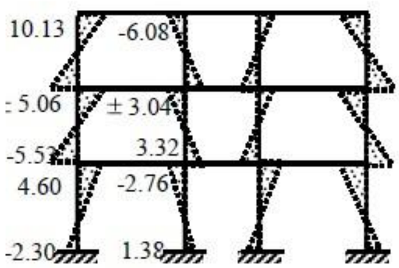
Beam SFD (k)



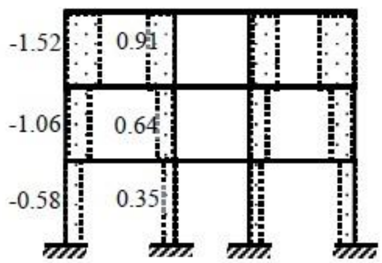
Column AFD (k)



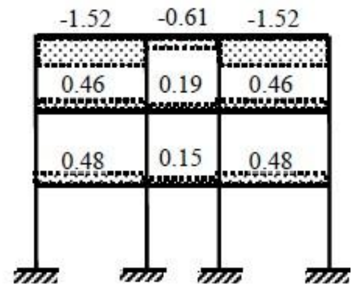
Beam BMD



Column BMD



Column SFD (k)



Beam AFD (k)

# UNIT IV: MATRIX METHOD OF ANALYSIS

## THE DIRECT STIFFNESS METHOD

### INTRODUCTION

All known methods of structural analysis are classified into two distinct groups:-

1. force method of analysisand
2. displacement method ofanalysis.

In module 2, the force method of analysis or the method of consistent deformation is discussed. An introduction to the displacement method of analysis is given in module 3, where in slope-deflection method and moment- distribution method are discussed. In this module the direct stiffness method is discussed. In the displacement method of analysis the equilibrium equations are written by expressing the unknown joint displacements in terms of loads by using load- displacement relations. The unknown joint displacements (the degrees of freedom of the structure) are calculated by solving equilibrium equations. The slope-deflection and moment-distribution methods were extensively used before the high speed computing era. After the revolution in computer industry, only direct stiffness method isused.

The displacement method follows essentially the same steps for both statically determinate and indeterminate structures. In displacement /stiffness method of analysis, once the structural model is defined, the unknowns (joint rotations and translations) are automatically chosen unlike the force method of analysis. Hence, displacement method of analysis is preferred to computer implementation. The method follows a rather a set procedure. The direct stiffness method is closely related to slope-deflectionequations.

The general method of analyzing indeterminate structures by displacement method may be traced to Navier (1785-1836). For example consider a four member truss as shown in Fig.23.1.The given truss is statically indeterminate to second degree as there are four bar forces but we have only two equations of equilibrium. Denote each member by a number, for example (1), (2), (3) and (4).

Let $\alpha_i$be the angle, the *i*-th member makes with the horizontal. Under the action of external loads $P_X$ and $P_y$, the joint $E$ displaces to $E'$. Let $u$ and $v$ be its vertical and horizontal displacements. Navier solved this problem as follows.

In the displacement method of analysis $u$ and $v$ are the only two unknowns for this structure. The elongation of individual truss members can be expressed in terms of these two unknown joint displacements. Next, calculate bar forces in the members by using force–displacement relation.
The unknown displacements may be calculated by solving the equilibrium equations. In displacement method of analysis, there will be exactly as many equilibrium equations as there are unknowns.

Let an elastic body is acted by a force F and the corresponding displacement be $u$ in the direction of force. In module 1, we have discussed force- displacementrelationship. The force ($F$) is related to the displacement ($u$) for the linear elastic material by therelation

$$F = ku \qquad (23.1)$$

where the constant of proportionality $k$ is defined as the stiffness of the structure and it has units of force per unit elongation. The above equation may also be written as

$$u = aF \qquad (23.2)$$

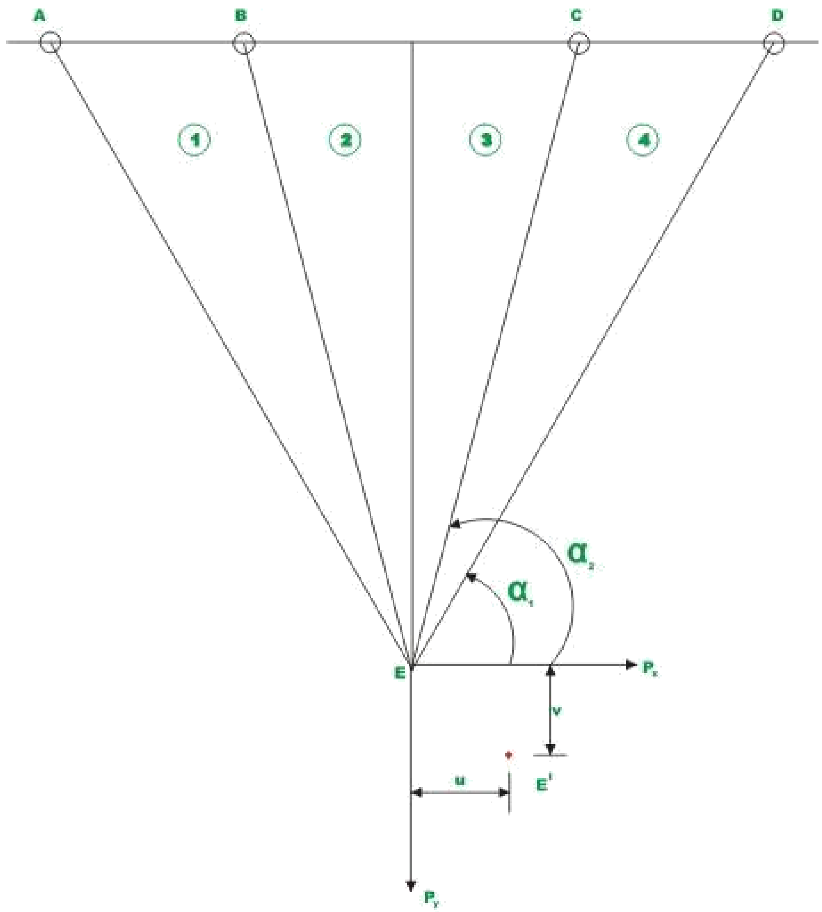



**Fig. 23.1 Four member truss**

The constant $a$is known as flexibility of the structure and it has a unit of displacement per unit force. In general the structures are subjected to $n$ forces at $n$ different locations on the structure. In such a case, to relate displacement at$i$to load at $j$, it is required to use flexibility coefficients with subscripts. Thus theflexibility coefficient $a_{ij}$is the deflection at $i$due to unit value of force applied at $j$ . Similarly the stiffness coefficient$k_{ij}$is defined as the force generated at$i$

due to unit displacement at $j$ with all other displacements kept at zero. To illustrate this definition, consider a cantilever beam which is loaded as shown in Fig.23.2. The two degrees of freedom for this problem are vertical displacement at $B$ and

rotation at $B$. Let them be denoted by  and  $(=\theta_1)$. Denote the vertical force P by and the tip moment $M$ by. Now apply a unit vertical force along calculate deflection  and . The vertical deflection is denoted by flexibility coefficient and rotation is denoted by flexibility coefficient . Similarly, by applying a unit force along  , one could calculate flexibility coefficient  and  . Thus  is the deflection at 1 corresponding to due to unit force applied at 2 in the direction of By using the principle of superposition, the displacements and are expressed as the sum of displacements due to loads and acting separately on the beam. Thus,

The above equation may be written in matrix notation as

$$\{u\} = [a]\{P\}$$

where, $\{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$; $\{a\} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$; and $\{P\} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$

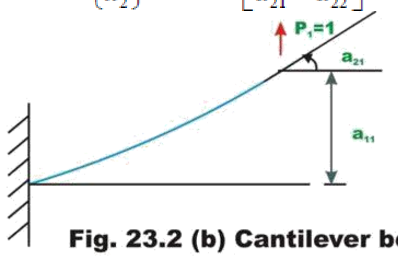



**Fig. 23.2 (b) Cantilever beam with unit load along $P_1$**


**Fig.23.2(a) Cantilever beam**

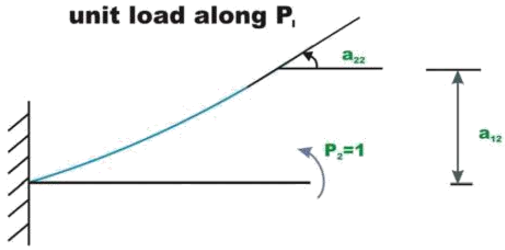



**Fig. 23.2 © Cantilever beam with unit moment along $P_2$**

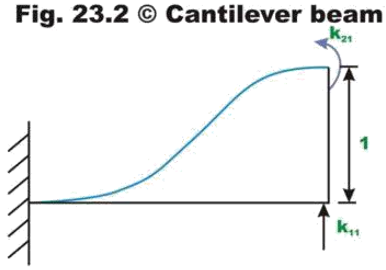



**Fig. 23.2(d) Cantilever beam with unit displacement along $u_1$**

The forces can also be related to displacements using stiffness coefficients. Apply a unit displacement along (see Fig.23.2d) keeping displacement as zero.

Calculate the required forces for this case as $k_{11}$and $k_{21}$ .Here, $k_{21}$ represents force developed along $P_2$ when a unit displacement along is introduced keeping=0. Apply a unit rotation along(vide Fig.23.2c),keeping□ 0

Calculate the required forces for this configuration $k_{12}$and $k_{22}$. Invoking the principle of superposition, the forces $P_1$ and $P_2$ are expressed as the sum of

forces developed due to displacements and acting separately on the beam. Thus,

$$\{P\}=[k]\{u\}$$

$$\text{where, } \{P\}=\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix};\ \{k\}=\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \text{;and } \{u\}=\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}.$$

$k$is defined as the stiffness matrix of the beam.

In this lesson, using stiffness method a few problems will be solved. However this approach is very rudimentary and is suited for hand computation. A more formal approach of the stiffness method will be presented in the next lesson.

## A SIMPLE EXAMPLE WITH ONE DEGREE OF FREEDOM

Consider a fixed–simply supported beam of constant flexural rigidity $EI$ and span $L$ which is carrying a uniformly distributed load of w kN/m as shown in Fig.23.3a. If the axial deformation is neglected, then this beam is kinematically indeterminate to first degree. The only unknown joint displacement is $\theta_B$.Thus the degrees of freedom for this structure is one (for a brief discussion on degrees of freedom, please see introduction to module 3).The analysis of above structure by stiffness method is accomplished in following steps:

Recall that in the flexibility /force method the redundants are released (i.e. made zero) to obtain a statically determinate structure. A similar operation in the stiffness method is to make all the unknown displacements equal to zero by altering the boundary conditions. Such an altered structure is known as kinematically determinate structure as all joint displacements are known in this case. In the present case the restrained structureisobtained by preventing the rotation at B as shown in Fig.23.3b. Apply all the external loads on the kinematically determinate structure. Duetorestraint at $B$, a moment $M_B$ is developed at $B$. In the stiffness method weadopt the following sign convention. Counterclockwise moments and counterclockwise rotations are taken as positive, upward forces and displacements are taken as positive.


**Fig.23.3(a) Cantilever beam**

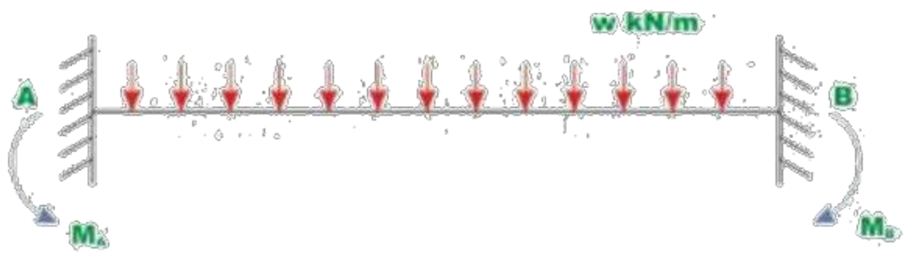



**Fig. 23.3b Kinematically determinate beam**


**Fig. 23.3 ©**

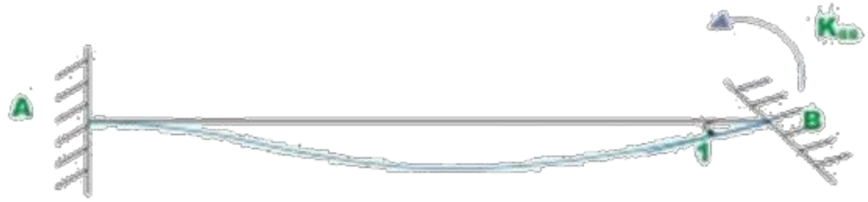



**Fig. 23.3 (d) Computation of stiffness co-efficients**

## TWO DEGREES OF FREEDOM STRUCTURE

Consider a plane truss as shown in Fig.23.4a.There is four members in the truss and they meet at the common point at *E*. The truss is subjected to external loadsandacting at *E*. In the analysis, neglect the self weight of members. Thereare two unknown displacements at joint *E* and are denoted byand.Thus the structure is kinematically indeterminate to second degree. The applied forces and unknown joint displacements are shown in the positive directions. The members are numbered from (1), (2), (3) and (4) as shown in the figure. The length and

axial rigidity of *i*-th member is $l_i$and $EA_i$respectively. Now it is soughtto evaluate andby stiffness method. This is done in following steps:

In the first step, make all the unknown displacements equal to zero by altering the boundary conditions as shown in Fig.23.4b. On thisrestrained/kinematically determinate structure, apply all the external loads except the joint loads and calculate the reactions corresponding to unknown joint displacements and. Since, in the present case, there are no external loads other than the joint loads, the reactions $(R_L)_1$and $(R_L)_2$ will be equal to zero. Thus,

In the next step, calculate stiffness coefficients $k_{11}$, $k_{21}$, $k_{12}$ and$k_{22}$ .This is done as follows. First give a unit displacement along $u_1$ holdingdisplacement along to zero and calculate reactions at *E* correspondingto unknown displacements and in thekinematically determinate structure. They are denoted by $k_{11}$,$k_{21}$. The joint stiffness $k_{11}$,$k_{21}$of thewhole truss is composed of individual member stiffness of the truss. This isshowninFig.23.4c.Nowconsiderthemember*AE*.Undertheactionofunit displacement along, the joint *E* displaces to. Obviously the new length is not equal to length *AE*. Let us denote the new length of themembersby$l_1 + \Delta l_1$, where $l$, is the change in length of the member AE′. The member *AE* ′also makes an angle with the horizontal. This isjustified as $l_1$ is small. From the geometry, the change in length of the members *AE*′is

(23.11a)

The elongation $\Delta l_1$ is related to the force in the member $AE'$, $F_{AE'}$ by

$$\Delta l_1 = \frac{F_{AE'} l_1}{A_1 E} \qquad (23.11b)$$

Thus from (23.11a) and (23.11b), the force in the members $AE$□ is

$$F'_{AE} = \frac{EA_1}{l_1} \cos\theta_1 \qquad (23.11c)$$

This force acts along the member axis. This force may be resolved alongand directions . Thus the horizontal component of force

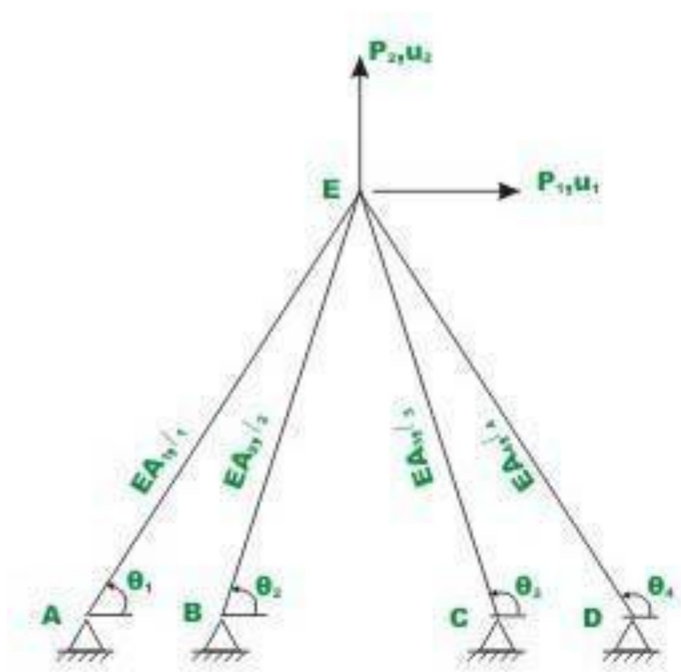



**Fig 23.4a A four - member truss**

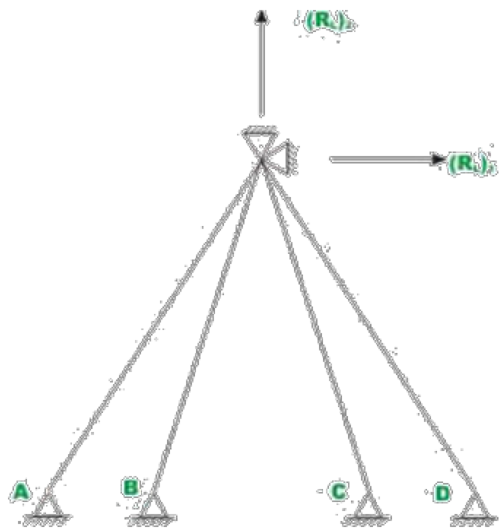



**Fig. 23.4b Kinematically determinate structure**

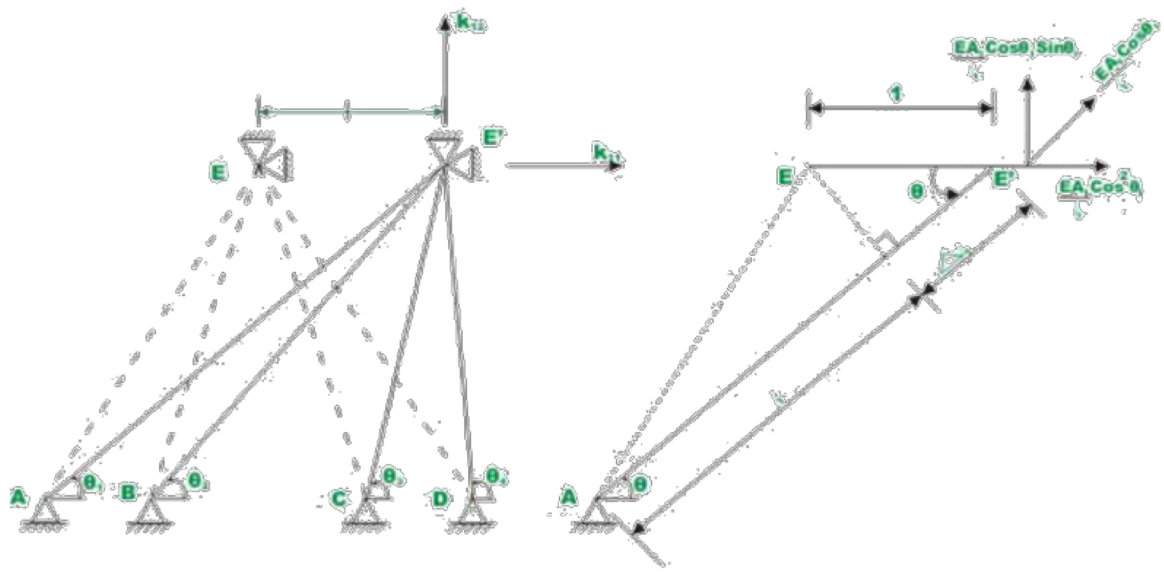



**Fig. 23.4c Unit displacement along $u_1$**

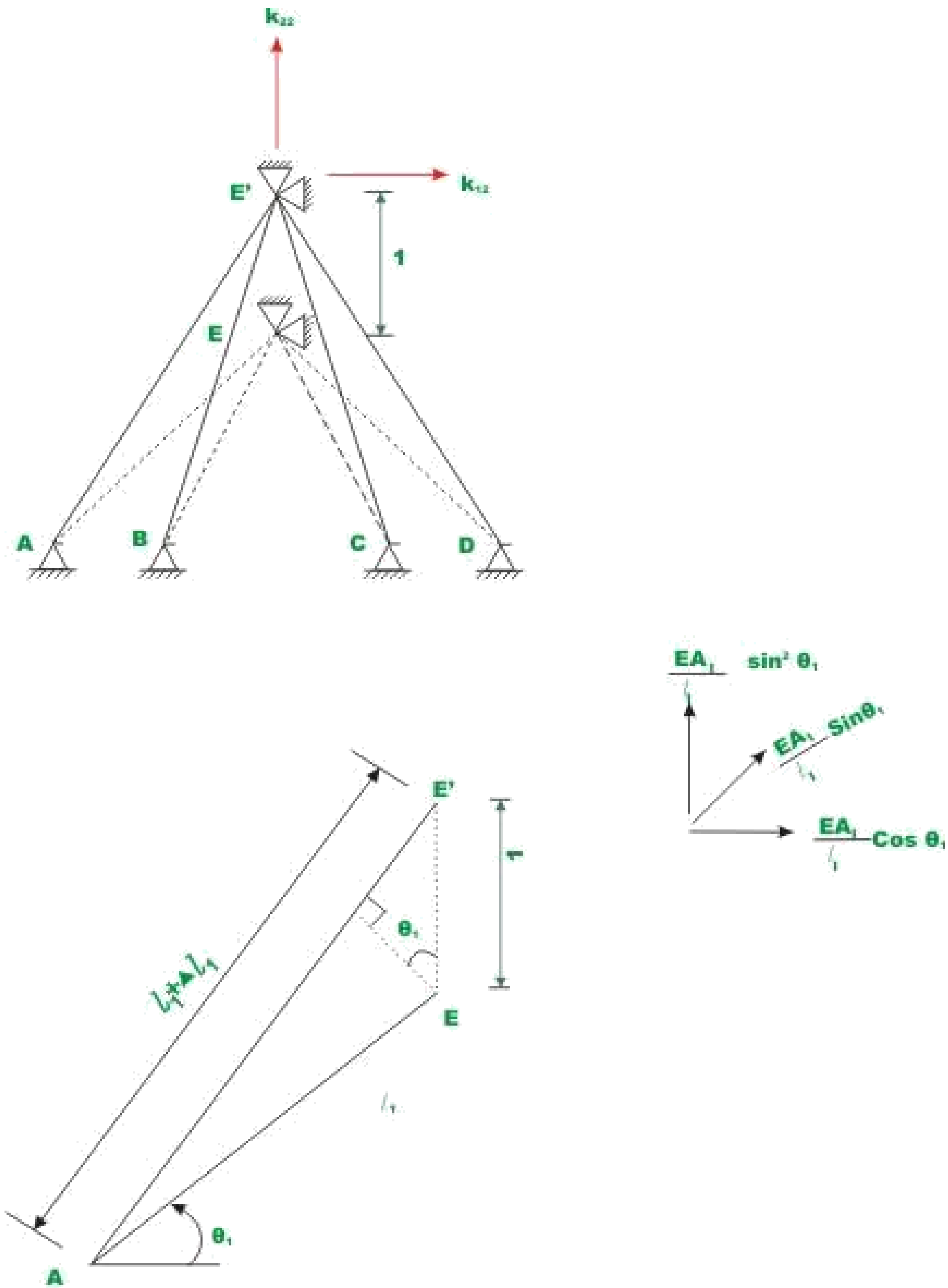



Fig.23.4d Unit displacement along $u_2$

Expressions of similar form as above may be obtained for all members. The sum of all horizontal components of individual forces gives us the stiffness coefficient

$k_{11}$ and sum of all vertical component of forces give us the required stiffness coefficient $k_{21}$ .

$$k_{11} = \frac{EA_1}{l_1}\cos^2\theta_1 + \frac{EA_2}{l_2}\cos^2\theta_2 + \frac{EA_3}{l_3}\cos^2\theta_3 + \frac{EA_4}{l_4}\cos^2\theta_4$$

$$k_{11} = \sum_{i=1}^{4} \frac{EA_i}{l_i}\cos^2\theta_i \qquad (23.15)$$

Similarly, $k_{12} = \sum_{i=1}^{4} \frac{EA_i}{l_i}\sin\theta_i \cos\theta_i$ (23.16)

B. Joint forces in the original structure corresponding to unknown displacements $u_1$ and $u_2$ are

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \qquad (23.17)$$

Now the equilibrium equations at joint *E* states that the forces in the original structure are equal to the superposition of (i) reactions in the kinematically restrained structure corresponding to unknown joint displacements and (ii) reactions in the restrained structure due to unknown displacements themselves. This may be expressed as,

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix};$$

$$\{R_L\} = \begin{Bmatrix} (R_L)_1 \\ (R_L)_2 \end{Bmatrix}$$

$$[k] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad (23.20)$$

For example take $P_1 = P_2 = P$, $L_i = \dfrac{L}{\sin\theta_i}$, $A_1 = A_2 = A_3 = A_4 = A$ and $\theta_1 = 35^\circ$, $\theta_2 = 70^\circ$, $\theta_3 = 105^\circ$ and $\theta_4 = 140^\circ$

Then.

$$\{F\} = \begin{Bmatrix} P \\ P \end{Bmatrix} \qquad (23.21)$$

$$\{R_L\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$k_{11} = \sum \frac{EA}{L} \cos^2\theta_i \sin\theta_i = 0.9367 \frac{EA}{L}$$

$$k_{12} = \sum \frac{EA}{L} \sin^2\theta_i \cos\theta_i = 0.0135 \frac{EA}{L}$$

$$k_{21} = \sum \frac{EA}{L} \sin^2 \theta_i \cos\theta_i = 0.0135 \frac{EA}{L}$$

$$k_{22} = \sum \frac{EA}{L} \sin^3 \theta_i = 2.1853 \frac{EA}{L} \qquad (23.22)$$

$$\begin{Bmatrix} P \\ P \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 0.9367 & 0.0135 \\ 0.0135 & 2.1853 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Solving which, yields

$$u_1 = 1.0611 \frac{L}{EA}$$

$$u_2 = 0.451 \frac{L}{EA}$$

## Example

Analyze the plane frame shown in Fig.23.5a by the direct stiffness method. Assume that the flexural rigidity for all members is the same. Neglect axial displacements.

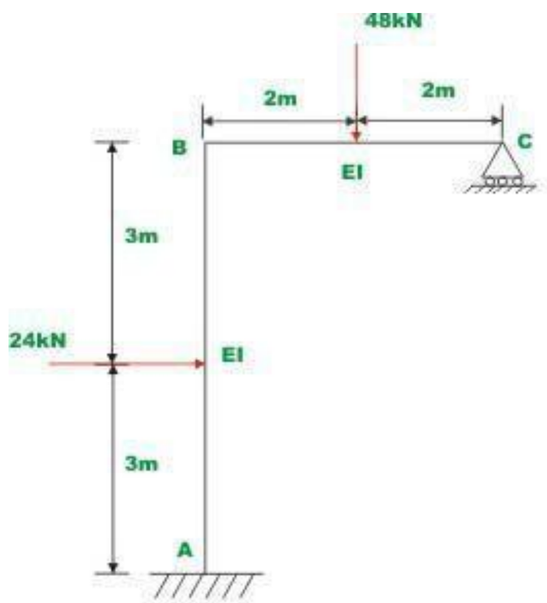



**Fig 23.5a Plane - frame of Example 23.1**

## Solution

In the first step identify the degrees of freedom of the frame .The given frame has three degrees of freedom (see Fig.23.5b):

B. Two rotations as indicated by $u_1$ and $u_2$and

C. One horizontal displacement of joint B and C as indicated by $u_3$ .

In the next step make all the displacements equal to zero by fixing joints B and C as shown in Fig.23.5c. On this kinematically determinate structure apply all the external loads and calculate reactions corresponding to unknown joint displacements .Thus,

$$\left(R_D^F\right)_1 = \frac{48\times2\times4}{16} + \left(-\frac{24\times3\times9}{36}\right)$$

(1)

$$= 24 - 18 = 6 \text{ kN.m}$$

$$\left(R_D^F\right)_2 = -24 \text{ kN.m}$$

$$\left(R_D^F\right)_3 = 12 \text{ kN.m} \qquad (2)$$

Thus,

$$\begin{Bmatrix} \left(R_D^F\right)_1 \\ \left(R_D^F\right)_2 \\ \left(R_D^F\right)_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ -24 \\ 12 \end{Bmatrix} \qquad (3)$$

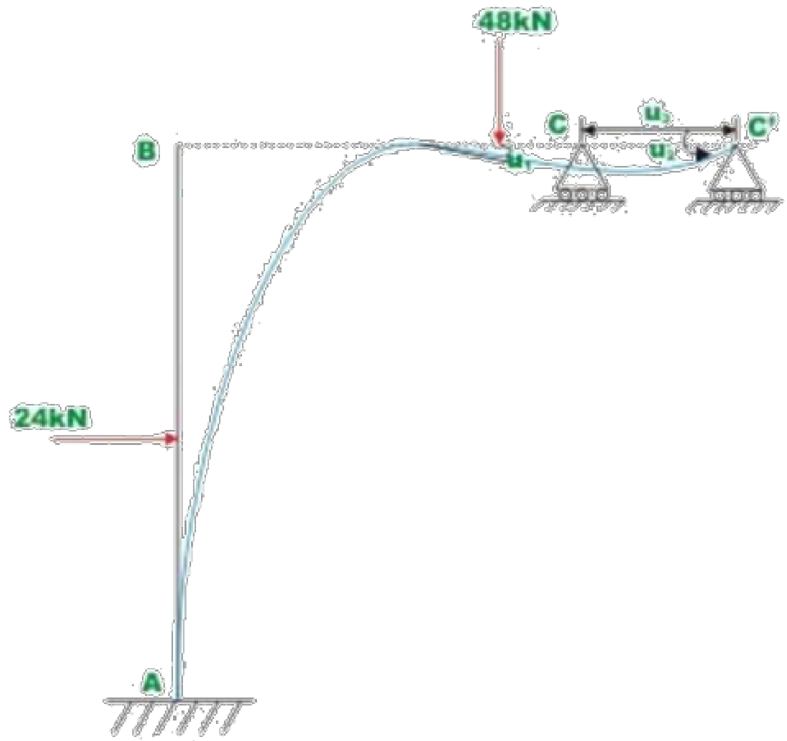



**Fig 23.5b Approximate deflected shape**

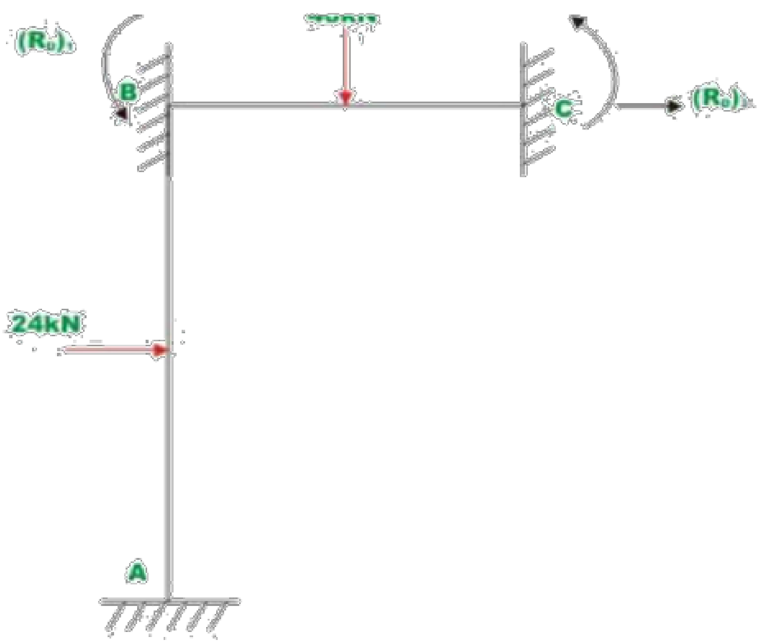



**Fig 23.5c Kinematically restrained structure**

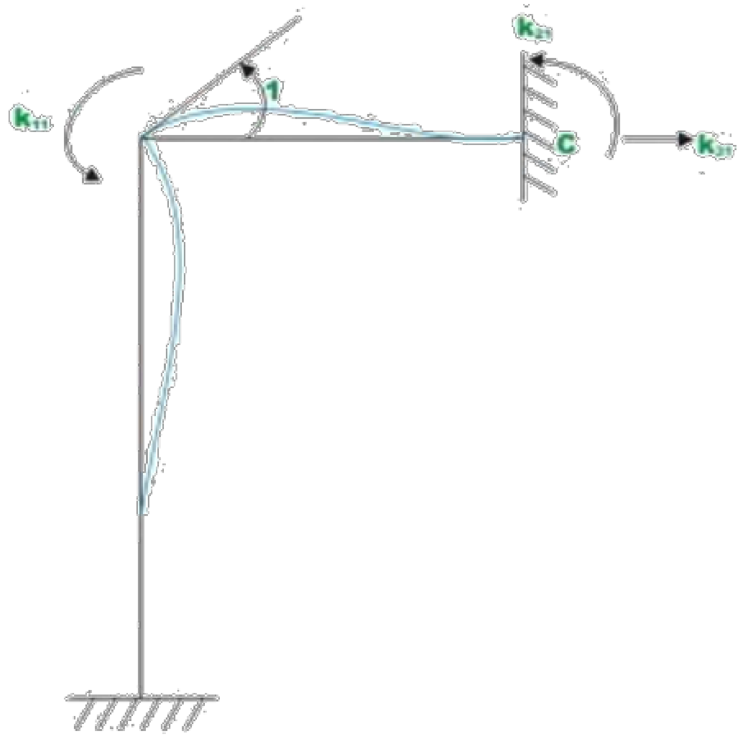



**Fig.23.5d Unit displacement along $u_1$**

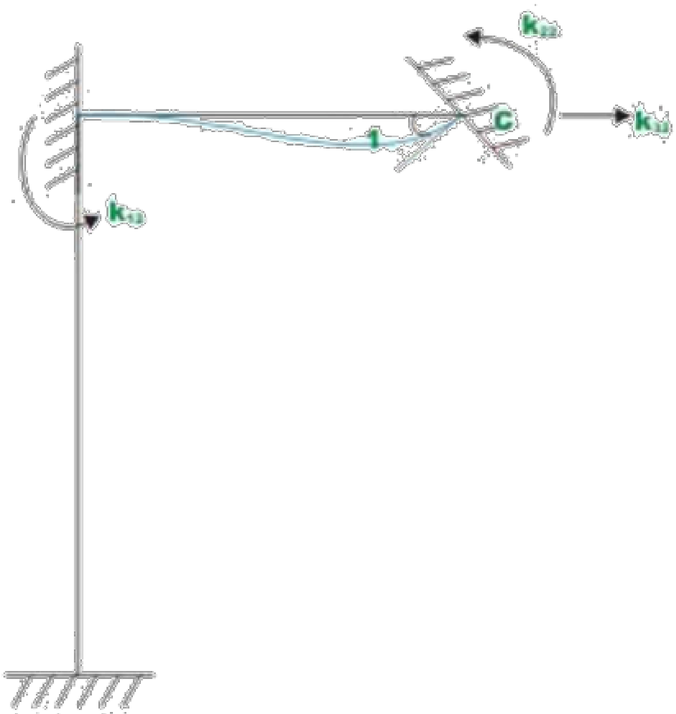



**Fig 23.5e Unit displacement along $u_2$**

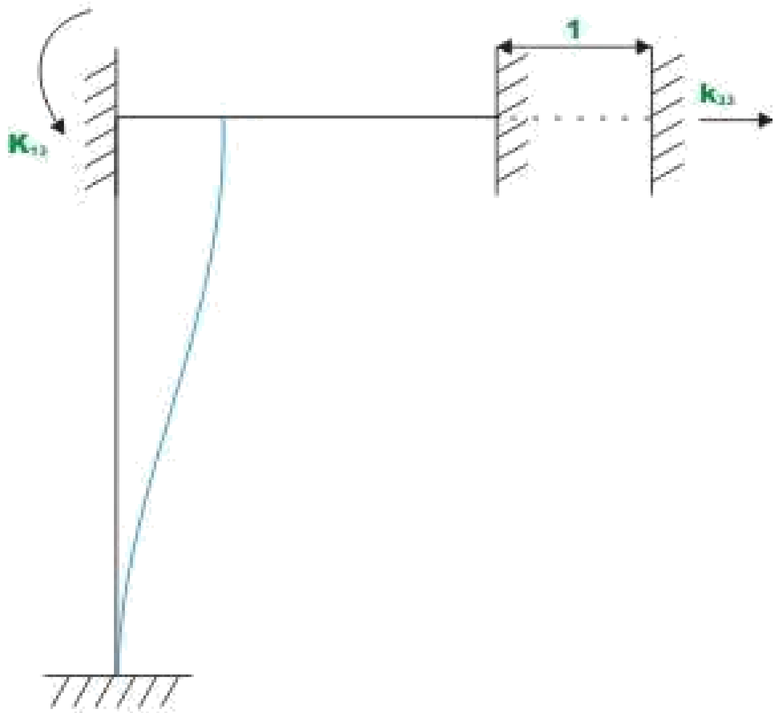



**Fig. 23.5f Unit displacement along $u_3$**

$$k_{11} = \frac{4EI}{4} + \frac{4EI}{6} = 1.667$$

$$k_{21} = \frac{2EI}{4} = 0.5EI$$

$$k_{31} = -\frac{6EI}{6 \times 6} = -0.166EI \qquad (4)$$

Similarly, apply a unit rotation along $u_2$ and calculate reactions corresponding to three degrees of freedom (see Fig.23.5e)

$$k_{12} = 0.5EI$$

$$k_{22} = EI$$

$$k_{32} = 0 \qquad (5)$$

Apply a unit displacement along $u_3$ and calculate joint reactions corresponding to unknown displacements in the kinematically determinate structure.

$$k_{13} = -\frac{6EI}{L^2} = -0.166E$$

$$k_{23} = 0$$

$$k_{33} = \frac{12EI}{6^3} = 0.056EI \qquad (6)$$

Finally applying the principle of superposition of joint forces, yields

Now, $$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ -24 \\ 12 \end{Bmatrix} + EI \begin{Bmatrix} 1.667 & 0.5 & -0.166 \\ 0.5 & 1 & 0 \\ -0.166 & 0 & 0.056 \end{Bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$ as there are no loads applied along $u_1$,$u_2$ and $u_3$.Thus the

unknown displacements are,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = -\frac{1}{EI} \begin{bmatrix} 1 & 0.5 & -0.166 \\ 0.5 & 1 & 0 \\ -0.166 & 0 & 0.056 \end{bmatrix}^{-1} \begin{Bmatrix} 6 \\ -24 \\ -24 \end{Bmatrix} \qquad (7)$$

Solving

$$u_1 = \frac{18.996}{EI}$$

$$u_2 = \frac{14.502}{EI}$$

$$u_3 = -\frac{270.587}{EI} \qquad (8)$$

## Example 24.1

Analyse the two member truss shown in Fig. 24.12a. Assume $EA$ to be constant for all members. The length of each member is $5m$.

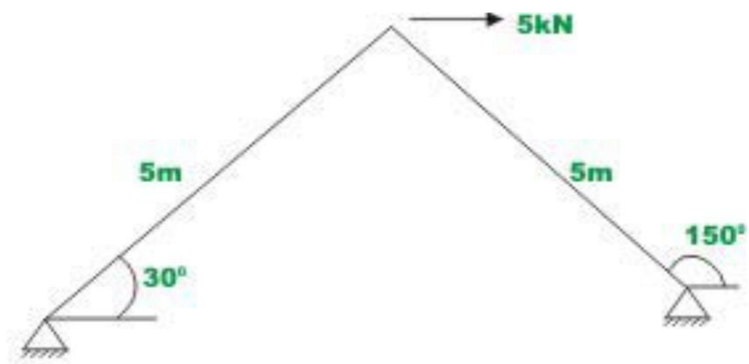



**Fig 24.12(a) Example 24.1**

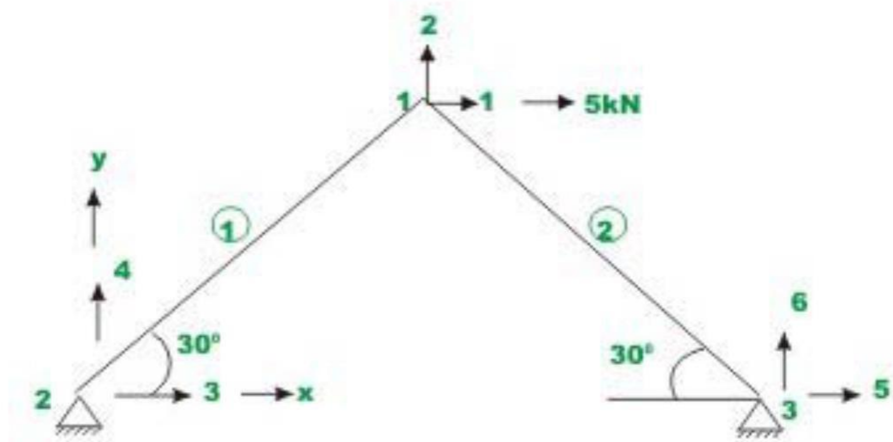



**Fig 24.12(b) Members and node numbering**

The co-ordinate axes, the number of nodes and members are shownin

Fig.24.12b. The degrees of freedom at each node are also shown. By inspection it is clear that the displacement $u_3 = u_4 = u_5 = u_6 = 0$. Also the external loads are

$$p_1 = 5 \quad \text{kN} \quad ; p_2 = 0 \quad \text{kN}. \tag{1}$$

Now member stiffness matrix for each member in global co-ordinate system is $(\theta_1 = 30°)$.

$$[k^1] = \frac{EA}{5}\begin{bmatrix} 0.75 & 0.433 & -0.75 & -0.433 \\ 0.433 & 0.25 & -0.433 & -0.25 \\ -0.75 & -0.433 & 0.75 & 0.433 \\ -0.433 & -0.25 & 0.433 & 0.25 \end{bmatrix} \tag{2}$$

$$[k^2] = \frac{EA}{5}\begin{bmatrix} 0.75 & -0.433 & -0.75 & 0.433 \\ -0.433 & 0.25 & 0.433 & -0.25 \\ -0.75 & 0.433 & 0.75 & -0.433 \\ 0.433 & -0.25 & -0.433 & 0.25 \end{bmatrix} \tag{3}$$

The global stiffness matrix of the truss can be obtained by assembling the two member stiffness matrices. Thus,

$$[K] = \frac{EA}{5}\left[\begin{array}{cc:cccc} 1.5 & 0 & -0.75 & -0.433 & -0.75 & 0.433 \\ 0 & 0.5 & -0.433 & -0.25 & 0.433 & -0.25 \\ \hdashline -0.75 & -0.433 & 0.75 & 0.433 & 0 & 0 \\ -0.433 & -0.25 & 0.433 & 0.25 & 0 & 0 \\ -0.75 & 0.433 & 0 & 0 & 0.75 & -0.433 \\ 0.433 & -0.25 & 0 & 0 & -0.433 & 0.25 \end{array}\right] \tag{4}$$

Again stiffness matrix for the unconstrained degrees of freedom is,

$$[K] = \frac{EA}{5}\begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix} \qquad (5)$$

Writing the load displacement-relation for the truss for the unconstrained degrees of freedom

$$\{p_k\} = [k_{11}]\{u_u\} \qquad (6)$$

$$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \frac{EA}{5}\begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} 5 \\ 0 \end{Bmatrix} = \frac{EA}{5}\begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$u_1 = \frac{16.667}{EA} \quad ; u_2 = 0$$

Support reactions are evaluated using equation (24.30).

$$\{p_u\} = [k_{21}]\{u_u\}$$

Substituting appropriate values in equation (9),

$$\{p_u\} = \frac{EA}{5}\begin{bmatrix} -0.75 & -0.433 \\ -0.433 & -0.25 \\ -0.75 & 0.433 \\ 0.433 & -0.25 \end{bmatrix}\frac{1}{AE}\begin{Bmatrix} 16.667 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \begin{pmatrix} -2.5 \\ -1.443 \\ -2.5 \\ 1.443 \end{pmatrix}$$

by equilibrium of joint

1. Also, $p_3+p_5+5=0$

Now force in each member is calculated as follows,

**Member 1**:$l=0.866$ ;$m=0.5$ ;$L=5m$.

$$\{p'\}=[k']\{u'\}$$

$$=[k'][T]\{u\} \quad (11)$$

$$\begin{Bmatrix} p'_1 \\ p'_2 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}\begin{Bmatrix} u_3 \\ v_4 \\ u_1 \\ v_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{L}\begin{bmatrix} l & m & -l & -m \end{bmatrix}\begin{Bmatrix} u_3 \\ v_4 \\ u_1 \\ v_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{L}[-0.866]\left\{\frac{16.667}{AE}\right\} = -2.88 \quad \text{kN}$$

**Member 2:** $l=-0.866$ ; $m=0.5$ ; $L=5\,m$ .

$$\begin{Bmatrix} p'_1 \\ p'_2 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}\begin{Bmatrix} u_5 \\ v_6 \\ u_1 \\ v_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{L}\begin{bmatrix} l & m & -l & -m \end{bmatrix}\begin{Bmatrix} u_3 \\ v_4 \\ u_1 \\ v_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{5}\begin{bmatrix} -0.866 \end{bmatrix}\begin{Bmatrix} \dfrac{16.667}{AE} \end{Bmatrix} = -2.88 \quad \text{kN}$$

\

## DIRECT STIFFNESS METHOD: BEAMS

In the earlier section, a few problems were solved using stiffness method from fundamentals. The procedure adopted therein is not suitable for computer implementation. In fact the load displacement relation for the entire structure was derived from fundamentals. This procedure runs into trouble when the structure is large and complex. However this can be much simplified provided we follow the procedure adopted for trusses. In the case of truss, the stiffness matrix of the entire truss was obtained by assembling the member stiffness matrices of individualmembers.

In a similar way, one could obtain the global stiffness matrix of a continuous beam from assembling member stiffness matrix of individual beam elements. Towards this end, we break the given beam into a number of beam elements. The stiffness matrix of each individual beam element can be written very easily. For example, consider a continuous beam *ABCD* as shown in Fig. 27.1a. The given continuous beam is divided into three beam elements as shown inFig.

27.1b. It is noticed that, in this case, nodes are located at the supports. Thus each span is treated as an individual beam. However sometimes it is required to consider a node between support points. This is done whenever the cross sectional area changes suddenly or if it is required to calculate vertical or rotational displacements at an intermediate point. Such a division is shown inFig.

27.1c. If the axial deformations are neglected then each node of the beam will have two degrees of freedom: a vertical displacement (corresponding to shear) and a rotation (corresponding to bending moment). In Fig. 27.1b, numbers enclosed in a circle represents beam numbers. The beam *ABCD* is divided into three beam members. Hence, there are four nodes and eight degrees of freedom. The possible displacement degrees of freedom of the beam are also shown in the figure. Let us use lower numbers to denote unknown degrees of freedom (unconstrained degrees of freedom) and higher numbers to denote known (constrained) degrees of freedom. Such a method of identification is adopted in this course for the ease of imposing boundary conditions directly on the structure stiffness matrix. However, one could number sequentially as shown in Fig. 27.1d. This is preferred while solving the problem on acomputer.

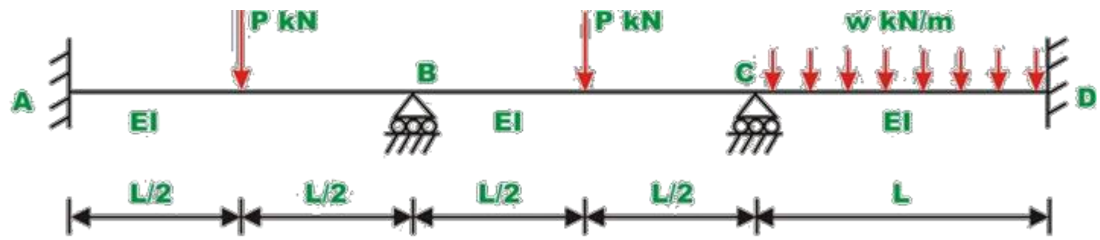



**Fig 27.1a Continuous beam**

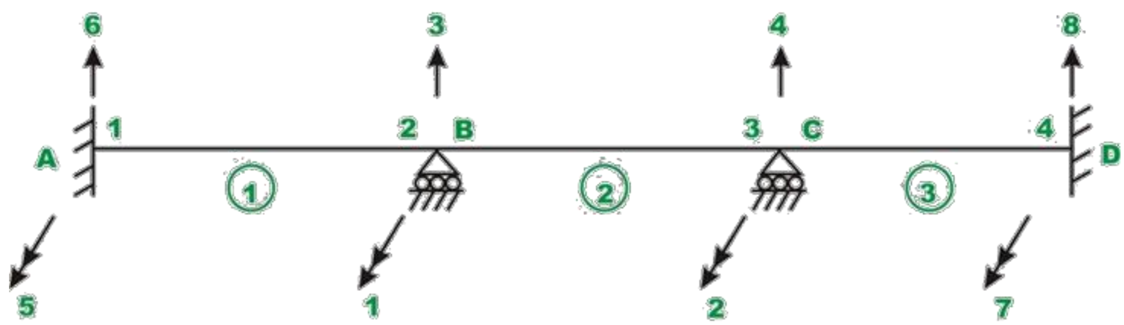



**Fig. 27.1b Member and node numbering**

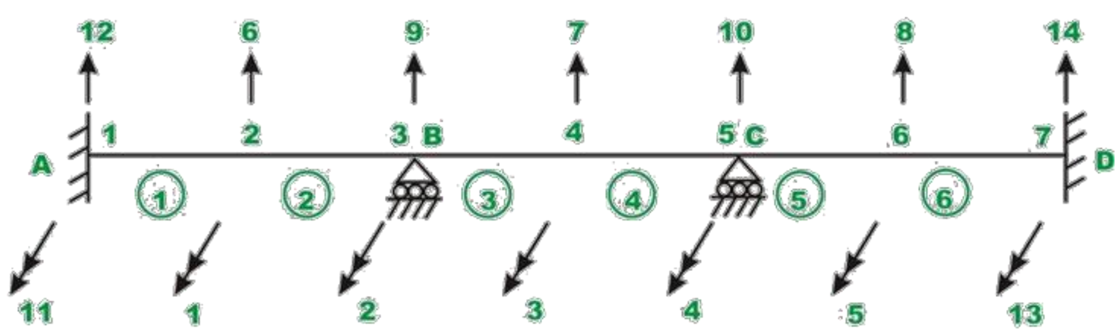



**Fig. 27.1c Member and node numbering**

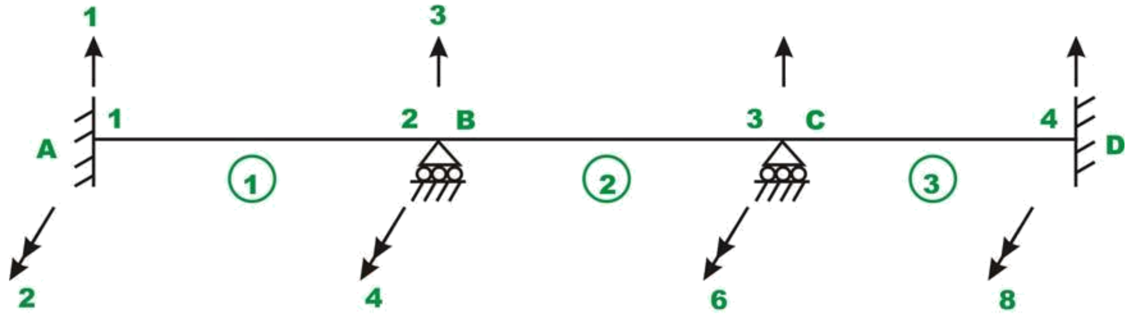



**Fig 27.1d Member and node numbering**

In the above figures, single headed arrows are used to indicate translational and double headed arrows are used to indicate rotational degrees of freedom.

## BEAM STIFFNESS MATRIX.

Fig. 27.2 shows a prismatic beam of a constant cross section that is fully restrained at ends in local orthogonal co-ordinate system $x'y'z'$ . The beam ends

are denoted by nodes $j$ and $k$ . The $x'$ axis coincides with the centroidal axis of the member with the positive sense being defined from $j$ to $k$ . Let $L$ be the length of the member, $A$area of cross section of the member and $I_{zz}$is the moment of inertia about $z'$axis.

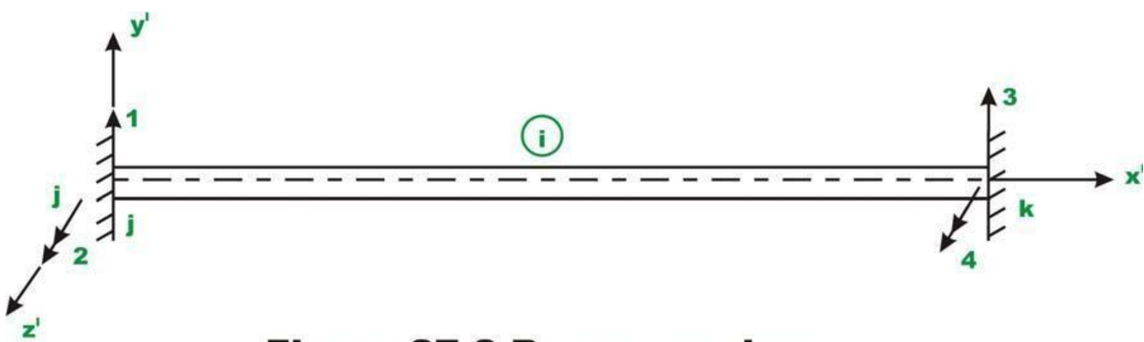
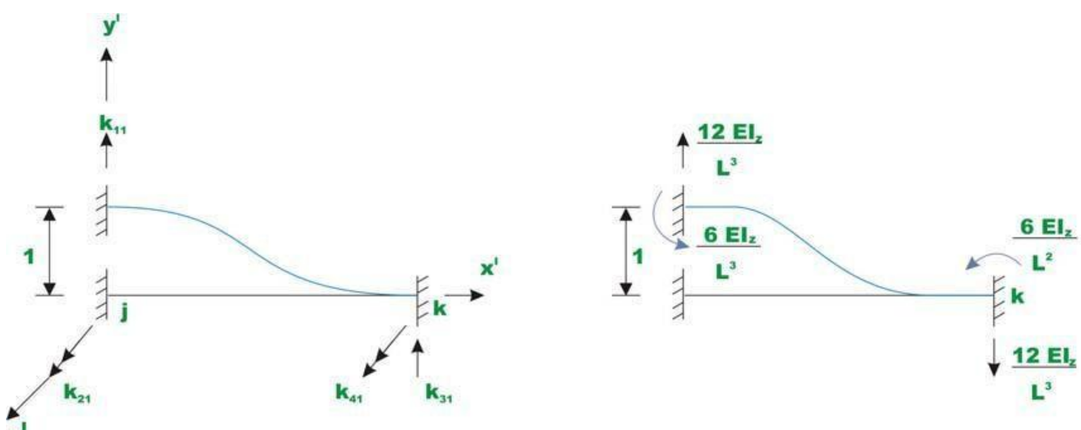


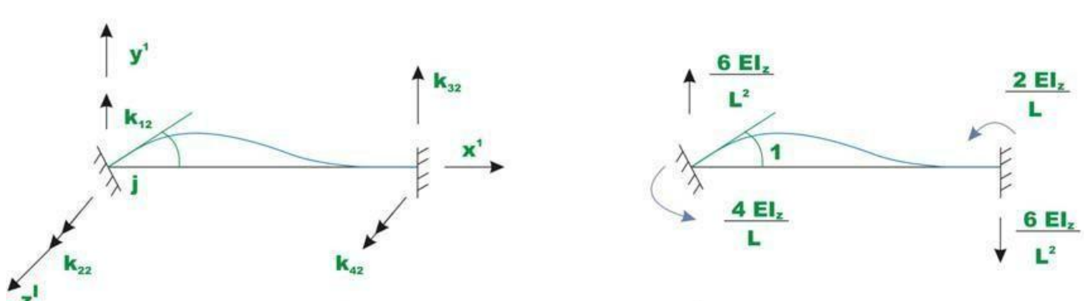

**Figure 27.2 Beam member**

Two degrees of freedom (one translation and one rotation) are considered at each end of the member. Hence, there are four possible degrees of freedom for this member and hence the resulting stiffness matrix is of the order $4 \cdot 4$ . In this method counterclockwise moments and counterclockwise rotations are taken as positive. The positive sense of the translation and rotation are also shown in the figure. Displacements are considered as positive in the direction of the co-ordinate axis. The elements of the stiffness matrix indicate the forces exerted on the member by the restraints at the ends of the member when unit displacements are imposed at each end of the member. Let us calculate the forces developed in the above beam member when unit displacement is imposed along each degree of freedom holding all other displacements to zero. Now impose a unit displacement along $y'$ axis at $j$ end of the member while holding all other displacements to zero as shown in Fig. 27.3a. This displacement causes both shear and moment in the beam. The restraint actions are also shown in the figure. By definition they are elements of the member stiffness matrix. In particular they form the first column of element stiffnessmatrix.

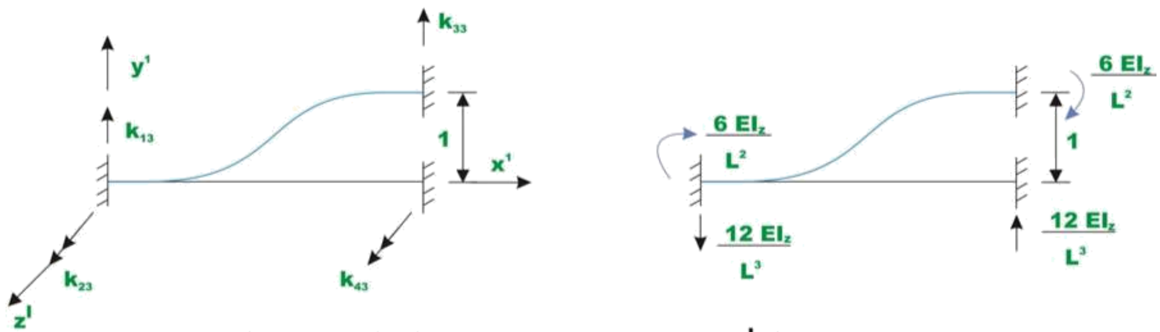
In Fig. 27.3b, the unit rotation in the positive sense is imposed at $j$ end of th beam while holding all other displacements to zero. The restraint actions are shown in the figure. The restraint actions at ends are calculated referring to tables given in lesson …



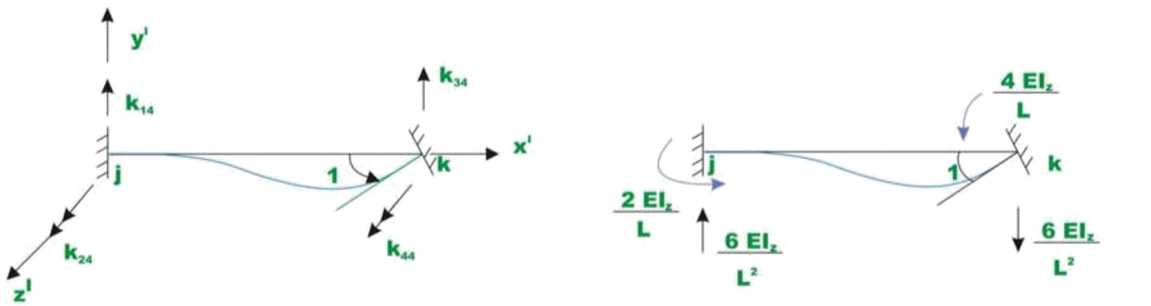

( a ) Unit translation along y' at end j




( b ) Unit rotation about z' at end j




**( c ) Unit displacement along y' at end k**




**( d ) Unit rotation about z' at end k**

**Fig. 27.3 Computation of beam stiffness matrix**

In Fig. 27.3c, unitdisplacementalong $y'$ axisatend $k$ is imposed and corresponding restraint actions are calculated. Similarly in Fig. 27.3d, unit rotation about $z'$ axis at end $k$ is imposed and corresponding stiffness coefficients are calculated. Hence the member stiffness matrix for the beam memberis

$$
\begin{array}{c} \\ [k] = \end{array}
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\left[\begin{array}{cc|cc}
\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\
\frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\
\hline
-\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\
\frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L}
\end{array}\right]
\begin{array}{c} \underline{1} \\ \underline{2} \\ \underline{3} \\ \underline{4} \end{array}
\end{array}
\qquad (27.1)
$$

The stiffness matrix is symmetrical. The stiffness matrix is partitioned to separate the actions associated with two ends of the member. For continuous beam problem, if the supports are unyielding, then only rotational degree of freedom shown in Fig. 27.4 is possible. In such a case the first and the third rows and columns will be deleted. The reduced stiffness matrix will be,

$$[k]=\begin{bmatrix} \frac{4EI_z}{L} & \frac{2EI_z}{L} \\ \frac{2EI_z}{L} & \frac{4EI_z}{L} \end{bmatrix} \tag{27.2}$$

Instead of imposing unit displacement along $y'$ at $j$ end of the member in Fig. 27.3a, apply a displacement $u'_1$ along $y'$ at $j$ end of the member as shown in Fig. 27.5a, holding all other displacements to zero. Let the restraining forces developed be denoted by $q_{11}, q_{21}, q_{31}$ and $q_{41}$.

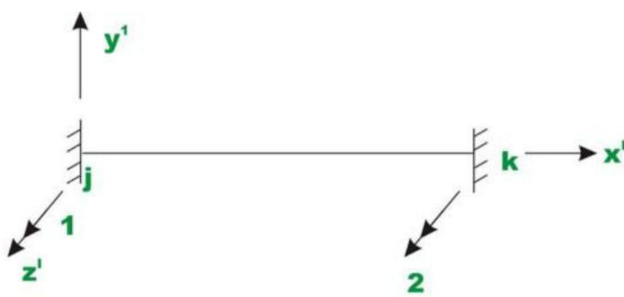
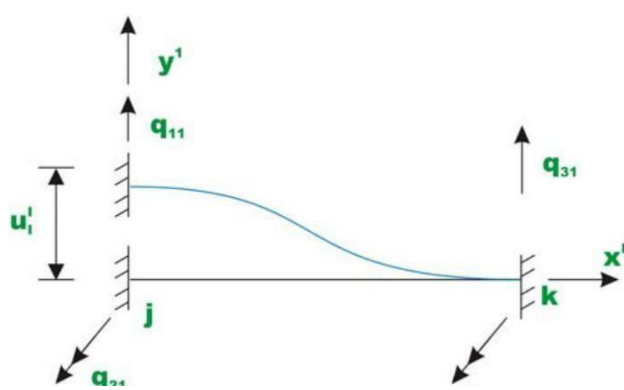



**Fig. 27.4**



Now, give displacements $u'_1, u'_2, u'_3$ and $u'_4$ simultaneously along displacement degrees of freedom 1,2,3 and 4 respectively. Let the restraining forces developed at member ends be $q_1, q_2, q_3$ and $q_4$ respectively as shown in Fig. 27.5b along

respective degrees of freedom. Then by the principle of superposition, the force displacement relationship can be written as,

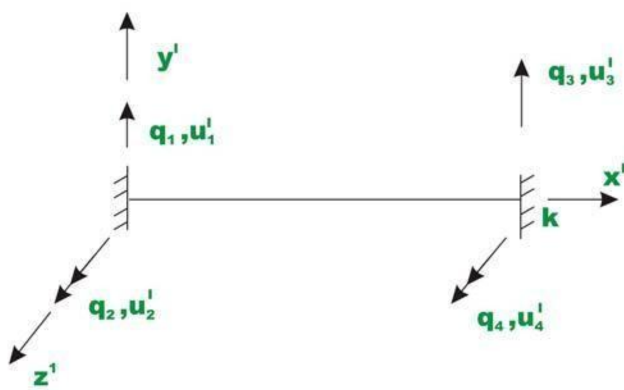



**Fig. 27.5 (b) Force - displacement relation**

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \\ u'_4 \end{bmatrix} \tag{27.4}$$

$$\{q\} = [k]\{u'\} \tag{27.5}$$

## BEAM (GLOBAL) STIFFNESS MATRIX.

The formation of structure (beam) stiffness matrix from its member stiffness matrices is explained with help of two span continuous beam shown in Fig. 27.6a. Note that no loading is shown on the beam. The orthogonal co-ordinate system *xyz* denotes the global co-ordinatesystem.

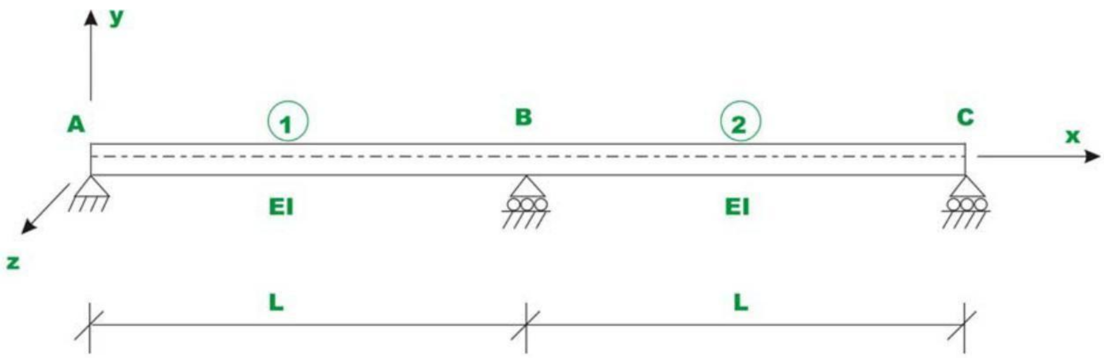



**Fig. 27.6a Continuous beam**

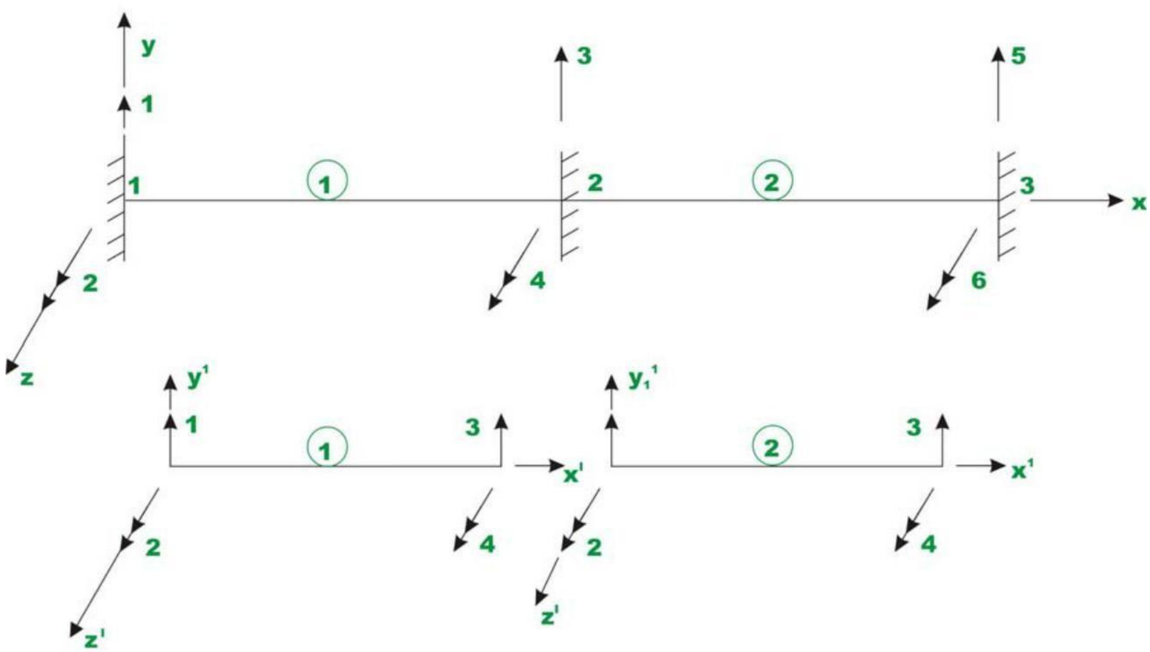



**Fig. 27.6 b**

For the case of continuous beam, the $x$ - and $x'$ - axes are collinear and other axes ( $y$and $y'$ , $z$ and $z'$ ) are parallel to each other. Hence it is not requiredto

transform member stiffness matrix from local co-ordinate system to global coordinate system as done in the case of trusses. For obtaining the global stiffness matrix, first assume that all joints are restrained. The node and member numbering for the possible degrees of freedom are shown in Fig 27.6b. The continuous beam is divided into two beam members. For this member there are six possible degrees of freedom. Also in the figure, each beam member with its displacement degrees of freedom (in local co ordinate system) is also shown.

Since the continuous beam has the same moment of inertia and span, the member stiffness matrix of element 1 and 2 are the same. They are,

$$\begin{array}{lcccc} Global\ d.o.f & 1 & 2 & 3 & 4 \\ Local\ d.o.f & 1 & 2 & 3 & 4 \end{array}$$

$$[k'] = \begin{bmatrix} k'_{11} & k'_{12} & k'_{13} & k'_{14} \\ k'_{21} & k'_{22} & k'_{23} & k'_{24} \\ k'_{31} & k'_{32} & k'_{33} & k'_{34} \\ k'_{41} & k'_{42} & k'_{43} & k'_{44} \end{bmatrix} \begin{matrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{matrix} \qquad (27.6a)$$

$$\begin{array}{lcccc} Global\ d.o.f & 3 & 4 & 5 & 6 \\ Local\ d.o.f & 1 & 2 & 3 & 4 \end{array}$$

$$[k^2] = \begin{bmatrix} k^2_{11} & k^2_{12} & k^2_{13} & k^2_{14} \\ k^2_{21} & k^2_{22} & k^2_{23} & k^2_{24} \\ k^2_{31} & k^2_{32} & k^2_{33} & k^2_{34} \\ k^2_{41} & k^2_{42} & k^2_{43} & k^2_{44} \end{bmatrix} \begin{matrix} 1 & 3 \\ 2 & 4 \\ 3 & 5 \\ 4 & 6 \end{matrix} \qquad (27.6b)$$

The local and the global degrees of freedom are also indicated on the top and side of the element stiffness matrix. This will help us to place the elements of the element stiffness matrix at the appropriate locations of the global stiffness matrix. The continuous beam has six degrees of freedom and hence the stiffness matrix is of the order $6$ X $6$ . Let $[K]$denotes the continuous beam stiffness matrix of order $6$ X $6$ . From Fig. 27.6b, $[K]$may be written as,

$$\textit{Member } AB\ (1)$$

$$[K]=\left[\begin{array}{cc|cc|cc} k_{11}^{1} & k_{12}^{1} & k_{13}^{1} & k_{14}^{1} & 0 & 0 \\ k_{21}^{1} & k_{22}^{1} & k_{23}^{1} & k_{24}^{1} & 0 & 0 \\ \hline k_{31}^{1} & k_{32}^{1} & k_{33}^{1}+k_{11}^{2} & k_{34}^{1}+k_{12}^{2} & k_{13}^{2} & k_{14}^{2} \\ k_{41}^{1} & k_{42}^{1} & k_{43}^{1}+k_{21}^{2} & k_{44}^{1}+k_{22}^{2} & k_{23}^{2} & k_{24}^{2} \\ \hline 0 & 0 & k_{31}^{2} & k_{32}^{2} & k_{33}^{2} & k_{34}^{2} \\ 0 & 0 & k_{41}^{2} & k_{42}^{2} & k_{43}^{2} & k_{44}^{2} \end{array}\right] \qquad (27.7)$$

$$\textit{Member } BC\ (2)$$

The $4 \times 4$ upper left hand section receives contribution from member 1 $(AB)$ and

$4 \times 4$ lower right hand section of global stiffness matrix receives contribution frommember 2. The element of the global stiffness matrix corresponding to global degrees of freedom 3 and 4 [overlapping portion of equation $(27.7)$] receives element from both members 1 and 2.

## Formation of load vector.

Consider a continuous beam $ABC$ as shown in Fig. 27.7.

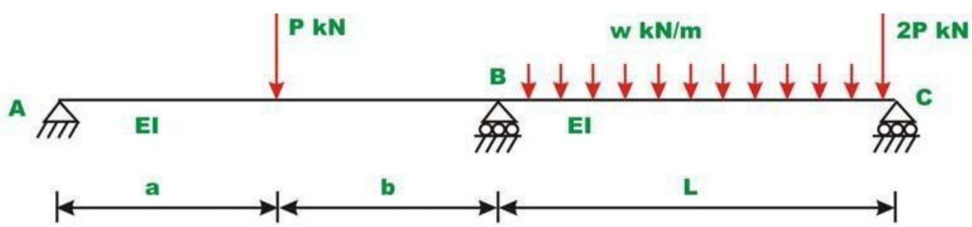
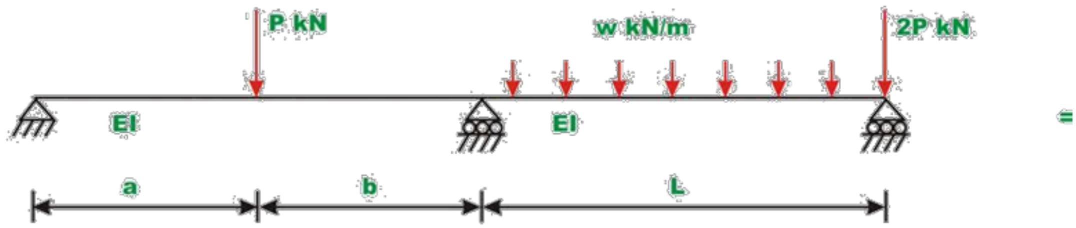
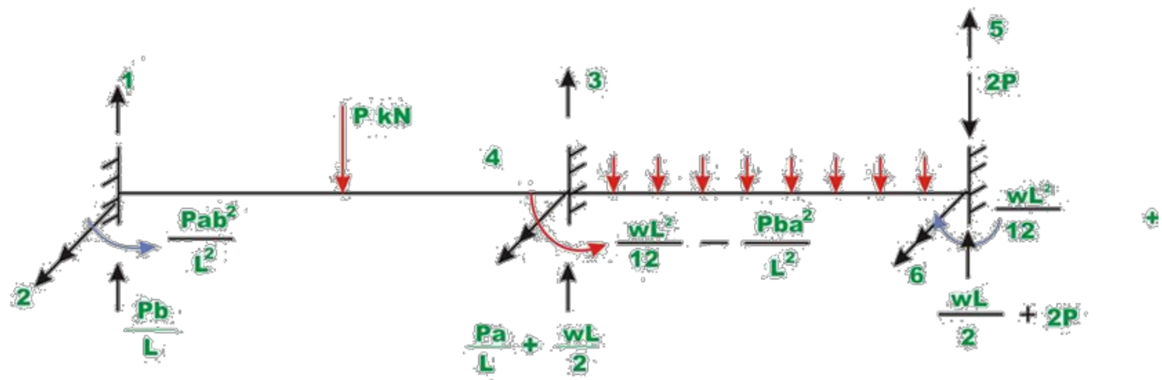



**Fig.27.7**

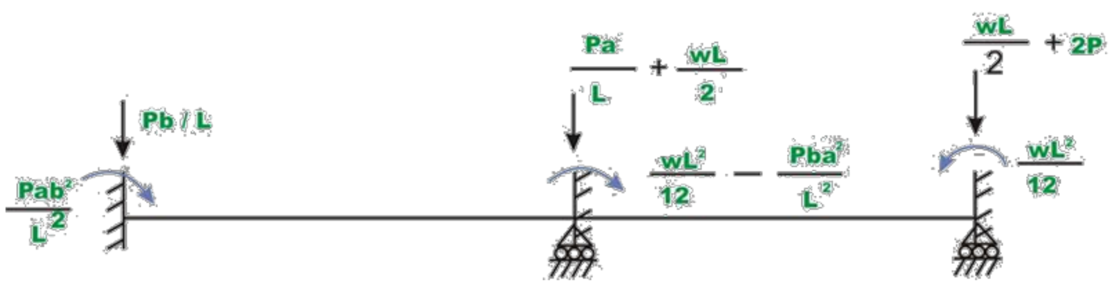
We have two types of load: member loads and joint loads. Joint loads could be handled very easily as done in case of trusses. Note that stiffness matrix of each member was developed for end loading only. Thus it is required to replace the member loads by equivalent joint loads. The equivalent joint loads must be evaluated such that the displacements produced by them in the beam should be the same as the displacements produced by the actual loading on the beam. This is evaluated by invoking the method of superposition.




( a ) Actual beam with loading




( b ) Reaction in the restrained beam




( c ) Equivalent joint loads

**Fig. 27.8**

The loading on the beam shown in Fig. 27.8(a), is equal to the sum of Fig.

27.8(b) and Fig. 27.8(c). In Fig. 27.8(c), the joints are restrained against displacements and fixed end forces are calculated. In Fig. 27.8(c) these fixed end actions are shown in reverse direction on the actual beam without any load.

Since the beam in Fig. 27.8(b) is restrained (fixed) against any displacement, the displacements produced by the joint loads in Fig. 27.8(c) must be equal to the displacement produced by the actual beam in Fig. 27.8(a). Thus the loads shown in Fig. 27.8(c) are the equivalent joint loads .Let, $p_1, p_2, p_3, p_4, p_5$ and $p_6$ be the

equivalent joint loads acting on the continuous beam along displacement degrees of freedom 1,2,3,4,5 and 6 respectively as shown in Fig. 27.8(b). Thus th global load vector is,

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{Pb}{L} \\ -\dfrac{Pab^2}{L^2} \\ -\left(\dfrac{Pa}{L}+\dfrac{wL}{2}\right) \\ -\left(\dfrac{wL^2}{12}-\dfrac{Pba^2}{L^2}\right) \\ -\left(\dfrac{wL}{2}+2P\right) \\ \dfrac{wL^2}{12} \end{Bmatrix} \tag{27.8}$$

## Solution of equilibrium equations

After establishing the global stiffness matrix and load vector of the beam, the load displacement relationship for the beam can be writtenas,

$$\{P\} = [K]\{u\} \tag{27.9}$$

where $\{P\}$ is the global load vector, $\{u\}$ is displacement vector and $[K]$ is the

global stiffness matrix. This equation is solved exactly in the similar manner as discussed in the lesson 24. In the above equation some joint displacements are known from support conditions. The above equation may be written as

$$\begin{Bmatrix} \{p_k\} \\ \{p_u\} \end{Bmatrix} = \begin{bmatrix} [k_{11}] & [k_{12}] \\ [k_{21}] & [k_{22}] \end{bmatrix} \begin{Bmatrix} \{u_u\} \\ \{u_k\} \end{Bmatrix} \tag{27.10}$$

displacements. And $\{p_u\}$, $\{u_u\}$ denote respectively vector of unknown forces and unknown displacements respectively. Now expanding equation(27.10),

$$\{p_k\} = [k_{11}]\{u_u\} + [k_{12}]\{u_k\} \qquad (27.11a)$$

$$\{p_u\} = [k_{21}]\{u_u\} + [k_{22}]\{u_k\} \qquad (27.11b)$$

Since $\{u_k\}$ is known, from equation 27.11(a), the unknown joint displacements

can be evaluated. And support reactions are evaluated from equation (27.11b), after evaluating unknown displacement vector.

Let $R_1, R_3$ and $R_5$ be the reactions along the constrained degrees of freedom as

shown in Fig. 27.9a. Since equivalent joint loads are directly applied at the supports, they also need to be considered while calculating the actual reactions.

Thus,

$$\begin{Bmatrix} R_1 \\ R_3 \\ R_5 \end{Bmatrix} = -\begin{Bmatrix} p_1 \\ p_3 \\ p_5 \end{Bmatrix} + [K_{21}]\{u_u\} \qquad (27.12)$$

The reactions may be calculated as follows. The reactions of the beam shown in Fig. 27.9a are equal to the sum of reactions shown in Fig. 27.9b, Fig. 27.9c and Fig. 27.9d.

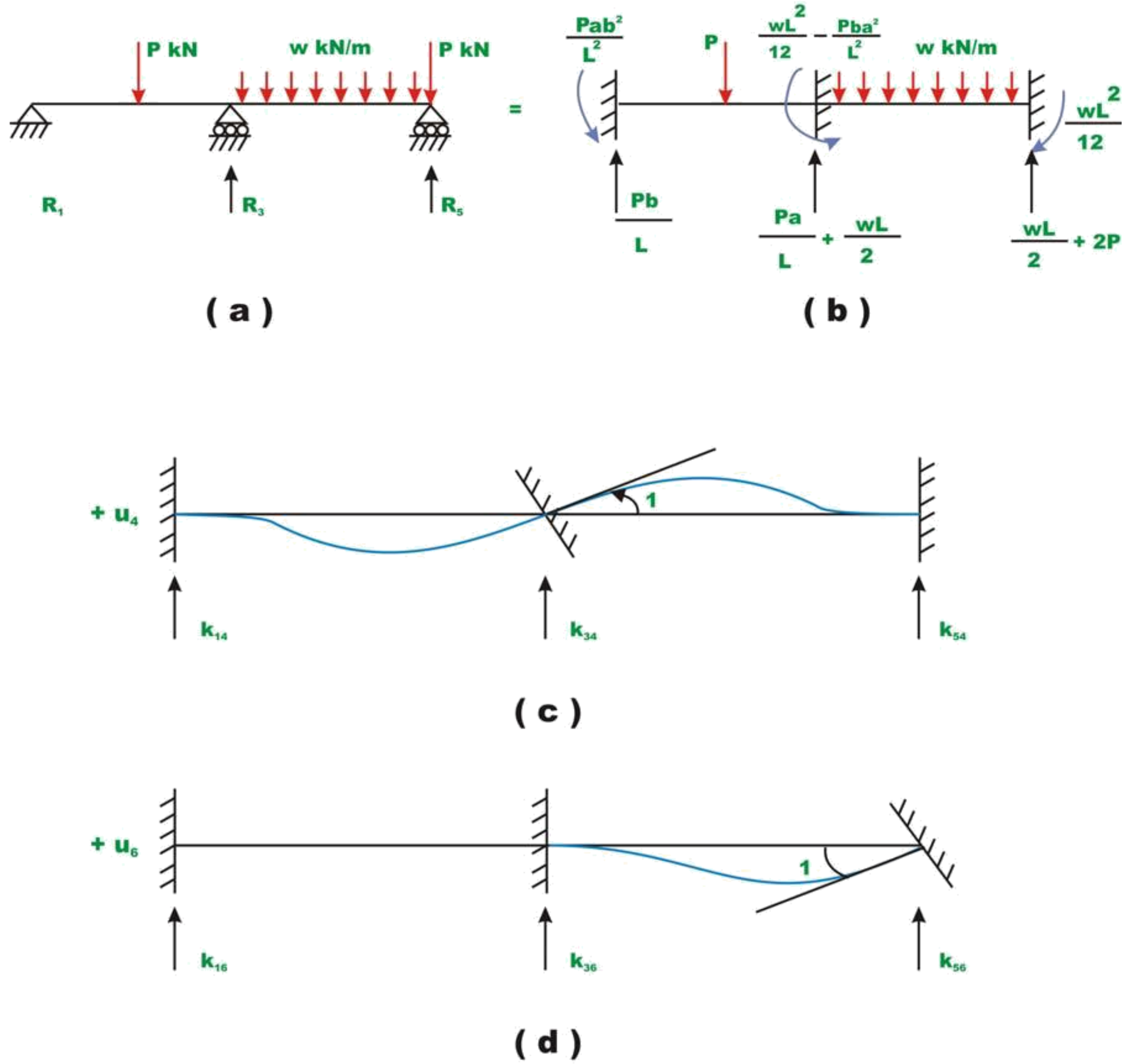

P kN
w kN/m
P kN
R1
R3
R5
(a)
=
Pab²/L²
P
wL²/12 – Pba²/L²
w kN/m
wL²/12
Pb/L
Pa/L + wL/2
wL/2 + 2P
(b)
+ u4
1
k14
k34
k54
(c)
+ u6
1
k16
k36
k56
(d)

**Fig. 27.9**

From the method of superposition,

$$R_1 = \frac{Pb}{L} + K_{14}u_4 + K_{16}u_6 \tag{27.13a}$$

$$R_3 = \frac{Pa}{L} + K_{34}u_4 + K_{36}u_6 \tag{27.13b}$$

$$R_5 = \frac{wL}{2} + 2P + K_{54}u_4 + K_{56}u_6 \tag{27.13c}$$

or

$$\begin{Bmatrix} R_1 \\ R_3 \\ R_5 \end{Bmatrix} = \begin{Bmatrix} Pb/L \\ Pa/L \\ \frac{wl}{2} + 2P \end{Bmatrix} + \begin{bmatrix} K_{14} & K_{16} \\ K_{34} & K_{36} \\ K_{54} & K_{56} \end{bmatrix} \begin{Bmatrix} u_4 \\ u_6 \end{Bmatrix} \tag{27.14a}$$

Equation (27.14a) may be written as,

$$\begin{Bmatrix} R_1 \\ R_3 \\ R_5 \end{Bmatrix} = -\begin{Bmatrix} Pb/L \\ Pa/L \\ \frac{wl}{2} + 2P \end{Bmatrix} + \begin{bmatrix} K_{14} & K_{16} \\ K_{34} & K_{36} \\ K_{54} & K_{56} \end{bmatrix} \begin{Bmatrix} u_4 \\ u_6 \end{Bmatrix} \tag{27.14b}$$

Member end actions $q_1, q_2, q_3, q_4$ are calculated as follows. For example consider the first element 1.

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} \frac{Pb}{L} \\ \frac{Pab^2}{L^2} \\ \frac{Pa}{L} \\ -\frac{Pa^2b}{L^2} \end{Bmatrix} + [K]_{element1} \begin{Bmatrix} 0 \\ u_2 \\ 0 \\ u_4 \end{Bmatrix} \tag{27.16}$$

In the next lesson few problems are solved to illustrate the method so far discussed.

In the last lesson, the procedure to analyse beams by direct stiffness method has been discussed. No numerical problems are given in that lesson. In this lesson, few continuous beam problems are solved numerically by direct stiffness method.

**Example 1**

Analyse the continuous beam shown in Fig. 28.1a. Assume that the supports are unyielding. Also assume that $EI$ is constant for all members.

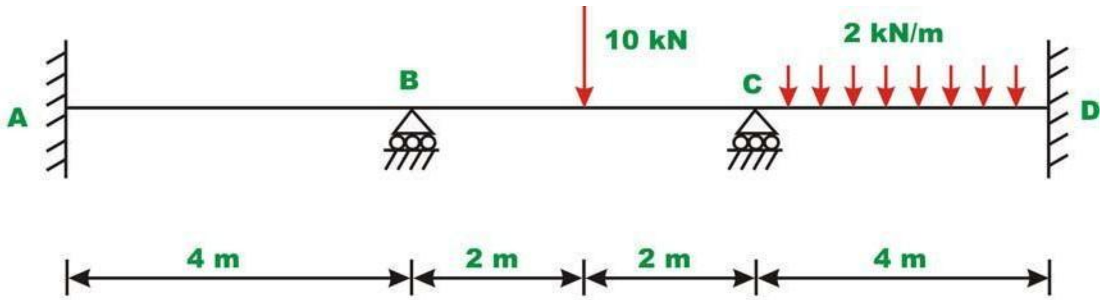



**Fig. 28.1a**

The numbering of joints and members are shown in Fig. 28.1b. The possible global degrees of freedom are shown in the figure. Numbers are put for the unconstrained degrees of freedom first and then that for constrained displacements.

The given continuous beam is divided into three beam elements Two degrees of freedom (one translation and one rotation) are considered at each end of the member. In the above figure, double headed arrows denote rotations and single headed arrow represents translations. In the given problem some displacements are zero, i.e., $u_3=u_4=u_5=u_6=u_7=u_8=0$ from support conditions.

In the case of beams, it is not required to transform member stiffness matrix from local co-ordinate system to global co-ordinate system, as the two co-ordinate system are parallel to each other.

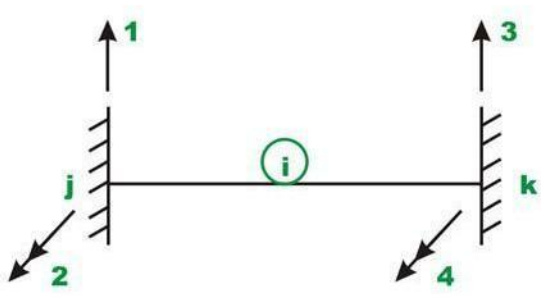



**Figure 28.1c**

First construct the member stiffness matrix for each member. This may be done from the fundamentals. However, one could use directly the equation (27.1) given in the previous lesson and reproduced below for the sakeconvenience.

$$[k] = \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \end{array}$$

$$[k] = \left[ \begin{array}{cc:cc} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ \hdashline -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{array} \right] \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \quad (27.1)$$

The degrees of freedom of a typical beam member are shown in Fig. 28.1c. Here equation (1) is used to generate element stiffness matrix.

**Member 1**: $L = 4m$, node points 1-2.

The member stiffness matrix for all the members are the same, as the length and flexural rigidity of all members is the same.

$$\begin{array}{ccccc} \textit{Global d.o.f} & 6 & 5 & 3 & 1 \end{array}$$

Me

$$[k'] = EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \begin{array}{c} 6 \\ 5 \\ 3 \\ 1 \end{array} \quad (2)$$

On the member stiffness matrix, the corresponding global degrees of freedom are indicated to facilitate assembling.

$$\begin{array}{ccccc} \textit{Global d.o.f} & 3 & 1 & 4 & 2 \end{array}$$

$$[k^2] = EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \begin{array}{c} 3 \\ 1 \\ 4 \\ 2 \end{array} \quad (3)$$

**Member 3**: $L = 4m$, node points 3-4.

$$Global\ d.o.f \quad 4 \quad 2 \quad 8 \quad 7$$

$$[k^3] = EI_{zz}\begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \begin{matrix} 4 \\ 2 \\ 8 \\ 7 \end{matrix} \quad (4)$$

The assembled global stiffness matrix of the continuous beam is of the order 8 x 8. The assembled global stiffness matrix may be written as,

$$[K] = EI_{zz}\begin{bmatrix} 2.0 & 0.5 & 0.0 & -0.375 & 0.5 & 0.375 & 0 & 0 \\ 0.5 & 2.0 & 0.375 & 0 & 0 & 0 & 0.5 & -0.375 \\ 0 & 0.375 & 0.375 & -0.1875 & -0.375 & -0.1875 & 0 & 0 \\ -0.375 & 0 & -0.1875 & 0.375 & 0 & 0 & 0.375 & -0.1875 \\ 0.5 & 0 & -0.375 & 0 & 1.0 & 0.375 & 0 & 0 \\ 0.375 & 0 & -0.1875 & 0 & 0.375 & 0.1875 & 0 & 0 \\ 0 & 0.5 & 0 & 0.375 & 0 & 0 & 1.0 & -0.375 \\ 0 & -0.375 & 0 & -0.1875 & 0 & 0 & -0.375 & 0.1875 \end{bmatrix}$$

(5)

Now it is required to replace the given members loads by equivalent joint loads. The equivalent loads for the present case is shown in Fig. 28.1d. The displacement degrees of freedom are also shown in Fig. 28.1d.

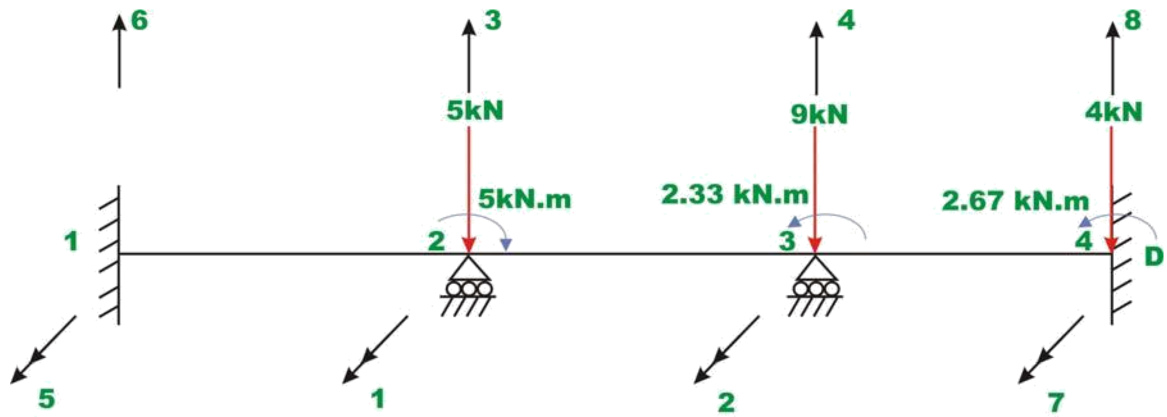



**Fig. 28.1 (d) Equivalent joint loads**

hus the global load vector corresponding to unconstrained degree of freedom is,

$$\{p_k\} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} -5 \\ 2.33 \end{Bmatrix} \qquad (6)$$

Writing the load displacement relation for the entire continuous beam,

$$\begin{Bmatrix} -5 \\ 2.33 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = EI_{zz} \left[\begin{array}{cc|cccccc} 2.0 & 0.5 & 0.0 & -0.375 & 0.5 & 0.375 & 0 & 0 \\ 0.5 & 2.0 & 0.375 & 0 & 0 & 0 & 0.5 & -0.375 \\ \hline 0 & 0.375 & 0.375 & -0.187 & -0.375 & -0.187 & 0 & 0 \\ -0.375 & 0 & -0.187 & 0.375 & 0 & 0 & 0.375 & -0.187 \\ 0.5 & 0 & -0.375 & 0 & 1.0 & 0.375 & 0 & 0 \\ 0.375 & 0 & -0.187 & 0 & 0.375 & 0.187 & 0 & 0 \\ 0 & 0.5 & 0 & 0.375 & 0 & 0 & 1.0 & -0.375 \\ 0 & -0.375 & 0 & -0.187 & 0 & 0 & -0.375 & 0.187 \end{array}\right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} \quad (7)$$

where $\{p\}$ is the joint load vector, $\{u\}$ is displacement vector. $u_1$ and

We know that $u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = 0$ . Thus solving for unknowns

$u_2$, yields

$$\begin{Bmatrix} -5 \\ 2.33 \end{Bmatrix} = EI_{zz} \begin{bmatrix} 2.0 & 0.5 \\ 0.5 & 2.0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad (8)$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{3.75EI_{zz}} \begin{bmatrix} 2.0 & -0.5 \\ -0.5 & 2.0 \end{bmatrix} \begin{Bmatrix} -5 \\ 2.333 \end{Bmatrix} \quad (9)$$

$$= \frac{1}{EI_{zz}} \begin{Bmatrix} -2.977 \\ 1.909 \end{Bmatrix}$$

Thus displacements are,

$$u_1 = \frac{-2.977}{EI_{zz}} \quad \text{and} \quad u_2 = \frac{1.909}{EI_{zz}} \quad (10)$$

The unknown joint loads are given by,

The actual reactions at the supports are calculated as,

$$\begin{Bmatrix} R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \end{Bmatrix} = \begin{Bmatrix} p_3^F \\ p_4^F \\ p_5^F \\ p_6^F \\ p_7^F \\ p_8^F \end{Bmatrix} + \begin{Bmatrix} p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = \begin{Bmatrix} 5 \\ 9 \\ 0 \\ 0 \\ -2.67 \\ 4 \end{Bmatrix} + \begin{Bmatrix} 0.715 \\ 1.116 \\ -1.488 \\ -1.116 \\ 0.955 \\ -0.715 \end{Bmatrix} = \begin{Bmatrix} 5.716 \\ 10.116 \\ -1.489 \\ -1.116 \\ -1.715 \\ 3.284 \end{Bmatrix} \qquad (12)$$

$$= \begin{Bmatrix} 0.715 \\ 1.116 \\ -1.488 \\ -1.116 \end{Bmatrix}$$

## Member end actions for element 1

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -2.977 \end{Bmatrix}$$

$$= \begin{Bmatrix} -1.116 \\ -1.488 \\ 1.116 \\ -2.977 \end{Bmatrix} \qquad (13)$$

## Member end actions for element 2

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = +EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} 0 \\ -2.977 \\ 0 \\ 1.909 \end{Bmatrix}$$

$$= \begin{Bmatrix} 4.6 \\ 2.98 \\ 5.4 \\ -4.58 \end{Bmatrix} \qquad (14)$$

## Member end actions for element 3

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 4.0 \\ 2.67 \\ 4.0 \\ -2.67 \end{Bmatrix} + EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} 0 \\ 1.909 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 4.72 \\ 4.58 \\ 3.28 \\ -1.72 \end{Bmatrix} \qquad (15)$$

**Example 2**

Analyse the continuous beam shown in Fig. 28.2a. Assume that the supports are unyielding. Assume $EI$ to be constant for all members.

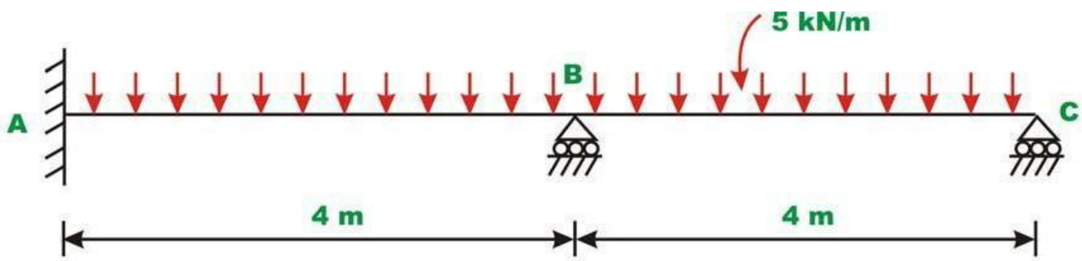



**Fig. 28.2a**

The numbering of joints and members are shown in Fig. 28.2b. The global degrees of freedom are also shown in the figure.

The given continuous beam is divided into two beam elements. Two degrees of freedom (one translation and one rotation) are considered at each end of the member. In the above figure, double headed arrows denote rotations and single headed arrow represents translations. Also it is observed that displacements $u_3 = u_4 = u_5 = u_6 = 0$from support conditions.

First construct the member stiffness matrix for each member.

**Member 1**:$L = 4m$, node points 1-2.

The member stiffness matrix for all the members are the same, as the length and flexural rigidity of all members is the same.

$$\begin{matrix} Global\ d.o.f & 6 & 5 & 3 & 1 & \\ \end{matrix}$$

$$[k'] = EI_{zz}\begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix}\begin{matrix} 6 \\ 5 \\ 3 \\ 1 \end{matrix} \qquad (1)$$

On the member stiffness matrix, the corresponding global degrees of freedom are indicated to facilitateassembling.

**Member 2:** $L = 4m$, node points 2-3.

$$\begin{array}{c} \text{Global d.o.f} \quad 3 \quad\quad 1 \quad\quad 4 \quad\quad 2 \end{array}$$

$$\left[k^2\right] = EI_{zz}\begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix}\begin{matrix} 3 \\ 1 \\ 4 \\ 2 \end{matrix} \qquad (2)$$

The assembled global stiffness matrix of the continuous beam is of order $6 \cdot 6$ .

The assembled global stiffness matrix may be written as,

$$\left[K\right] = EI_{zz}\begin{bmatrix} 2 & 0.5 & 0 & -0.375 & 0.5 & 0.375 \\ 0.5 & 1.0 & 0.375 & -0.375 & 0 & 0 \\ 0 & 0.375 & 0.375 & -0.1875 & -0.375 & -0.1875 \\ -0.375 & -0.375 & -0.1875 & 0.1875 & 0 & 0 \\ 0.5 & 0 & -0.375 & 0 & 1.0 & 0.375 \\ 0.375 & 0 & -0.1875 & 0 & 0.375 & 0.1875 \end{bmatrix} \qquad (3)$$

Now it is required to replace the given members loads by equivalent joint loads. The equivalent loads for the present case is shown in Fig. 28.2c. The displacement degrees of freedom are also shown in figure.

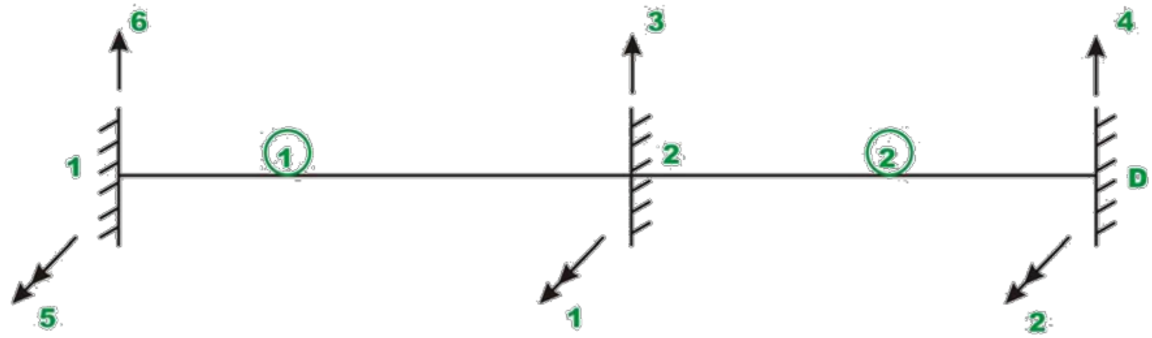



**Fig. 28.2b Node and member**

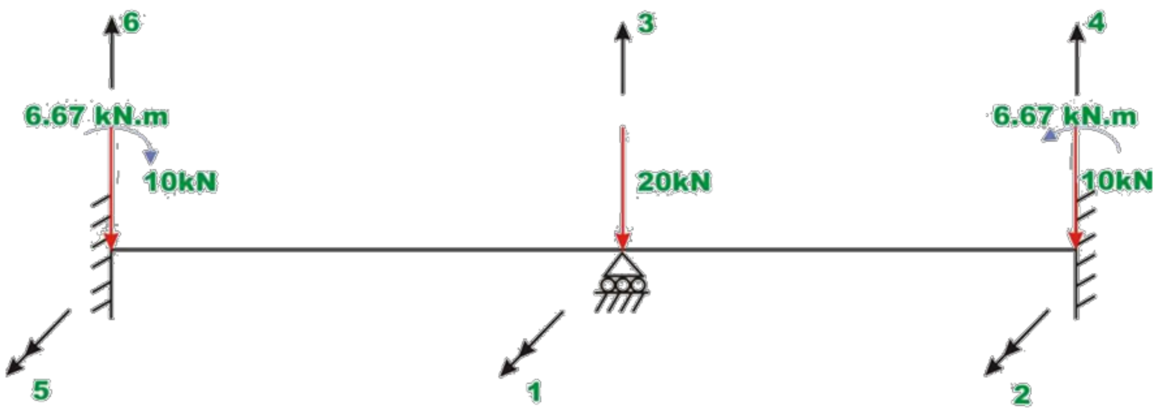



**Fig. 28.2c Equivalent joint loads**

Thus the global load vector corresponding to unconstrained degree of freedom is,

$$\{p_k\} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 6.67 \end{Bmatrix} \qquad (4)$$

Writing the load displacement relation for the entire continuous beam,

$$\begin{Bmatrix} 0 \\ 6.67 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = EI_{zz} \begin{bmatrix} 2 & 0.5 & 0 & -0.375 & 0.5 & 0.375 \\ 0.5 & 1.0 & 0.375 & -0.375 & 0 & 0 \\ 0 & 0.375 & 0.375 & -0.1875 & -0.375 & -0.1875 \\ -0.375 & -0.375 & -0.1875 & 0.1875 & 0 & 0 \\ 0.5 & 0 & -0.375 & 0 & 1.0 & 0.375 \\ 0.375 & 0 & -0.1875 & 0 & 0.375 & 0.1875 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} \quad (5)$$

We know that $u_3=u_4=u_5=u_6=0$ . Thus solving for unknowns $u_1$ and $u_2$ , yields

$$\begin{Bmatrix} 0 \\ 6.67 \end{Bmatrix} = EI_{zz} \begin{bmatrix} 2.0 & 0.5 \\ 0.5 & 1.0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad (6)$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{1.75EI_{zz}} \begin{bmatrix} 1.0 & -0.5 \\ -0.5 & 2.0 \end{bmatrix} \begin{Bmatrix} 0 \\ 6.67 \end{Bmatrix}$$

$$= \frac{1}{EI_{zz}} \begin{Bmatrix} -1.905 \\ 7.62 \end{Bmatrix} \quad (7)$$

Thus displacements are,

$$u_1 = \frac{-1.905}{EI_{zz}} \quad \text{and} \quad u_2 = \frac{7.62}{EI_{zz}}$$

The unknown joint loads are given by,

$$\begin{Bmatrix} p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = EI_{zz} \begin{bmatrix} 0 & 0.375 \\ -0.375 & -0.375 \\ 0.5 & 0 \\ 0.375 & 0 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} -1.905 \\ 7.62 \end{Bmatrix}$$

**Example 30.1**

Analyze the rigid frame shown in Fig 30.4a by direct stiffness matrix method.

Assume $E = 200 GPa; I_{ZZ} = 1.33 \cdot 10^{-4} m^4$ and $A = 0.04 m^2$. The flexural rigidity $EI$ and axial rigidity $EA$ are the same for both thebeams.

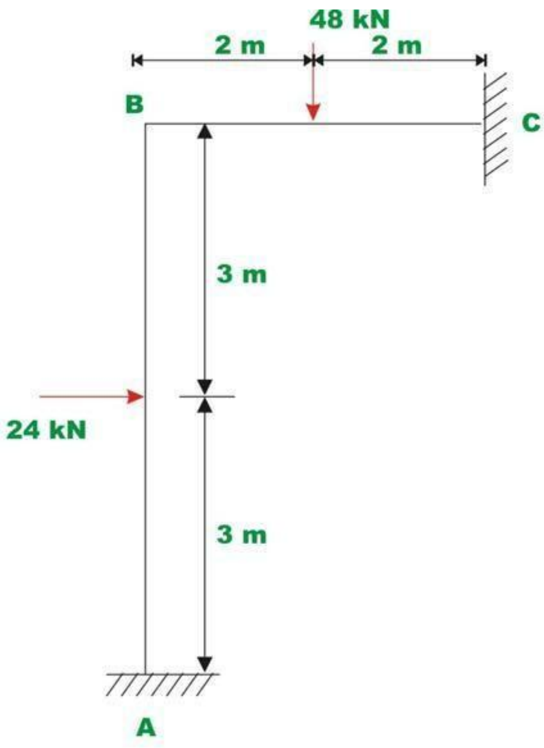



**Fig. 30.4a Rigid Frame.**

**Solution:**

The plane frame is divided in to two beam elements as shown in Fig. 30.4b. The numbering of joints and members are also shown in Fig. 30.3b. Each node has three degrees of freedom. Degrees of freedom at all nodes are also shown in the figure. Also the local degrees of freedom of beam element are shown in the figure asinset.

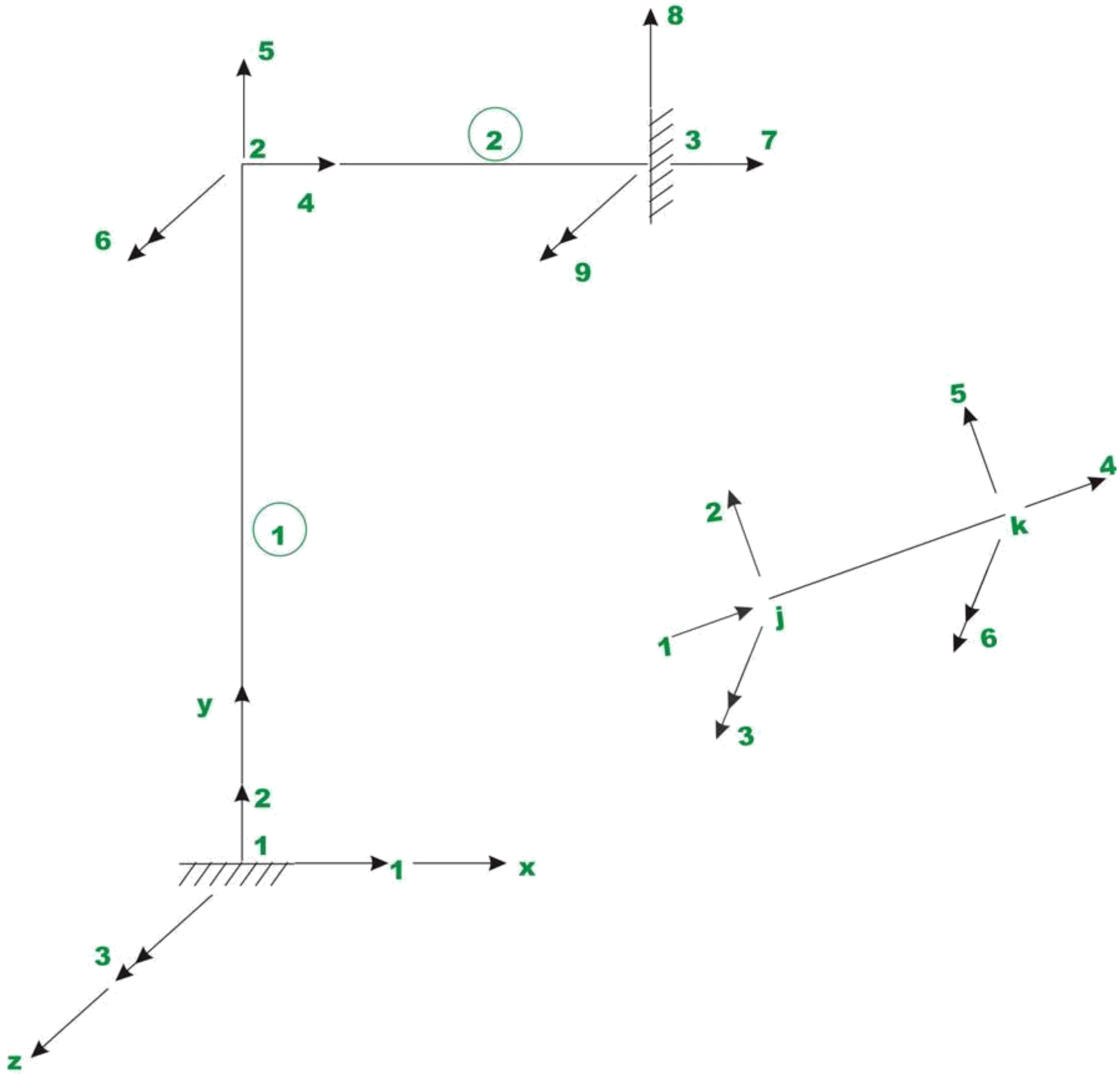



**Fig. 30.4b Node and member numbering.**

Formulate the element stiffness matrix in local co-ordinate system and then transform it to global co-ordinate system. The origin of the global co-ordinate system is taken at node 1. Here the element stiffness matrix in global co-ordinates is only given.

**Member 1:** $L=6m$; $\theta=90°$ node points 1-2; $l=0$ and $m=1$.

$$\left[k^1\right]=[T]^T[k']\ [T]$$

$$\left[k^1\right]=\begin{array}{c} \\ \end{array}\begin{bmatrix} 1.48\times10^3 & 0 & 4.44\times10^3 & 1.48\times10^3 & 0 & 4.44\times10^3 \\ 0 & 1.333\times10^6 & 0 & 0 & -1.333\times10^6 & 0 \\ 4.44\times10^3 & 0 & 17.78\times10^3 & 4.44\times10^3 & 0 & 8.88\times10^3 \\ 1.48\times10^3 & 0 & 4.44\times10^3 & 1.48\times10^3 & 0 & 4.44\times10^3 \\ 0 & -1.333\times10^6 & 0 & 0 & 1.333\times10^6 & 0 \\ 4.44\times10^3 & 0 & 8.88\times10^3 & 4.44\times10^3 & 0 & 17.78\times10^3 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

(columns 1, 2, 3, 4, 5, 6)

(1)

**Member 2:** $L=4\ m$ ; $\theta=0°$ ; node points 2-3 ; $l=1$ and $m=0$.

$$\left[k^2\right]=[T]^T[k'][T]$$

$$=\begin{bmatrix} 2.0\times10^6 & 0 & 0 & -2.0\times10^6 & 0 & 0 \\ 0 & 5\times10^3 & 10\times10^3 & 0 & -5\times10^3 & 10\times10^3 \\ 0 & 10\times10^3 & 26.66\times10^3 & 0 & -10\times10^3 & 8.88\times10^3 \\ -2.0\times10^6 & 0 & 0 & 2.0\times10^6 & 0 & 0 \\ 0 & -5\times10^3 & -10\times10^3 & 0 & 5\times10^3 & -10\times10^3 \\ 0 & 10\times10^3 & 8.88\times10^3 & 0 & -10\times10^3 & 26.66\times10^3 \end{bmatrix}\begin{matrix} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{matrix}$$

(columns 4, 5, 6, 7, 8, 9)

(2)

The assembled global stiffness matrix $[K]$ is of the order $9\times9$. Carrying out assembly in the usual manner, we get,

$$
[K]=\left[\begin{array}{ccc|ccc|ccc}
1.48 & 0 & -4.44 & -1.48 & 0 & -4.44 & 0 & 0 & 0 \\
0 & 1333.3 & 0 & 0 & -1333.3 & 0 & 0 & 0 & 0 \\
-4.44 & 0 & 17.77 & 4.44 & 0 & 8.88 & 0 & 0 & 0 \\
\hline
-1.48 & 0 & 4.44 & 2001.5 & 0 & 4.44 & -2000 & 0 & 0 \\
0 & -1333.3 & 0 & 0 & 1338.3 & 10 & 0 & -5 & 10 \\
-4.44 & 0 & 8.88 & 4.44 & 10 & 44.44 & 0 & -10 & 13.33 \\
\hline
0 & 0 & 0 & -2000 & 0 & 0 & 2000 & 0 & 0 \\
0 & 0 & 0 & 0 & -5 & -10 & 0 & 5 & -10 \\
0 & 0 & 0 & 0 & 10 & 13.33 & 0 & -10 & 26.66
\end{array}\right] \quad (3)
$$

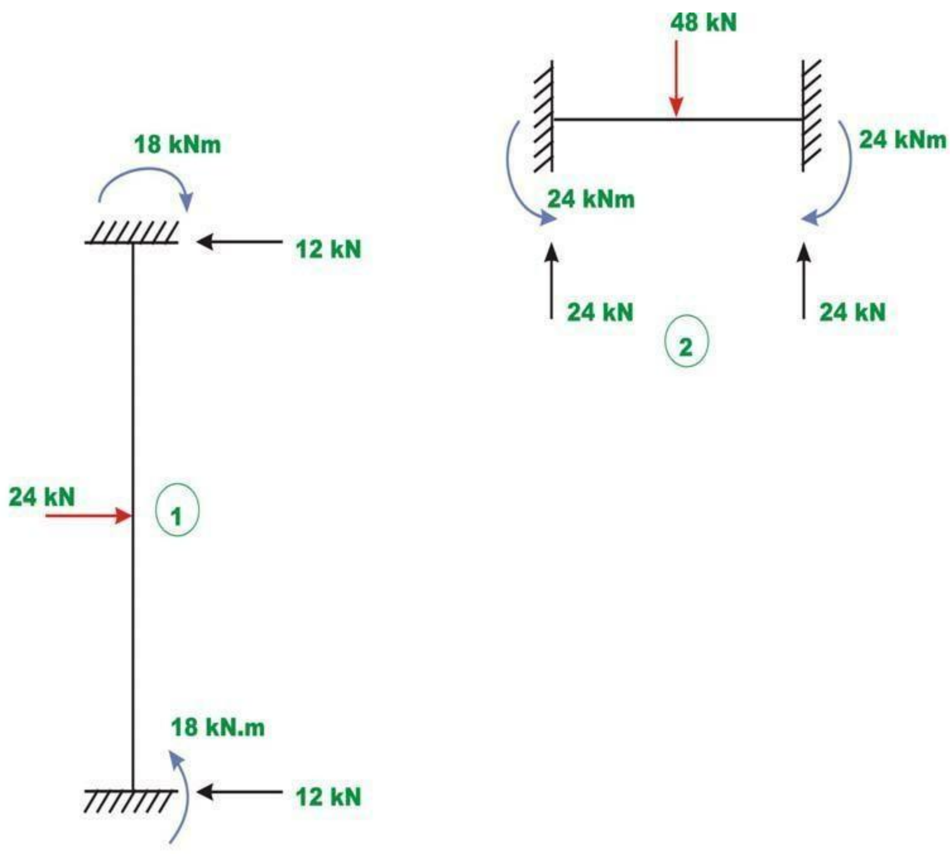



**Fig. 30.4c Fixed end action due to external load in element (1) and (2)**

The load vector corresponding to unconstrained degrees of freedom is (vide

30.4d),

$$\{p_k\} = \begin{Bmatrix} p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \begin{Bmatrix} 12 \\ -24 \\ -6 \end{Bmatrix} \qquad (4)$$

In the given frameconstraint degrees of freedom are $u_1, u_2, u_3, u_7, u_8, u_9$.

Eliminating rows and columns corresponding to constrained degrees of freedom from global stiffness matrix and writing load-displacement relationship for only unconstrained degree of freedom,

$$\begin{Bmatrix} 12 \\ -24 \\ -6 \end{Bmatrix} = 10^3 \begin{bmatrix} 2001.5 & 0 & 4.44 \\ 0 & 1338.3 & 10 \\ 4.44 & 10 & 44.44 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix} \qquad (5)$$

Solving we get,

$$\begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 6.28 \times 10^{-6} \\ -1.695 \times 10^{-5} \\ -0.13 \times 10^{-3} \end{Bmatrix} \qquad (6)$$

$u_4 = 6.28 \times 10^{-6}$ m., $\quad u_5 = -1.695 \times 10^{-5}$

Let $R_1, R_2, R_3, R_7, R_8, R_9$ be the support reactions along degrees of freedom $1, 2, 3, 7, 8, 9$ respectively (vide Fig. 30.4e). Support reactions are calculated by

## Example 30.2

Analyse the rigid frame shown in Fig 30.5a by direct stiffness matrix method.

Assume $E$=200 GPa ;$I_{ZZ}$=1.33·$10^{-5}$$m^4$ and $A$=0.01$m^2$. The flexural rigidity $EI$ and axial rigidity $EA$ are the same for all beams.

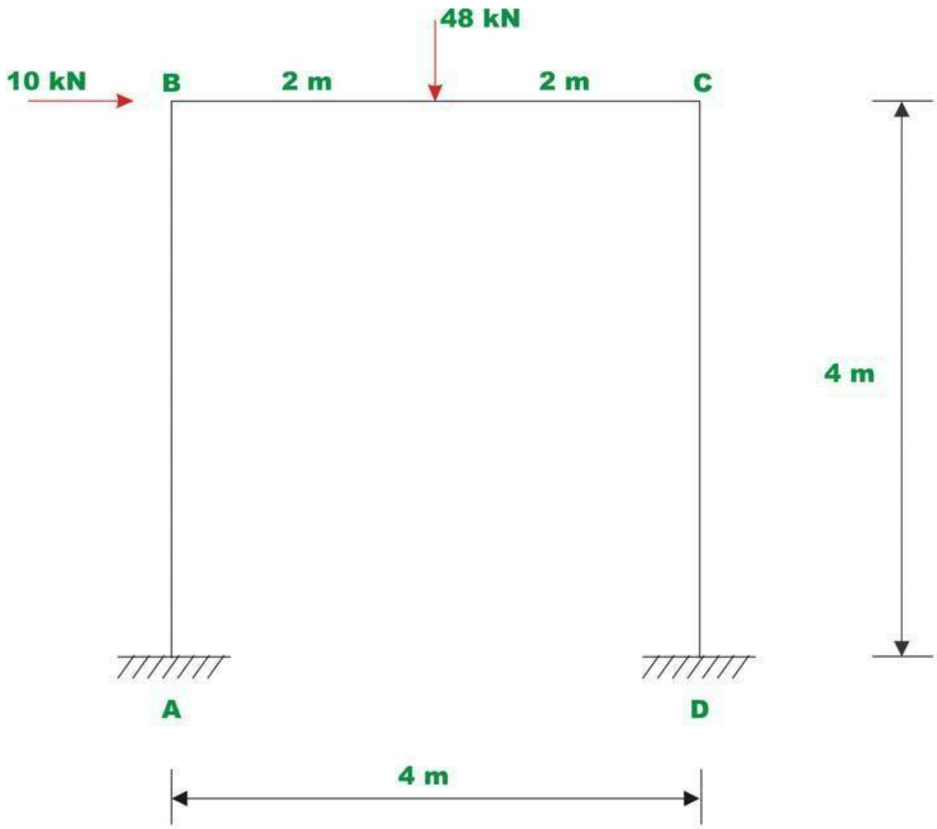



Fig. 30.5a Rigid Frame of Example 30.2

**Solution:**

The plane frame is divided in to three beam elements as shown in Fig. 30.5b. The numbering of joints and members are also shown in Fig. 30.5b. The possible degrees of freedom at nodes are also shown in the figure. The origin of the global co- ordinate system is taken at $A$(node1).

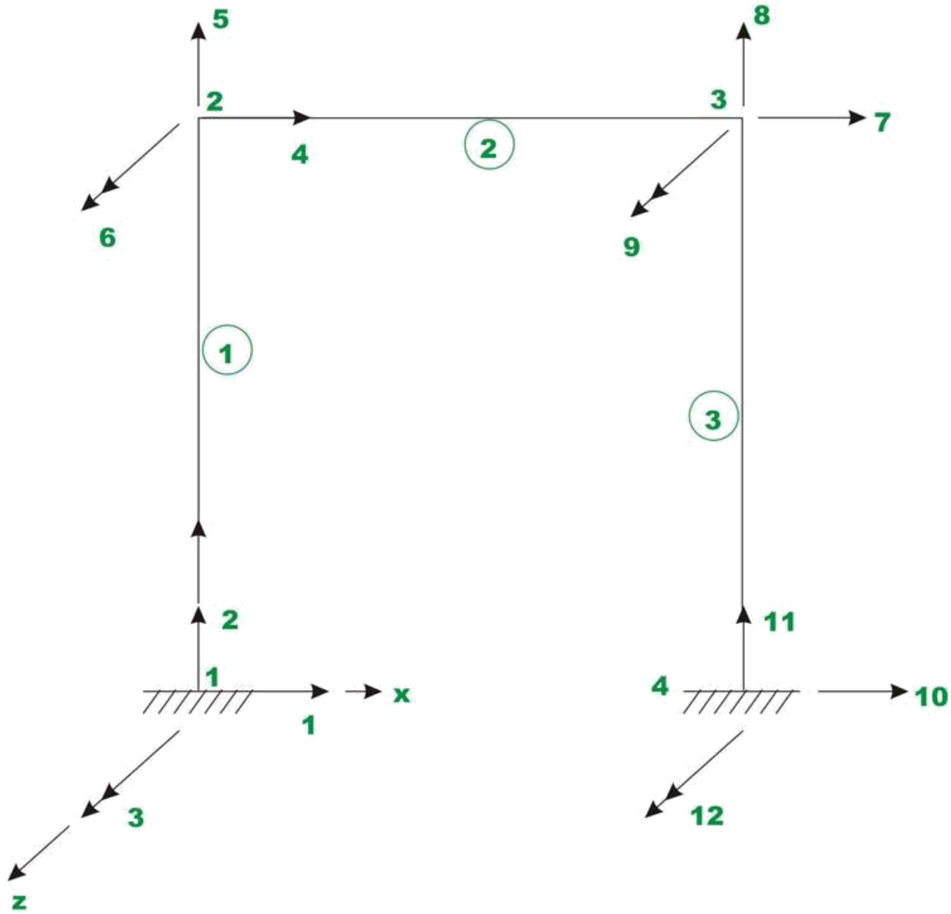



**Fig. 30.5b Node and Member numbering.**

Now formulate the element stiffness matrix in local co-ordinate system and then transform it to global co-ordinate system. In the present case three degrees of freedom are considered at each node.

**Member 1:** $L = 4\ m$ ; $\theta = 90°$ ; node points 1-2 ; $l = \frac{x_2 - x_1}{L} = 0$ and $m = \frac{y_2 - y_1}{L} = 1$.

The following terms are common for all elements.

$$\frac{AE}{L} = 5\times10^5\ \text{kN/m}; \quad \frac{6EI}{L^2} = 9.998\times10^2\text{kN}$$

$$\frac{12EI}{L^3} = 4.999\times10^2\ \text{kN/m}; \quad \frac{4EI}{L} = 2.666\times10^3\text{kN.m}$$

$$[k^1] = [T]^T[k'][T]$$

$$= \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.50\times10^3 & 0 & -1\times10^3 & -0.50\times10^3 & 0 & -1\times10^3 \\ 0 & 5\times10^5 & 0 & 0 & -5\times10^5 & 0 \\ -1\times10^3 & 0 & 2.66\times10^3 & 1\times10^3 & 0 & 1.33\times10^3 \\ -0.50\times10^3 & 0 & 1\times10^3 & 0.50\times10^3 & 0 & 1\times10^3 \\ 0 & -5\times10^5 & 0 & 0 & 5\times10^5 & 0 \\ -1\times10^3 & 0 & 1.33\times10^3 & 1\times10^3 & 0 & 2.66\times10^3 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

(1)

**Member 2:** $L = 4\ m$ ; $\theta = 0°$ node points 2-3; $l = 1$ and $m = 0$.

$$\left[k^2\right] = [T]^T [k'] [T]$$

$$= \begin{array}{c} \\ \end{array}\begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$

$$= \begin{bmatrix} 5.0\times10^5 & 0 & 0 & -5.0\times10^6 & 0 & 0 \\ 0 & 0.5\times10^3 & 1\times10^3 & 0 & -0.5\times10^3 & 1\times10^3 \\ 0 & 1\times10^3 & 2.666\times10^3 & 0 & -1\times10^3 & 1.33\times10^3 \\ -5.0\times10^6 & 0 & 0 & 5.0\times10^6 & 0 & 0 \\ 0 & -0.5\times10^3 & -1\times10^3 & 0 & 0.5\times10^3 & -1\times10^3 \\ 0 & 1\times10^3 & 1.33\times10^3 & 0 & -1\times10^3 & 2.666\times10^3 \end{bmatrix} \begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}$$

(2)

**Member 3:** $L = 4\ m$ ; $\theta = 270°$ ; node points 3-4 ; $l = \frac{x_2 - x_1}{L} = 0$ and $m = \frac{y_2 - y_1}{L} = -1$.

$$\left[k^3\right] = [T]^T [k'] [T]$$

$$\begin{array}{cccccc} 7 & 8 & 9 & 10 & 11 & 12 \end{array}$$

$$= \begin{bmatrix} 0.50\times10^3 & 0 & 1\times10^3 & -0.50\times10^3 & 0 & 1\times10^3 \\ 0 & 5\times10^5 & 0 & 0 & -5\times10^5 & 0 \\ 1\times10^3 & 0 & 2.66\times10^3 & -1\times10^3 & 0 & 1.33\times10^3 \\ -0.50\times10^3 & 0 & -1\times10^3 & 0.50\times10^3 & 0 & -1\times10^3 \\ 0 & -5\times10^5 & 0 & 0 & 5\times10^5 & 0 \\ 1\times10^3 & 0 & 1.33\times10^3 & -1\times10^3 & 0 & 2.66\times10^3 \end{bmatrix} \begin{array}{c} 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array}$$

(3)

The assembled global stiffness matrix $[K]$ is of the order $12 \times 12$. Carrying out assembly in the usual manner, we get,

$$[K] = 10^3 \times \left[\begin{array}{ccc|cccccc|ccc} 0.50 & 0 & -1.0 & -0.50 & 0 & -1.0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 500 & 0 & 0 & -500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.0 & 0 & 2.66 & 1.0 & 0 & 1.33 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -0.50 & 0 & 1.0 & 500.5 & 0 & 1.0 & -500 & 0 & 0 & 0 & 0 & 0 \\ 0 & -500 & 0 & 0 & 500.5 & 1.0 & 0 & -0.50 & 1.0 & 0 & 0 & 0 \\ -1.0 & 0 & 1.33 & 1.0 & 1.0 & 5.33 & 0 & -1.0 & 1.33 & 0 & 0 & 0 \\ 0 & 0 & 0 & -500 & 0 & 0 & 500.5 & 0 & 1.0 & -0.5 & 0 & 1.0 \\ 0 & 0 & 0 & 0 & -0.5 & -1.0 & 0 & 500.5 & -1.0 & 0 & -500 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 1.33 & 1.0 & -1.0 & 5.33 & -1.0 & 0 & 1.33 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0 & -1.0 & 0.5 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -500 & 0 & 0 & 500 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.0 & 0 & 1.33 & -1 & 0 & 2.66 \end{array}\right]$$

(4)

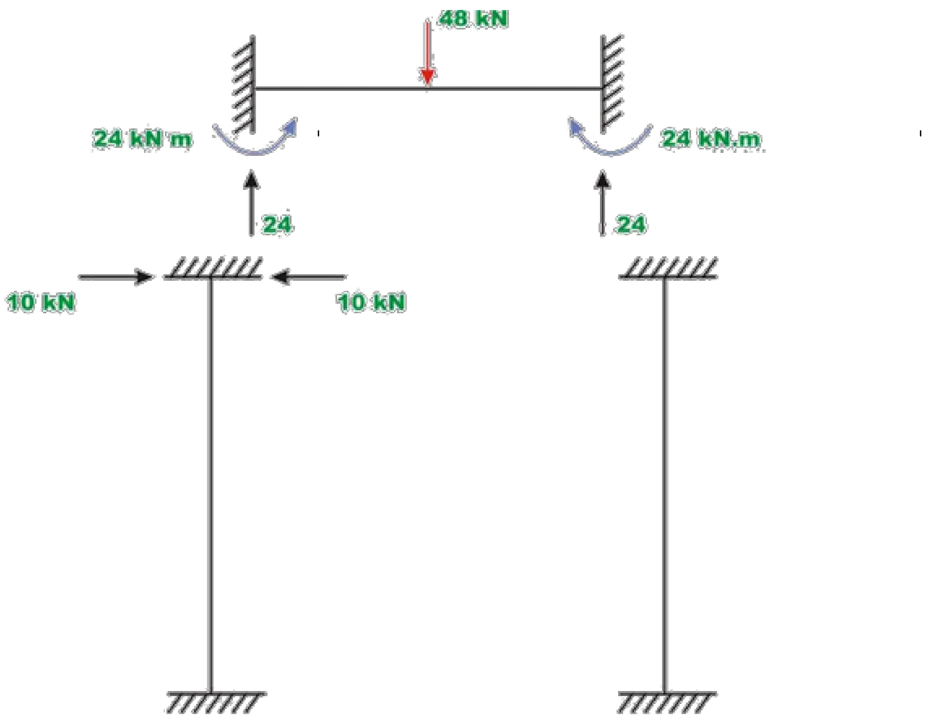



**Fig . 30.5c Fixed end action due to external load.**

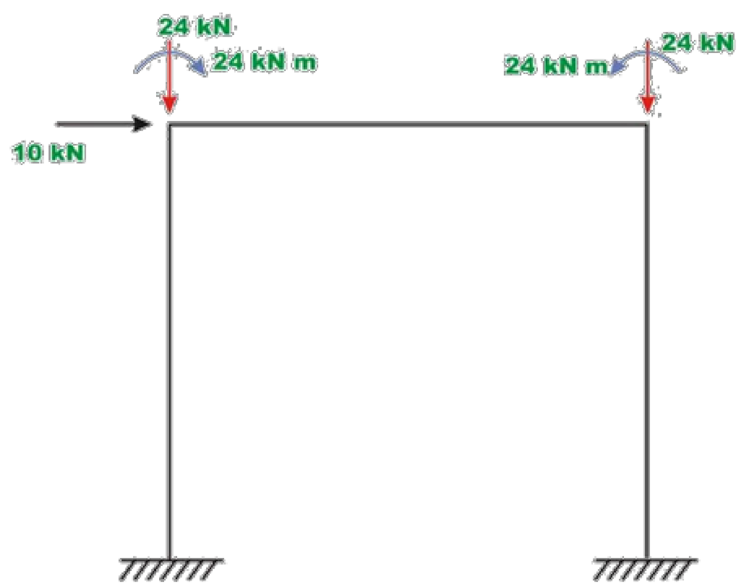



**Fig. 30.5d Equivalent joint loads.**

The load vector corresponding to unconstrained degrees of freedom is,

$$\{p_k\} = \begin{Bmatrix} p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \end{Bmatrix} = \begin{Bmatrix} 10 \\ -24 \\ -24 \\ 0 \\ -24 \\ 24 \end{Bmatrix} \tag{5}$$

In the given frame, constraint (known) degrees of freedom are $u_1, u_2, u_3, u_{10}, u_{11}, u_{12}$. Eliminating rows and columns corresponding to constrained degrees of freedom from global stiffness matrix and writing load displacement relationship,

$$\begin{Bmatrix} 10 \\ -24 \\ -24 \\ 0 \\ -24 \\ 24 \end{Bmatrix} = 10^3 \begin{bmatrix} 500.5 & 0 & 1.0 & -500 & 0 & 0 \\ 0 & 500.5 & 1.0 & 0 & -0.5 & 1.0 \\ 1.0 & 1.0 & 5.33 & 0 & -1.0 & 1.33 \\ -500 & 0 & 0 & 500.5 & 0 & 1 \\ 0 & -0.5 & -1 & 0 & 500.5 & -1 \\ 0 & 1 & 1.33 & 1 & -1 & 5.33 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} \tag{6}$$

Solving we get,

$$\begin{Bmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} = \begin{Bmatrix} 1.43 \times 10^{-2} \\ -3.84 \times 10^{-5} \\ -8.14 \times 10^{-3} \\ 1.43 \times 10^{-2} \\ -5.65 \times 10^{-5} \\ 3.85 \times 10^{-3} \end{Bmatrix} \tag{7}$$

Let $R_1, R_2, R_3, R_{10}, R_{11}, R_{12}$ be the support reactions along degrees of freedom 1, 2, 3,10,11,12 respectively. Support reactions are calculated by

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_{10} \\ R_{11} \\ R_{12} \end{Bmatrix} = \begin{Bmatrix} p_1^F \\ p_2^F \\ p_3^F \\ p_{10}^F \\ p_{11}^F \\ p_{12}^F \end{Bmatrix} + 10^3 \begin{matrix} 4 & 5 & 6 & 7 & 8 & 9 \end{matrix} \begin{bmatrix} -0.5 & 0 & -1.0 & 0 & 0 & 0 \\ 0 & -500 & 0 & 0 & 0 & 0 \\ 1.0 & 0 & 1.33 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.5 & 0 & -1.0 \\ 0 & 0 & 0 & 0 & -500 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 1.33 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_{10} \\ R_{11} \\ R_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0.99 \\ 19.71 \\ 3.43 \\ -10.99 \\ 28.28 \\ 19.42 \end{Bmatrix} = \begin{Bmatrix} 0.99 \\ 19.71 \\ 3.43 \\ -10.99 \\ 28.28 \\ 19.42 \end{Bmatrix} \quad (8)$$

# Unit 5: INFLUENCE LINES FOR INDETERMINATE BEAMS

**DEFINITION INFLUENCE LINES**

***Influence lines*** are important in thedesign of structures that resist large live loads.shear and moment diagrams are important in determining the maximum internal force in a structure.If a structure is subjected to a ***live*** or ***moving load***, the variation in shear andmoment is best described using ***influencelines***.

An ***influence line*** represents the variation of the reaction, shear, moment, or deflection at a ***specific point*** in a member as a concentrated force moves over the member.

Once the ***influence line*** is drawn, the location of the live load which will cause the greatest influence on the structure can be found very quickly.Therefore, ***influence lines*** are important in the design of a structure where the loads move along the span (bridges, cranes, conveyors, etc.).

Although the procedure for constructing an ***influence line*** is rather simple, it is importantto remember the difference between constructing an influence line and constructing a shear or moment diagram.***Influence lines*** represent the effect of amoving load **only at a specified point** on a member, whereas shear and moment diagrams represent the effect of fixed loads at **allpoints**along the member.

**Procedure** for determining the***influence line*** at a point**P**foranyfunction (reaction, shear, or moment).

1. Place a unit load (a load whose magnitude is equal to one) at a point, *x*, along the member.
2. Use the equations of equilibrium to find the value of the function (reaction, shear, or moment) at a specific point **P** due the concentrated load at*x*.
3. Repeat steps 1 and 2 for various values of *x* over the wholebeam.
4. Plot the values of the reaction, shear, or moment for themember.

## Construction of Influence Lines using Equilibrium Methods

The most basic method of obtaining influence line for a specific response parameter is to solve the static equilibrium equations for various locations of the unit load. The general procedure for constructing an influence line is described below.

1. Define the positive direction of the response parameter under consideration through a free body diagram of the wholesystem.

2..For a particular location of the unit load, solve for the equilibrium of the whole system and if required, as in the case of an internal force, also for a part of the member to obtain the response parameter for that location of the unit load.This gives the ordinate of the influence line at that particular location of theload.

3. Repeat this process for as many locations of the unit load as required to determine the shape of the influence line for the whole length of the member. It is often helpful if we can consider a generic location (or several locations) *x* of the unitload.

4. Joining ordinates for different locations of the unit load throughout the length of the member,we get the influence line for that particular responseparameter.

The following three examples show how to construct influence lines for a support reaction, a shear force and a bending moment for the simply supported beam *AB* .

**Example .1** Draw the influence line for $R_A$ (vertical reaction at *A* ) of beam *AB* in Fig. E6.1.

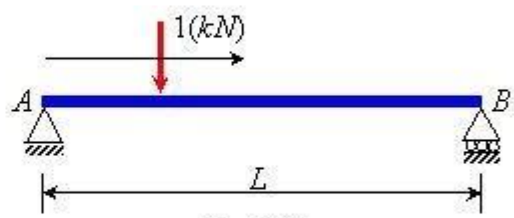
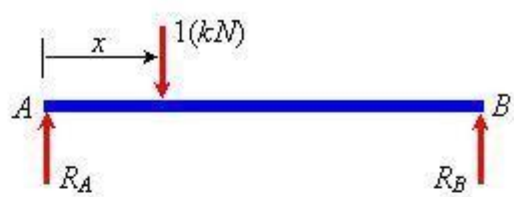



Fig. E6.1

**Solution:**

Free body diagram of *AB* :

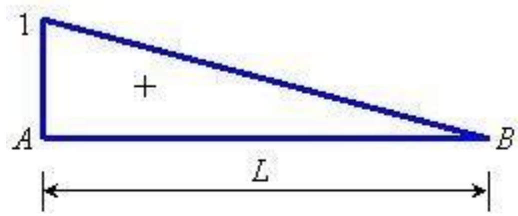



$$\sum F_y = 0 \Rightarrow R_A = 1 - R_B$$

$$\sum M(about B) = 0 \Rightarrow R_A(L) = 1(L - x)$$

$$\Rightarrow R_A = 1 - \frac{x}{L}$$

So the influencelineof $R_A$:




**Example 6.2** Draw the influence line for $V_C$(shear force at mid point) ofbeam *AB* in Fig. E6.2.

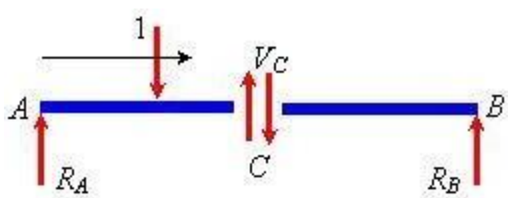
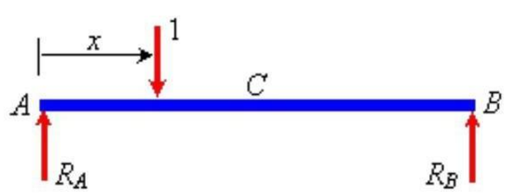


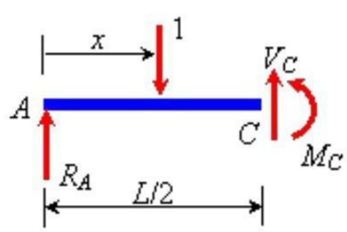

Fig. E6.2

**Solution:**




$$\sum M(about\, B) = 0 \Rightarrow R_A = 1 - \frac{x}{L}$$

$x < \frac{L}{2}$

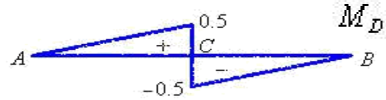


$$\sum F_y = 0 \Rightarrow V_C = 1 - R_A = \frac{x}{L}$$

For $x > \frac{L}{2}$


$$\sum F_y = 0 \Rightarrow V_C = -R_A = \frac{x}{L} - 1$$

So the influence line for $V_C$




**Example 6.3** Draw the influence line for $M_D$ (bending moment at $x = 2L/3$) for beam *AB* in Fig.E6.3.

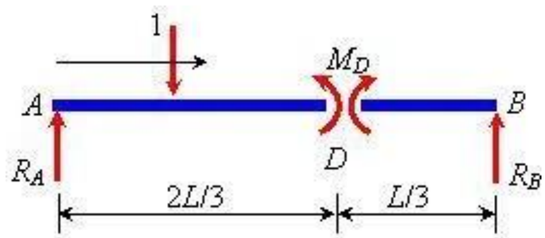
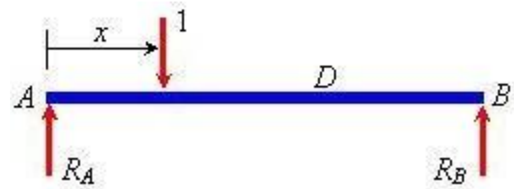
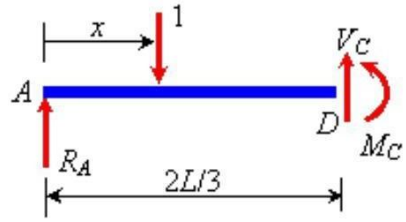



Fig. E6.3

**Solution:**



$$\sum M(about\ B) = 0 \Rightarrow R_A = 1 - \frac{x}{L}$$

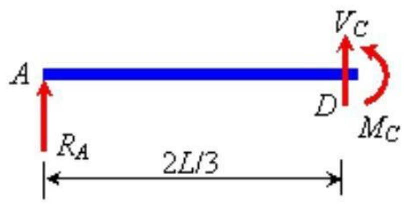



For $x < \frac{2L}{3}$

$$\sum M(about\ D) = 0$$

$$\Rightarrow M_D = R_A\left(\frac{2L}{3}\right) - 1\left(\frac{2L}{3} - x\right) = \left(1 - \frac{x}{L}\right)\left(\frac{2L}{3}\right) - \frac{2L}{3} + x$$

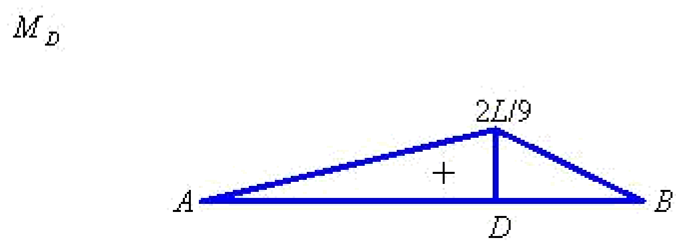
$$= -\frac{2x}{3} + x = \frac{x}{3}$$



For $x > \frac{2L}{3}$

$$\sum M(about\,D) = 0 \Rightarrow M_D = R_A\left(\frac{2L}{3}\right) = \frac{2L}{3} - \frac{2x}{3}$$

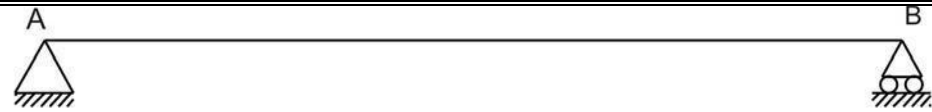
So, the influenceof $M_D$ :




### MÜLLER BRESLAU PRINCIPLE FOR QUALITATIVE INFLUENCE LINES

In 1886, Heinrich Müller Breslau proposed a technique to draw influence lines quickly. The Müller Breslau Principle states that the ordinate value of an influence line for any function on any structure is proportional to the ordinates of the deflected shape that is obtained by removing the restraint corresponding to the function from the structure and introducing a force that causes a unit displacement in the positive direction.

Let us say, our objective is to obtain the influence line for the support reaction at A for the beam shown in Figure 38.1.




**Simply supported beam**

First of all remove the support corresponding to the reaction and apply a force (Figure 38.2) in the positive direction that will cause a unit displacement in the direction of $R_A$. The resulting deflected shape will be proportional to the true influence line (Figure 38.3) for the support reaction at A.

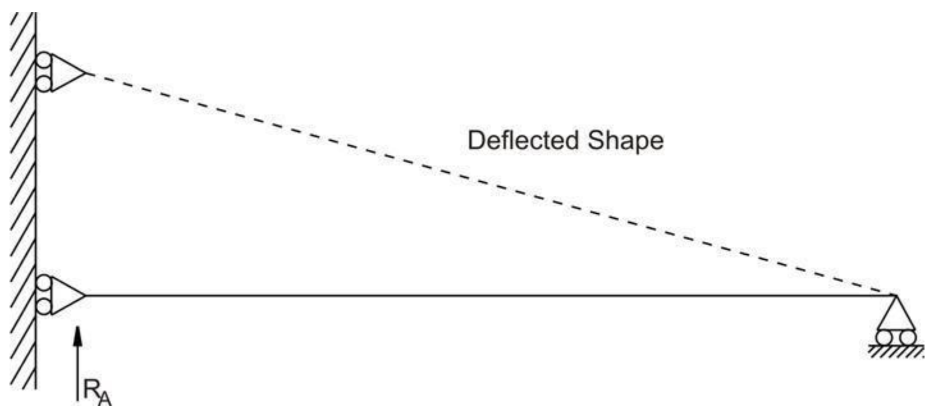



**Figure 38.2: Deflected shape of beam**

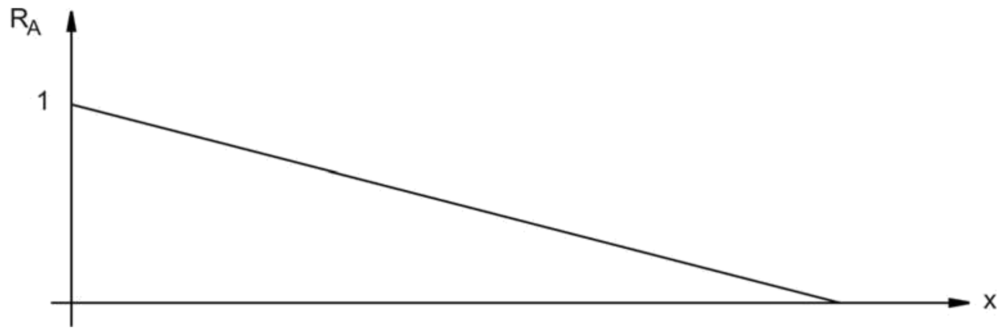



**Figure 38.3: Influence line for support reaction A**

The deflected shape due to a unit displacement at A is shown in Figure 38.2 and matches with the actual influence line shape as shown in Figure 38.3. Note that the deflected shape is linear, i.e., the beam rotates as a rigid body without any curvature. This is true only for statically determinate systems.

Similarly some other examples are given below.

Here we are interested to draw the qualitative influence line for shear at section

C of overhang beam as shown in Figure 38.4.


**Figure 38.4: Overhang beam**

As discussed earlier, introduce a roller at section C so that it gives freedom to the beam in vertical direction as shown in Figure 38.5.

Now apply a force in the positive direction that will cause a unit displacement in the direction of $V_C$. The resultant deflected shape is shown in Figure 38.5. Again, note that the deflected shape is linear. Figure 38.6 shows the actual influence, which matches with the qualitative influence.

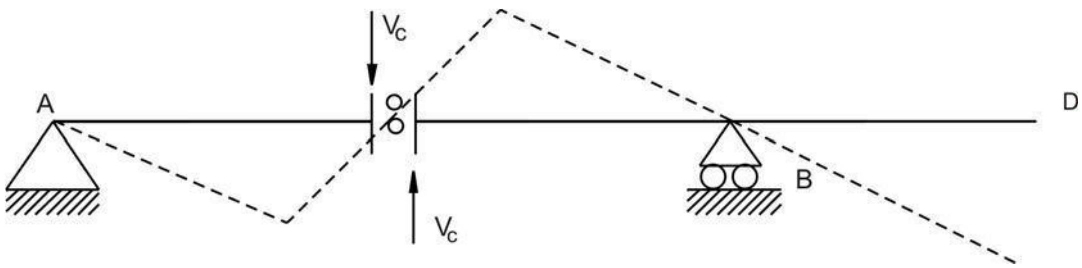



**Figure 38.5: Deflected shape of beam**

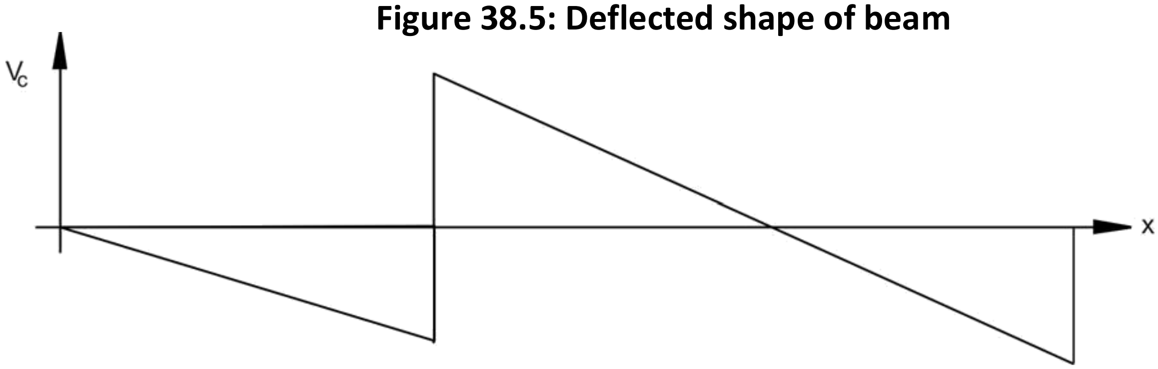



**Figure 38.6: Influence line for shear at section C**

In this second example, we are interested to draw a qualitative influence line for moment at C for the beam as shown in Figure 38.7.

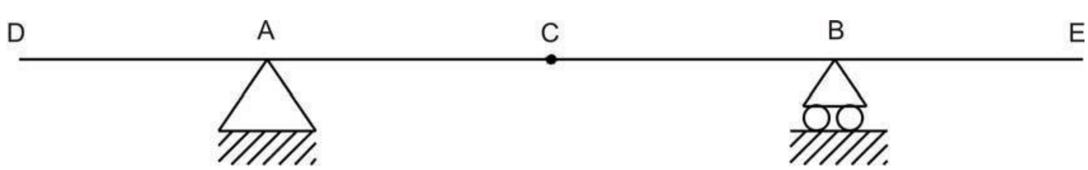


**Figure 38.7: Beam structure**

In this example, being our objective to construct influence line for moment, we will introduce hinge at C and that will only permit rotation at C. Now apply moment in the positive direction that will cause a unit rotation in the direction of

$M_c$. The deflected shape due to a unit rotation at C is shown in Figure 38.8 and matches with the actual shape of the influence line as shown in Figure 38.9.

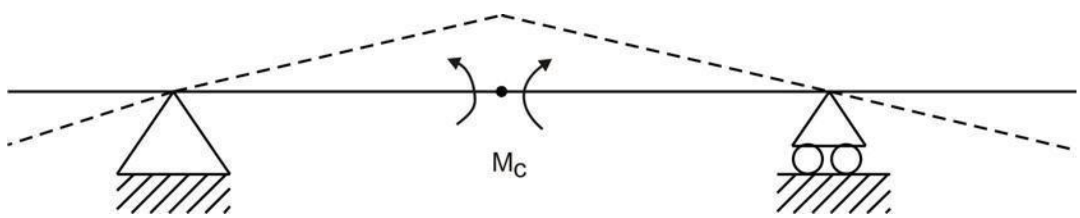



**Figure 38.8: Deflected shape of beam**

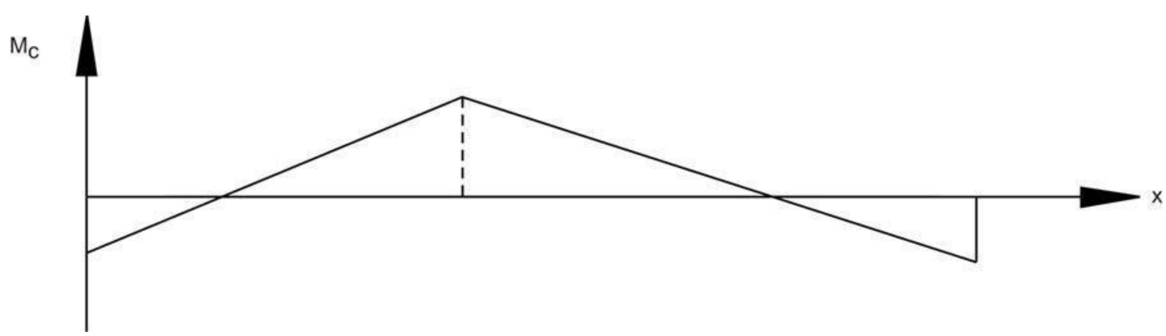



**Figure 38.9: Influence line for moment at section C**

Maximum shear in beam supporting UDLs

If UDL is rolling on the beam from one end to other end then there are two possibilities. Either Uniformly distributed load is longer than the span or uniformly distributed load is shorter than the span. Depending upon the length of the load and span, the maximum shear in beam supporting UDL will change. Following section will discuss about these two cases. It should be noted that for maximum values of shear, maximum areas should be loaded.

### UDL LONGER THAN THE SPAN

Let us assume that the simply supported beam as shown in Figure 38.10 is loaded with UDL of w moving from left to right where the length of the load is longer than the span. The influence lines for reactions $R_A$, $R_B$ and shear at section C located at x from support A will be as shown in Figure 38.11, 38.12 and 38.13 respectively. UDL of intensity w per unit for the shear at supports A and B will be given by

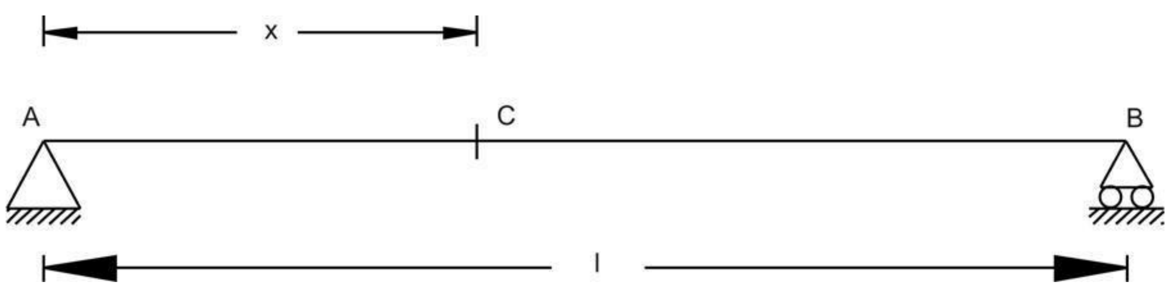



**Figure 38.10: Beam Structure**

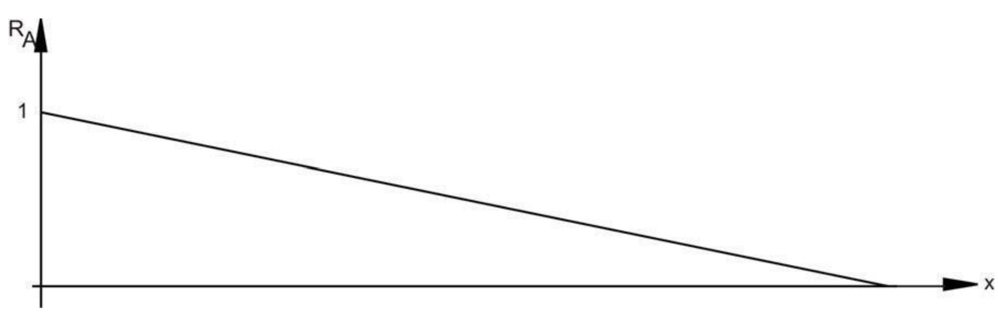



**Figure 38.11: Influence line for support reaction at A**

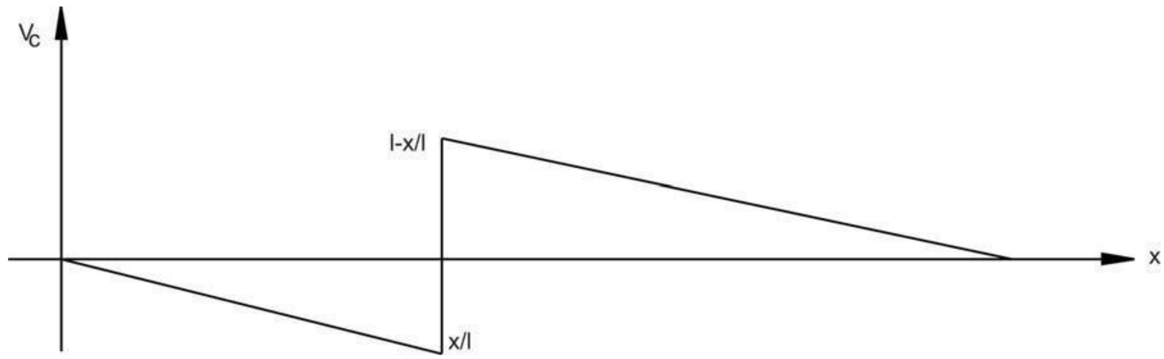



**Figure 38.13: Influence line for shear at section C**

$$R_A = w \times \frac{1}{2} \times l \times 1 = \frac{wl}{2}$$

$$R_B = -w \times \frac{1}{2} \times l \times 1 = \frac{-wl}{2}$$

Suppose we are interested to know shear at given section at C. As shown in Figure 38.13, maximum negative shear can be achieved when the head of the load is at the section C. And maximum positive shear can be obtained when the tail of the load is at the section C. As discussed earlier the shear force is computed by intensity of the load multiplied by the area of influence line diagram covered by load. Hence, maximum negative shear is given by $= -\frac{1}{2} \times x \times \frac{x}{l} \times w = -\frac{wx^2}{2l}$

and maximum positive shear is given by

$$= \frac{1}{2} \times \left(\frac{l-x}{l}\right) \times (l-x) \times w = -\frac{w(l-x)^2}{2l}$$

### UDL SHORTER THAN THE SPAN

When the length of UDL is shorter than the span, then as discussed earlier, maximum negative shear can be achieved when the head of the load is at the section. And maximum positive shear can be obtained when the tail of the load is at the section. As discussed earlier the shear force is computed by the load intensity multiplied by the area of influence line diagram covered by load. The example is demonstrated in previous lesson.

## Maximum bending moment at sections in beams supporting UDLs.

Like the previous section discussion, the maximum moment at sections in beam supporting UDLs can either be due to UDL longer than the span or due to ULD shorter than the span. Following paragraph will explain about computation of moment in these twocases.

### UDL longer than the span

Let us assume the UDL longer than the span is traveling from left end to right hand for the beam as shown in Figure 38.14. We are interested to know maximum moment at C located at x from the support A. As discussed earlier, the maximum bending moment is given by the load intensity multiplied by the area of influence line (Figure 38.15) covered. In the present case the load will cover the completed span and hence the moment at section C can be givenby

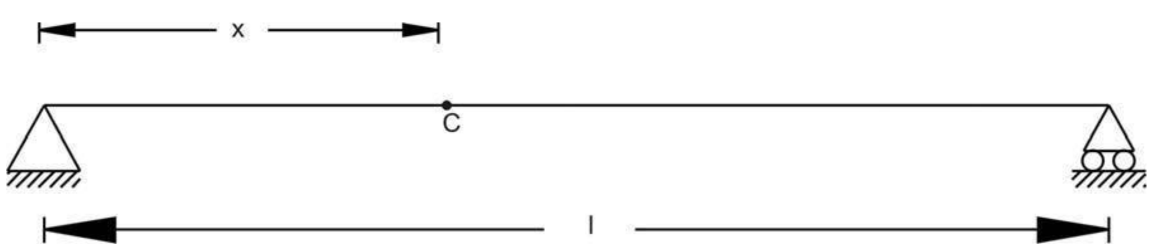


**Figure 38.14: Beam structure**

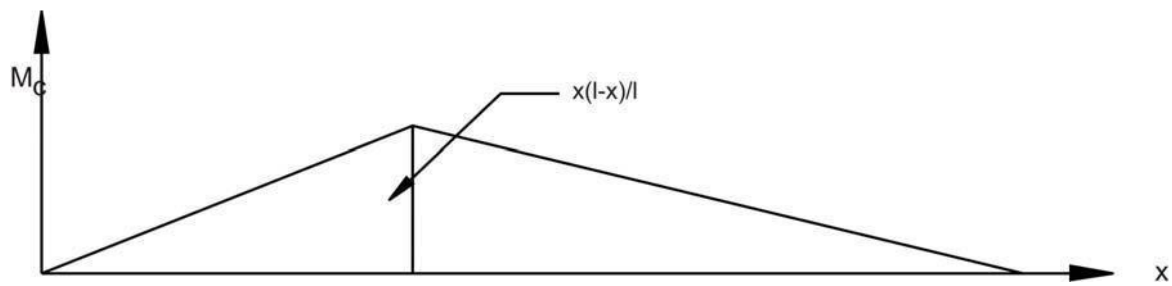



**Figure 38.15: Influence line for moment at section C**

$$w \times \frac{1}{2} \times l \times \frac{x(l-x)}{l} = -\frac{wx(l-x)}{2}$$

Suppose the section C is at mid span, then maximum moment is given by

$$\frac{w \times \frac{l}{2} \times \frac{l}{2}}{2} = \frac{wl^2}{8}$$

## UDL shorter than the span

As shown in Figure 38.16, let us assume that the UDL length y is smaller than the span of the beam AB. We are interested to find maximum bending moment at section C located at x from support A. Let say that the mid point of UDL is located at D as shown in Figure 38.16 at distance of zfrom support A. Take moment with reference to A and it will bezero.

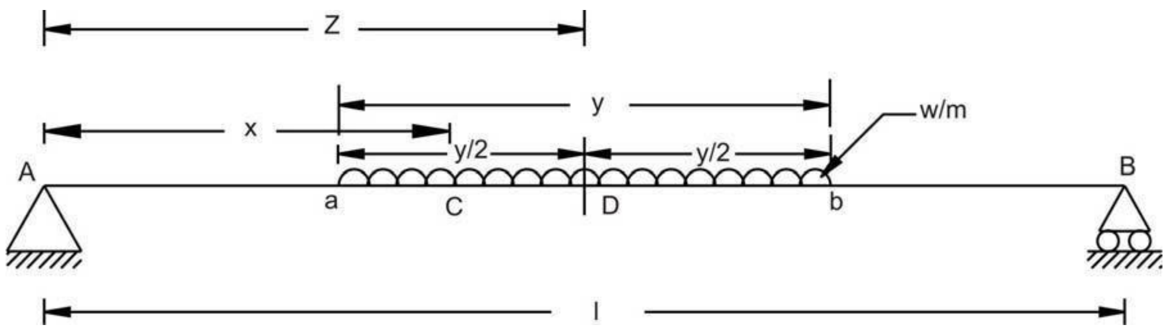



**Figure 38.16: Beam loaded with UDL shorter in length than span**

Hence, the reaction at B is given by

$$R_B = w \times y \times \frac{z}{l} = -\frac{wx(l-x)}{2}$$

And moment at C will be

$$M_C = R_B(l-x) - \frac{w}{2}(z + \frac{y}{2} - x)^2$$

Substituting value of reaction B in above equation, we can obtain

$$M_C = \frac{wyz}{l}(l-x) - \frac{w}{2}(z + \frac{y}{2} - x)^2$$

To compute maximum value of moment at C, we need to differentiate above given equation with reference to z and equal to zero.

$$\frac{dM_c}{dz} = \frac{wy}{l}(l-x) - w(z + \frac{y}{2} - x) = 0$$

Therefore,

$$\frac{y}{l}(l-x)=(z+\frac{y}{2}-x)$$

Using geometric expression, we can state that

$$\frac{ab}{AB}=\frac{Cb}{CB}$$

$$\therefore \frac{CB}{Cb}=\frac{AB}{ab}=\frac{AB-CB}{ab-Cb}=\frac{AC}{aC}$$

$$\therefore \frac{aC}{Cb}=\frac{AC}{CB}$$

The expression states that for the UDL shorter than span, the load should be placed in a way so that the section divides it in the same proportion as it divides the span. In that case, the moment calculated at the section will give maximum momentvalue.

## INDETERMINATE TRUSSES

A difficulty arises in determining the number of releases required to return the completely stiff equivalent of a truss to its original state.

Consider the completely stiff equivalent of a plane truss shown in Fig. 16.9(a); we are not concerned here with the indeterminacy or otherwise of the support system which is therefore omitted. In the actual truss each member is assumed to be capable of resisting axial load only so that there are two releases for each member, one of shear and one of moment, a total of 2*M* releases. Thus, if we insert a hinge at the end of each member as shown in Fig. 16.9(b) we have achieved the required number, 2*M*, of releases. However, in this configuration, each joint would be free to rotate as a mechanism through an infinitesimally small angle, independently of the members; the truss is then excessively pin-jointed. This situation can be prevented by removing one hinge at each joint as shown, for example at joint B in Fig. 16.9(c). The member BC then prevents rotation of the joint at B. Furthermore, the presence of a hinge at B in BA and at B in BE ensures that there is no moment at B in BC so that the conditions for a truss are satisfied.

From the above we see that the total number, 2*M*, of releases is reduced by 1 for each node. Thus the required number of releases in a plane truss is

$$r = 2M - N \qquad (16.4)$$

so that Eq. (16.3) becomes

$$n_s = 3(M - N + 1) - (2M - N)$$

or

$$n_s = M - 2N + 3 \qquad (16.5)$$

Equation (16.5) refers only to the internal indeterminacy of a truss so that the degree of indeterminacy of the support system is additional. Also, returning to the simple triangular truss of Fig. 16.7(a) we see that its degree of internal indeterminacy is, from Eq. (16.5), given by

$$n_s = 3 - 2 \times 3 + 3 = 0$$

A similar situation arises in a space truss where, again, each member is required to resist axial load only so that there are 5*M* releases for the complete truss. This could be achieved by inserting ball joints at the ends of each member. However, we would then be in the same kind of position as the plane truss of

Fig. 16.9(b) in that each joint would be free to rotate through infinitesimally small angles about each of the three axes (the members in the plane truss can only rotate about one axis) so that three constraints are required at each node, a total of $3N$ constraints. Therefore the number of releases is given by

$$r = 5M - 3N$$

so that Eq. (16.2) becomes

$$n_s = 6(M - N + 1) - (5M - 3N)$$

or

$$n_s = M - 3N + 6 \qquad (16.6)$$

For statically determinate plane trusses and space trusses, i.e. $n_s = 0$, Eqs (16.5) and (16.6), respectively, becomes

$$M = 2N - 3 \quad M = 3N - 6 \qquad (16.7)$$

which are the results deduced in Section 4.4 (Eqs (4.1) and (4.2)).

## KINEMATIC INDETERMINACY

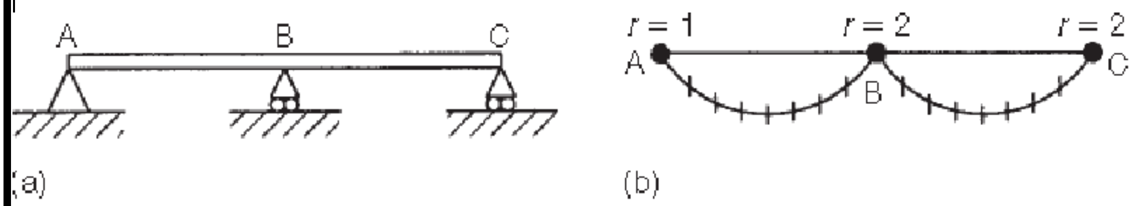
We have seen that the degree of statical indeterminacy of a structure is, in fact, the number of forces or stress resultants which cannot be determined usingthe equations of staticalequilibrium

**Analysis of Statically Indeterminate Structures**

Another form of the indeterminacy of a structure is expressed in terms of its *degrees of freedom*; this is known as the *kinematic indeterminacy,* $n_k$, of a structure and is of particular relevance in the stiffness method of analysis where the unknowns are the displacements.

A simple approach to calculating the kinematic indeterminacy of a structure is to sum the degrees of freedom of the nodes and then subtract those degrees of freedom that are prevented by constraints such as support points. It is therefore important to remember that in three-dimensional structures each node possesses 6 degrees of freedom while in plane structures each node possess three degrees of freedom.

**EXAMPLE 16.1** Determine the degrees of statical and kinematic indeterminacy ofthe beam ABC shown in Fig. 16.10(a).




## Statically Indeterminate Trusses

A truss may be internally and/or externally statically indeterminate. For a truss that is externally statically indeterminate, the support reactions may be found by the methods described in Section 16.4. For a truss that is internally statically indeterminate the flexibility method may be employed as illustrated in the following examples.

**Example 16.8** Determine the forces in the members of the truss shown in Fig. 16.18(a); the cross-sectional area, $A$, and Young's modulus, $E$, are the same for all members.

The truss in Fig. 16.18(a) is clearly externally statically determinate but, from Eq. (16.5), has a degree of internal statical indeterminacy equal to 1 ($M = 6$, $N = 4$). We therefore release the truss so that it becomes statically determinate by 'cutting' one of the members, say BD, as shown in Fig. 16.18(b). Due to the actual loads ($P$ in this case) the cut ends of the member BD will separate or come together, depending on whether the force in the member (before it was cut) was tensile or compressive; we shall assume that it was tensile.

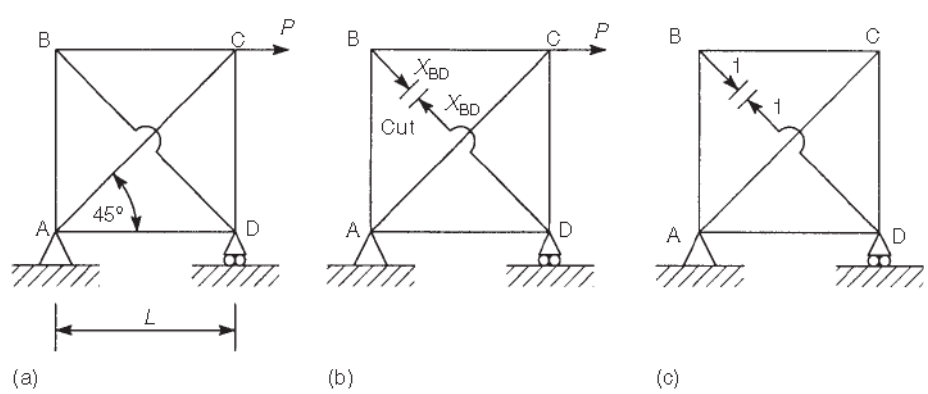



FIGURE 16.18 Analysis of a statically indeterminate truss

We are assuming that the truss is linearly elastic so that the relative displacement of the cut ends of the member BD (in effect the movement of B and D away from or towards each other along the diagonal BD) may be found using, say, the unit load method as illustrated in Exs 15.6 and 15.7. Thus we determine the forces $F_{a,j}$, in the members produced by the actual loads. We then apply equal and opposite unit loads to the cut ends of the member BD as shown in Fig. 16.18(c) and calculate the forces, $F_{1,j}$ in the members. The displacement of B relative to D, $\Delta_{BD}$, is then given by

$$\Delta_{BD} = \sum_{j=1}^{n} \frac{F_{a,j}F_{1,j}L_j}{AE} \quad \text{(see Eq. (viii) in Ex. 15.7)}$$

The forces, $F_{a,j}$, are the forces in the members of the released truss due to the actual loads and are not, therefore, the actual forces in the members of the complete truss. We shall therefore redesignate the forces in the members of the released truss as $F_{0,j}$. The expression for $\Delta_{BD}$ then becomes

$$\Delta_{BD} = \sum_{j=1}^{n} \frac{F_{0,j}F_{1,j}L_j}{AE} \qquad \text{(i)}$$

In the actual structure this displacement is prevented by the force, $X_{BD}$, in the redundant member BD. If, therefore, we calculate the displacement, $a_{BD}$, in the direction of BD produced by a unit value of $X_{BD}$, the displacement due to $X_{BD}$ will be $X_{BD}a_{BD}$. Clearly, from compatibility

$$\Delta_{BD} + X_{BD}a_{BD} = 0 \quad \text{(ii)}$$

from which $X_{BD}$ is found. Again, as in the case of statically indeterminate beams, $a_{BD}$ is a flexibility coefficient. Having determined $X_{BD}$, the actual forces in the members of the complete truss may be calculated by, say, the method of joints or the method of sections.

In Eq. (ii), $a_{BD}$ is the displacement of the released truss in the direction of BD produced by a unit load. Thus, in using the unit load method to calculate this displacement, the

TABLE 16.1

| *Member* | $L_j$ (m) | $F_{0,j}$ | $F_{1,j}$ | $F_{0,j}F_{1,j}L_j$ | $F_{1,j}^2L_j$ | $F_{a,j}$ |
|---|---|---|---|---|---|---|
| AB | $L$ | 0 | −0.71 | 0 | $0.5L$ | $+0.40P$ |
| BC | $L$ | 0 | −0.71 | 0 | $0.5L$ | $+0.40P$ |
| CD | $L$ | $-P$ | −0.71 | $0.71PL$ | $0.5L$ | $-0.60P$ |
| BD | $1.41L$ | – | 1.0 | – | $1.41L$ | $-0.56P$ |
| AC | $1.41L$ | $1.41P$ | 1.0 | $2.0PL$ | $1.41L$ | $+0.85P$ |
| AD | $L$ | 0 | −0.71 | 0 | $0.5L$ | $+0.40P$ |
| | | | | $\sum = 2.71PL$ | $\sum = 4.82L$ | |

actual member forces ($F_{1,j}$) and the member forces produced by the unit load ($F_{l,j}$) are the same. Therefore, from Eq. (i)

$$a_{BD} = \sum_{j=1}^{n} \frac{F_{1,j}^2 L_j}{AE} \quad \text{(iii)}$$

The solution is completed in Table 16.1.

From Table 16.1

$$\Delta_{BD} = \frac{2.71PL}{AE} \qquad a_{BD} = \frac{4.82L}{AE}$$

Substituting these values in Eq. (i) we have

$$\frac{2.71PL}{AE} + X_{BD}\frac{4.82L}{AE} = 0$$

from which

$$X_{BD} = -0.56P \quad \text{(i.e. compression)}$$

The actual forces, $F_{a,j}$, in the members of the complete truss of Fig. 16.18(a) are now calculated using the method of joints and are listed in the final column of Table 16.1.

We note in the above that $\Delta_{BD}$ is positive, which means that $\Delta_{BD}$ is in the direction of the unit loads, i.e. B approaches D and the diagonal BD in the released structure decreases in length. Therefore in the complete structure the member BD, which prevents this shortening, must be in compression as shown; also $a_{BD}$ will always be positive since it contains the term $F_{1,j}^2$. Finally, we note that the cut member BD is included in the calculation of the displacements in the released structure since its deformation, under a unit load, contributes to $a_{BD}$.

**CASTIGLIANO'S SECOND THEOREM :–**

"The partial derivative of the total strain energy stored with respect to a particular force gives the corresponding deformation at that point."

Mathematically,

$$\frac{\partial U}{\partial P} = \Delta$$

and

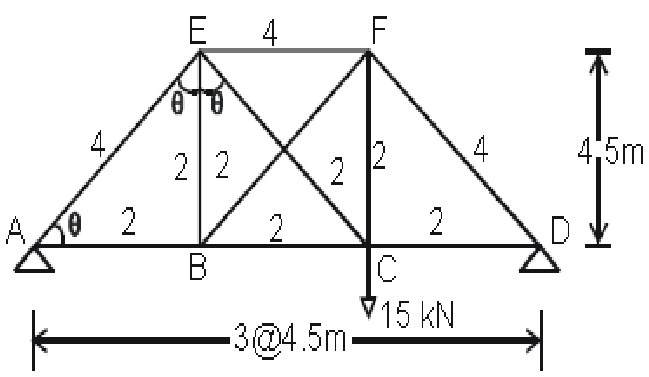
$$\frac{\partial U}{\partial M} = \theta$$

The credit for developing method of least work goes to Alberto Castiglianos who worked as an engineer in Italian Railways. This method was presented in a thesis in partial fulfillment of the requirement for the award of diploma engineering of associate engineer. He published a paper for finding deflections which is called Castiglianos first theorem and in consequence thereof, method of least work which is also known as Castiglianos second theorem. Method of least work also mentioned earlier in a paper by an Italian General Menabrea who was not able to give a satisfactory proof. Leonard Euler had also used the method about 50 years ago for derivation of equations for buckling of columns wherein, Daniel Bernolli gave valuable suggestion to him.

Method of least work or Castiglianos second theorem is a very versatile method for the analysis of indeterminate structures and specially to trussed type structures. The method does not however, accounts for erection stresses, temperature stresses or differential support sinking. The reader is advised to use some other method for the analysis of such indeterminate structures like frames and continuos beams.

It must be appreciated in general, for horizontal and vertical indeterminate structural systems, carrying various types of loads, there are generally more than one structural actions present at the same time including direct forces, shear forces, bending moments and twisting moments. In order to have a precise analysis all redundant structural actions and hence strain energies must be considered which would make the method laborious and cumbersome. Therefore, most of engineers think it sufficient to consider only the significant strain energy. The reader should know that most of structural analysis approaches whether classical or matrix methods consider equilibrium of forces and displacement/strain compatibility of members of a system.

The procedure for analysis has already been given. Utilizing that procedure, analyze the following truss by the method of least work. Areas in ( ) carry the units of $10^{-3}$ $m^2$ while the value of E can be taken as $200 \times 10^6$ KN/$m^2$.




where $i$ = total degree of indeterminacy

$b$ = number of bars.

$r$ = total number of reactive components which the support can provide.

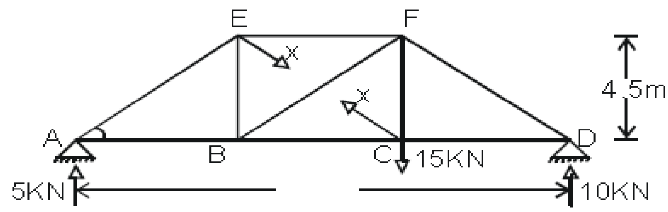
$b + r = 2j$

$10 + 3 > 2 \times 6$ $\quad 13 > 12$ so $i = 1$ . First degree internal indeterminancy.

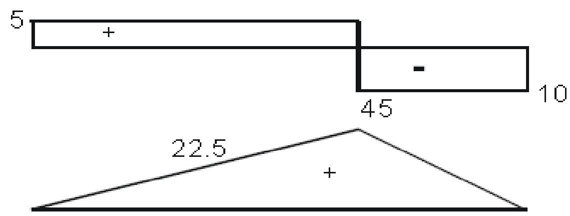
$U = \frac{F^2 L}{2\,AE}$ Strain energy due to direct forces induced due to applied loads in a BDS Truss.

$$\frac{\partial U}{\partial X} = F.\frac{\partial F}{\partial X}.\frac{L}{AE} = 0$$

***Note:–*** We select the redundants in such a way that the stability of the structure is not effected. Selecting member EC as redundant.




F-diagram B.D.S. under the action of applied loads & redundant.




S.F.D. due to applied load only.

B.M.D. due to applied load only.

Method of moments and shears has been used to find forces in BDS due to applied loads. A has been made. Forces vertical in members in terms of redundant X may be determined by the meth joints as before. From table.

$$\Sigma F.\frac{\partial F}{\partial x}.\frac{L}{AE} = 0 \qquad = -331.22 \times 10^{-6} + 51.49 \times 10^{-6}X$$

$$\text{or} \quad -331.22 + 51.49X = 0$$

$$\boxed{X = +6.433 \text{ KN}}$$

The final member forces are obtained as below by putting value of X in column 5 of the table

| Member | Force (KN) |
|---|---|
| AB | + 5 |
| BC | +5.45 |
| CD | + 10 |
| EF | – 9.55 |
| BE | + 0.45 |
| CF | + 10.45 |
| CE | + 6.43 |
| BF | – 0.64 |
| AE | – 7.07 |
| DF | – 14.14 |