LECTURE NOTES

ON

LINEAR ALGEBRA AND ORDINARY DIFFERENTIAL EQUATIONS

I B. Tech I semester

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UNIT-I
THEORY OF MATRICES

Solution for linear systems

Matrix : A system of mn numbers real (or) complex arranged in the form of an ordered set of 'm' rows, each row consisting of an ordered set of 'n' numbers between [ ] (or) ( ) (or) || || is called a matrix of order m xn.

Eg: $\begin{bmatrix} a_{11} & a_{12}.........a_{1n} \\ a_{21} & a_{12}.........a_{2n} \\ .... & \\ .... & \\ a_{m1} & a_{m2}.......a_{mn} \end{bmatrix}_{m \times n} = [a_{ij}]_{mxn}$ where $1\le i\le m$, $1\le j\le n$.

some types of matrics :

1. **square matrix :** A square matrix A of order nxn is sometimes called as a n- rowed matrix A (or) simply a square matrix of order n

   eg : $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is 2nd order matrix

2. Rectangular matrix : A matrix which is not a square matrix is called a rectangular matrix,

$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ is a 2x3 matrix

3. Row matrix : A matrix of order 1xm is called a row matrix

eg: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1x3}$

4. Column matrix : A matrix of order nx1 is called a column matrix

Eg: $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}_{3x1}$

5. **Unit matrix :** if A= $[a_{ij}]_{nxn}$ such that a_{ij} = 1 for i = j and a_{ij} = 0 for i≠j, then A is called a unit matrx.

Eg: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6. **Zero matrix :** it $A = [a_{ij}]_{mxn}$ such that $a_{ij} = 0\ \forall$ I and j then A is called a zero matrix (or) null matrix

   Eg: $O_{2x3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

7. **Diagonal elements in a matrix** $A = [a_{ij}]_{nxn}$, the elements a_{ij} of A for which i = j. i.e. $(a_{11}, a_{22}....a_{nn})$ are called the diagonal elements of A

   Eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ diagonal elements are 1,5,9

   Note : the line along which the diagonal elements lie is called the principle diagonal of A

8. **Diagonal matrix :** A square matrix all of whose elements except those in leading diagonal are zero is called diagonal matrix.

   If $d_1, d_2..... d_n$ are diagonal elements of a diagonal matrix A, then A is written as A = diag $(d_1, d_2....d_n)$

   Eg : A = diag (3,1,-2)= $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

9. **Scalar matrix :** A diagonal matrix whose leading diagonal elements are equal is called a scalar matrix. Eg : A= $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

10. **Equal matrices :** Two matrices $A = [a_{ij}]$ and $b = [b_{ij}]$ are said to be equal if and only if (i) A and B are of the same type(order) (ii) $a_{ij} = b_{ij}$ for every i&j

11. **The transpose of a matrix :** The matrix obtained from any given matrix A, by inter changing its rows and columns is called the transpose of A. It is denoted by A^1 (or) A^T.

    If $A = [a_{ij}]_{mxn}$ then the transpose of A is $A^1 = [b_{ji}]_{nxm}$, where $b_{ji} = a_{ij}$ Also $(A^1)^1 = A$

Note : A^1 and B^1 be the transposes of A and B repectively, then

(i) $(A^1)^1 = A$

(ii) $(A+B)^1 = A^1+B^1$

(iii) $(KA)^1 = KA^1$, K is a scalar

(iv) $(AB)^1 = B^1A^1$

12. The conjugate of a matrix : The matrix obtained from any given matrix A, on replacing its elements by corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\bar{A}$

Note : if $\bar{A}$ and $\bar{B}$ be the conjugates of A and B respectively then,

(i) $\overline{(\bar{A})} = A$

(ii) $\overline{(A+B)} = \bar{A}+\bar{B}$

(iii) $\overline{(KA)} = \bar{K}\bar{A}$, K is a any complex number

(iv) $\overline{(AB)} = \bar{B}\,\bar{A}$

Eg ; if A= $\begin{bmatrix} 2 & 3i & 2-5i \\ -i & 0 & 4i+3 \end{bmatrix}_{2x3}$ *then* $\bar{A} = \begin{bmatrix} 2 & -3i & 2+5i \\ i & 0 & -4i+3 \end{bmatrix}_{2x3}$

13. <u>The conjugate Transpose of a matrix</u>

The conjugate of the transpose of the matrix A is called the conjugate transpose of A and is denoted by A^θ Thus $A^\theta = (\bar{A^1})$ where A^1 is the transpose of A. Now $A = [a_{ij}]_{m\times n} \Rightarrow A^\theta = [b_{ij}]_{n\times m}$, where bij = $\bar{a}$ ij i.e. the $(i,j)^{th}$ element of A^θ conjugate complex of the $(j, i)^{th}$ element of A

Eg: if A = $\begin{bmatrix} 5 & 3-i & -2i \\ 0 & 1+i & 4-i \end{bmatrix}_{2X3}$ then $A^\theta = \begin{bmatrix} 5 & 0 \\ 3+i & 1-i \\ 2i & 4+i \end{bmatrix}_{3x2}$

Note: $A^{\theta} = \overline{A}^{1} = (\overline{A})^{1}$ and $(A^{\theta})^{\theta}$ =A

14.

(i) Upper Triangular matrix : A square matrix all of whose elements below the leading diagonal are zero is called an Upper triangular matrix. i.e, $a_{ij=0 \text{ for } i>j}$

Eg; $\begin{bmatrix} 1 & 3 & 8 \\ 0 & 4 & -5 \\ 0 & 0 & 2 \end{bmatrix}$ is an **Upper triangular matrix**

(ii) **Lower triangular matrix ;** A square matrix all of whose elements above the leading diagonal are zero is called a lower triangular matrix. i.e, $a_{ij=0}$ for i< j

Eg: $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 2 & 0 \\ 7 & 3 & 6 \end{bmatrix}$ is an **Lower triangular matrix**

(iii) **Triangular matrix:** A matrix is said to be triangular matrix it is either an upper triangular matrix or a lower triangular matrix

15. Symmetric matrix : A square matrix A =[a_{ij}] is said to be symmetric if $a_{ij} = a_{ji}$ for every i and j

Thus A is a symmetric matrix iff A^1 = A

Eg: $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix

16. Skew – Symmetric : A square matrix A = [a_{ij}] is said to be skew – symmetric if $a_{ij} = -a_{ji}$ for every i and j.

Eg : $\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ is a skew – symmetric matrix

Thus A is a skew – symmetric iff $A= -A^1$ (or) $-A= A^1$

Note: Every diagonal element of a skew – symmetric matrix is necessarily zero.

Since $a_{ij} = -a_{ij} \Rightarrow a_{ij} = 0$

17. Multiplication of a matrix by a scalar.

Let 'A' be a matrix. The matrix obtain by multiplying every element of A by a scalar K, is called the product of A by K and is denoted by KA (or) AK

Thus : $A + [a_{ij}]_{m\times n}$ then $KA = [ka_{ij}]_{m\times n} = k[a_{ij}]_{m\times n}$

18. Sum of matrices :

Let $A = [a_{ij}]_{m\times n}$,$B = [b_{ij}]_{m\times n}$ be two matrices. The matrix $C = [c_{ij}]_{m\times n}$ where $c_{ij} = a_{ij}+b_{ij}$ is called the sum of the matrices A and B.

The sum of A and B is denoted by A+B. Thus $[a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [a_{ij}+b_{ij}]_{m\times n}$ and $[a_{ij}+b_{ij}]_{m\times n} = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}$

19. The difference of two matrices : If A, B are two matrices of the same type then A+(-B) is taken as A – B

Theorem 1: Every square matrix can be expressed as the sum of a symmetric and skew – symmetric matrices in one and only way

Proof : let A be any square matrix. We can write

$A= ½ (A+A^1)+ ½ (A-A^1)=P+Q$ (say).

Where $P = ½ (A+A^1)$

$Q = ½ (A-A^1)$

We have $P^1 = \{½ (A+A^1)\}^1 = ½ (A+A^1)^1$ since $[(KA)^1 = KA^1]$

$= ½ [A^1+(A^1)^1]= ½ [A+A^1]=P$

P is symmetric matrix.

Now , $Q^1 = [½ (A-A^1)]^1 = ½ (A-A^1)^1$

$= ½ [A^1-(A^1)^1] = ½ (A^1-A)$

$= - ½ (A-A^1)= -Q$

Q is a skew – symmetric matrix.

Thus square matrix = symmetric + skew – symmetric.

Then to prove the sum is unique.

It possible, let A = R+S be another such representation of A where R is a symmetric and S is a skew – symmetric matrix.

$R^1 = R$ and $S^1 = -S$

Now $A^1 = (R+S)^1 = R^1+S^1 = R-S$ and

$½ (A+A^1) = ½ (R+S+R-S) = R$

$½ (A-A^1) = ½ (R+S-R+S) = S$

$\Rightarrow R = P$ and $S=Q$

Thus, the representation is unique.

Theorem2: Prove that inverse of a non – singular symmetric matrix A is symmetric.

Proof : since A is non – singular symmetric matrix A^{-1} exists and $A^T = A$……………(1)

Now, we have to prove that A^{-1} is symmetric we have $(A^{-1})^T = (A^T)^{-1} = A^{-1}$ (by (1)) Since $(A^{-1})^T = A^{-1}$ therefore, A^{-1} is symmetric.

Theorem3 : If A is a symmetric matrix, then prove that adj A is also symmetric

Proof : Since A is symmetric, we have $A^T = A$ … (1)

Now, we have $(adjA)^T = adj\ A^T$ [since $adj\ A^1 = (AdjA)^1$]

$= adj\ A$ [by (1)]

$(adjA)^T = adjA$ therefore, adjA is a symmetric matrix.

20. Matrix multiplication: Let A = $[a_{ik}]_{m\times n}$, B = $[b_{kj}]_{nxp}$ then the matrix C = $[c_{ij}]_{mxp}$ where c_{ij} is called the product of the matrices A and B in that order and we write C = AB.

The matrix A is called the pre-factor & B is called the post – factor

Note : If the number of columns of A is equal to the number of rows in B then the matrices are said to be conformable for multiplication in that order.

Theorem 4 : Matrix multiplication is associative i.e. If A,B,C are matrices then (AB) C= A(BC)

Proof : Let A= $[a_{ij}]_{m\times n}$ B = $[b_{jk}]_{n\times p}$ and C= $[C_{kl}]_{p\times q}$

Then AB = $[u_{ik}]_{m\times p}$ where $u_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$ -------(1)

Also BC = $[v_{jl}]_{n\times q}$ where $v_{jl} = \sum_{k=1}^{p} b_{jk}\, c_{kl}$ ------(2)

Now, A(BC) is an mxq matrix and (AB)C is also an mxq matrix.

let A(BC) = $[w_{il}]_{m\times q}$ where w_{il} is the $(i,j)^{th}$ element of A(BC)

Then $w_{il} = \sum_{j=1}^{n} a_{ij}\, v_{jl}$

$$= \sum_{j=1}^{n}\left[a_{ij}\left\{\sum_{k=1}^{p} b_{jk}c_{kl}\right\}\right] \qquad \text{by equation (2)}$$

$$= \sum_{k=1}^{p} \left[\left\{ \sum_{j=1}^{n} a_{ij}\, b_{jk} \right\} c_{kl} \right]$$

(Since Finite summations can be interchanged)

$$= \sum_{k=1}^{p} u_{ik} c_{kl} \qquad \text{(from (1))}$$

= The $(i,j)^{th}$ element of (AB)C

A(BC) = (AB)C

21. Positive integral powers of a square matrix:

Let A be a square matrix. Then A^2 is defined A.A

Now, by associative law $A^3 = A^2.A = (AA)A$

$= A(AA) = AA^2$

Similarly we have $A^{m-1}A = A\,A^{m-1} = A^m$ where m is a positive integer

Note : $I^n = I$

$O^n = 0$

Note 1: Multiplication of matrices is distributiue w.r.t. addition of matrices.

i.e, A(B+C) = AB + AC

(B+C)A = BA+CA

Note 2: If A is a matrix of order mxn then $AI_n = I_nA = A$

22. **Trace of A square matrix :** Let $A = [a_{ij}]_{n\times n}$ the trace of the square matrix A is defined as $\sum_{i=1}^{n} a_{ii}$. And is denoted by 'tr A'

Thus trA = $\sum_{i=1}^{n} a_{ii}$ = $a_{11}+a_{22}+$ …….a_{nn}

Eg : A = $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ then trA = a+b+c

<u>Properties :</u> If A and B are square matrices of order n and λ is any scalar, then

(i) tr (λ A) = λ tr A

(ii) tr (A+B) = trA + tr B

(iii) tr(AB) = tr(BA)

23. Idempotent matrix : If A is a square matrix such that $A^2 = A$ then 'A' is called idempotent matrix

24. Nilpotent Matrix : If A is a square matrix such that $A^m=0$ where m is a +ve integer then A is called nilpotent matrix.

Note : If m is least positive integer such that $A^m = 0$ then A is called nilpotent of index m

25. Involutary : If A is a square matrix such that $A^2 = I$ then A is called involuntary matrix.

26. Orthogonal Matrix : A square matrix A is said to be orthogonal if $AA^1 = A^1A = I$

Theorem 5: If A, B are orthogonal matrices, each of order n then AB and BA are orthogonal matrices.

Proof : Since A and B are both orthogonal matrices.

$AA^T = A^TA = I$ -------- 1

$BB^T = B^TB = I$ -------- 2

Now $(AB)^T = B^TA^T$

Consider $(AB)^T (AB) = (B^TA^T) (AB)$

$= B^T(A^TA)B$

$= B^TIB$ (by 1)

$= B^TB$

$= I$ (by 2)

$\therefore$ AB is orthogonal

Similarly we can prove that BA is also orthogonal

Theorem 6 : Prove that the inverse of an orthogonal matrix is orthogonal and its transpose is also orthogonal.

Proof : Let A be an orthogonal matrix

Then $A^T.A = AA^T = I$

Consider $A^TA = I$

Taking inverse on both sides $(A^T.A)^{-1} = I^{-1}$

$A^{-1}(A^T)^{-1} = I$

$A^{-1}(A^{-1})^T = I$

$\therefore A^{-1}$ is orthogonal

Again $A^T.A = I$

Taking transpose on both sides $(A^T.A)^T = I^T$

$A^T(A^T)^T = I$

Hence A^T is orthogonal

Examples:

1. Show that A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ **is orthogonal.**

Sol: Given A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Consider $A.A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\therefore$ A is orthogonal matrix.

2. Prove that the matrix $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ is orthogonal.

Sol: Given A = $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

Then $A^T = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

Consider $A.A^T = \frac{1}{9}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A.A^T = I$

Similarly $A^T .A = I$

Hence A is orthogonal Matrix

3. Determine the values of a,b,c when $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ **is orthogonal.**

Sol: - For orthogonal matrix $AA^T = I$

So $AA^T = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}\begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = I$

$$\Rightarrow \begin{bmatrix} 4b^2 + c^2 & 2b^2 - c^2 & -2b^2 + c^2 \\ 2b^2 - c^2 & a^2 + b^2 + c^2 & a^2 - b^2 - c^2 \\ -2b^2 + c^2 & a^2 - b^2 - c^2 & a^2 + b^2 + c^2 \end{bmatrix} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving $2b^2-c^2 = 0$, $a^2-b^2-c^2 = 0$

We get $c = \pm\sqrt{2}b$ $a^2 = b^2+2b^2 = 3b^2$

$\Rightarrow a = \pm\sqrt{3}b$

From the diagonal elements of I

$4b^2+c^2= 1 \Rightarrow 4b^2+2b^2=1$ $(c^2=2b^2)$

⇨ b = $\pm\frac{1}{\sqrt{6}}$

a= $\pm\sqrt{3}b$

= $\pm\frac{1}{\sqrt{2}}$

b= $\pm\frac{1}{\sqrt{6}}$

c = $\pm\sqrt{2}b$

= $\pm\frac{1}{\sqrt{3}}$

27. Determinant of a square matrix:

If A = $\begin{bmatrix} a_{11} & a_{12}.........a_{1n} \\ a_{21} & a_{22}.........a_{2n} \\ .... & \\ a_{i1} & a_{i2}.........a_{in} \\ a_{n1} & a_{n2}.......a_{nn} \end{bmatrix}_{nxn}$ then $|A| =$ $\begin{vmatrix} a_{11} & a_{12}.........a_{1n} \\ a_{21} & a_{22}.........a_{2n} \\ .... & \\ a_{i1} & a_{i2}.........a_{in} \\ a_{n1} & a_{n2}.......a_{nn} \end{vmatrix}$

28. Minors and cofactors of a square matrix

Let A =$[a_{ij}]_{n\times n}$ be a square matrix when form A the elements of ith row and jth column are deleted the determinant of (n-1) rowed matrix [Mij] is called the minor of aij of A and is denoted by $|M_{ij}|$

The signed minor $(-1)^{i+j}\,|M_{ij}|$ is called the cofactor of a_{ij} and is denoted by A_{ij}..

If A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$| A | = a_{11}\ |M_{11}| + a_{12}\ |M_{12}| + a_{13}\ |M_{13}|$ (or)

$= a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$

Eg: Find Determinant of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ **by using minors and co-factors.**

Sol: A = $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

det A = $1\begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} - 1\begin{vmatrix} 1 & -3 \\ -2 & -4 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix}$

=1(-12-12)-1(-4-6)+3(-4+6)

= -24+10+6 = -8

Similarly we find det A by using co-factors also.

Note 1: If A is a square matrix of order n then $|KA| = K^n\,|A|$, where k is a scalar.

Note 2: If A is a square matrix of order n, then $|A| = |A^T|$

Note 3: If A and B be two square matrices of the same order, then $|AB| = |A|\ |B|$

29. Inverse of a Matrix: Let A be any square matrix, then a matrix B, if exists such that AB = BA =I then B is called inverse of A and is denoted by A^{-1}.

Theorem 7: The inverse of a Matrix if exists is Unique

Proof: Let if possible B and C be the inverses of ‘A’.

Then AB = BA =I

AC = CA= I

consider B = BI =B(AC)

=(BA)C

=IC

⇨B=C

Hence inverse of a Matrix is Unique

Note:1 $(A^{-1})^{-1} = A$

Note 2: $I^{-1} = I$

30. Adjoint of a matrix:

Let A be a square matrix of order n. The transpose of the matrix got from A

By replacing the elements of A by the corresponding co-factors is called the adjoint of A and is denoted by adj A.

Note: For any scalar k, adj(kA) = k^{n-1} adj A

Note : The necessary and sufficient condition for a square matrix to posses inverse is that $|A| \neq 0$

Note: if $|A| \neq 0$ then $A^{-1} = \frac{1}{|A|}(adj\ A)$

3. Singular and Non-singular Matrices:

A square matrix A is said to be singular if $|A| = 0$. If $|A| \neq 0$ then A is said to be non-singular.

Note: 1. Only non-singular matrices posses inverses.

2. The product of non-singular matrices is also non-singular.

Theorem 9: If A, B are invertible matrices of the same order, then (i). $(AB)^{-1} = B^{-1}A^{-1}$

(ii). $(A^1)^{-1} = (A^{-1})^1$

Proof: (i). we have $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$

$= B^{-1}(I\ B)$

$= B^{-1}B$

$= I$

$(AB)^{-1} = B^{-1}A^{-1}$

(ii). $A^{-1}A = AA^{-1} = I$

Consider $A^{-1}A = I$

⇨ $(A^{-1}A)^1 = I^1$

⇨ $A^1.(A^{-1})^1 = I$

⇨ $(A^1)^{-1} = (A^{-1})^1$

Problems

1). Express the matrix A as sum of symmetric and skew – symmetric matrices. Where

$$A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$$

Sol: Given $A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix}$

Matrix A can be written as $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$

$\Rightarrow P = \frac{1}{2}(A+A^T) = \frac{1}{2}\left\{\begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & +2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix}\right\}$

$$= \frac{1}{2}\begin{bmatrix} 6 & 0 & 11 \\ 0 & 14 & 3 \\ 11 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 11/2 \\ 0 & 7 & 3/2 \\ 11/2 & 3/2 & 0 \end{bmatrix}$$

$Q = \frac{1}{2}(A-A^T)$

$= \frac{1}{2}\left\{\begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 5 \\ -2 & 7 & 4 \\ 6 & -1 & 0 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & -2 & 1/2 \\ 2 & 0 & -5/2 \\ -1/2 & 5/2 & 0 \end{bmatrix}_s$

A = P+Q where 'P' is symmetric matrix

'Q' is skew-symmetric matrix.

2. Find the adjoint and inverse of a matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

Soln: Adjoint of A = $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

Where Aij are the cofactors of the elements of a_{ij}.

Thus minors of aij are

$M_{11} = \begin{vmatrix} -2 & 1 \\ 2 & 1 \end{vmatrix} = -4$ $\quad M_{12} = \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} = -1$ $\quad M_{13} = \begin{vmatrix} 3 & -2 \\ 4 & 2 \end{vmatrix} = 14$

$M_{21} = \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = -4$ $\quad M_{22} = \begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} = -11$ $\quad M_{23} = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = -6$

$M_{31} = \begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} = 8$ $\quad M_{32} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = -8$ $\quad M_{31} = \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = -8$

Cofactors $A_{ij} = (-1)^{i+j} M_{ij}$

Adjoint of A = $\begin{bmatrix} -4 & 1 & 14 \\ 4 & -11 & 6 \\ 8 & 8 & -8 \end{bmatrix}^T = \begin{bmatrix} -4 & 4 & 8 \\ 1 & -11 & 8 \\ 14 & 6 & -8 \end{bmatrix}$

$|A| =$ -4-2(-1) +3(14) = 40

$$A^{-1} = \frac{1}{|A|}(adj\ A)$$

$$= \frac{1}{40}\begin{bmatrix} -4 & 4 & 8 \\ 1 & -11 & 8 \\ 14 & 6 & -8 \end{bmatrix}$$

MATRIX INVERSE METHOD

3). Solve the equations 3x+4y+5z = 18, 2x-y+8z =13 and 5x-2y+7z =20

Soln: The given equations in matrix form is AX = B

$$\begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

det A = 3(-7+16)-4(14-40)+5(-4+5) = 136

co-factor matrix is D = $\begin{bmatrix} (-7+16) & -(14-40) & (-4+5) \\ -(28+10) & (21-25) & -(-6-20) \\ (32+5) & -(24-10) & (-3-8) \end{bmatrix}$

D = $\begin{bmatrix} 9 & 26 & 1 \\ -38 & -4 & 26 \\ 37 & -14 & -11 \end{bmatrix}$

Adj A = D^T = $\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$

A^{-1} = 1/det A adj A = $\frac{1}{136}\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$

A x = B => x = A^{-1} B

$$= \frac{1}{136}\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}\begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

$$= \frac{1}{136}\begin{bmatrix} 162-494-740 \\ 468-52\ -280 \\ 18+\ 338-220 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{136}\begin{bmatrix} 408 \\ 136 \\ 136 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

Soln is x =3 , y=1, z=1

Sub – Matrix: Any matrix obtained by deleting some rows or columns or both of a given matrix is called is submatrix.

E.g: Let A = $\begin{bmatrix} 1 & 5 & 6 & 7 \\ 8 & 9 & 10 & 5 \\ 3 & 4 & 5 & -1 \end{bmatrix}$. Then $\begin{bmatrix} 8 & 9 & 10 \\ 3 & 4 & 5 \end{bmatrix}_{2x3}$ is a sub matrix of A obtained by deleting first row and 4^{th} column of A.

<u>**Minor of a Matrix**</u>**:** Let A be an mxn matrix. The determinant of a square sub matrix of A is called a minor of the matrix.

Note: If the order of the square sub matrix is 't' then its determinant is called a minor of order 't'.

Eg: A = $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}_{4X3}$ be a matrix

$\rightarrow B = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ is a sub-matrix of order '2'

$|B|$ = 2-3 = -1 is a minor of order '2'

$$\rightarrow C = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 5 & 6 & 7 \end{bmatrix}$$ is a sub-matrix of order '3'

detc= 2(7-12)-1(21-10)+(18-5)

= 2(-5)-1(11)+1(13)

= -10-11+13 = -8 is a minor of order '3'

***Rank of a Matrix:**

Let A be mxn matrix. If A is a null matrix, we define its rank to be '0'. If A is a non-zero matrix, we say that r is the rank of A if

(i) Every $(r+1)^{th}$ order minor of A is '0' (zero) &

(ii) At least one r^{th} order minor of A which is not zero.

Note: 1. It is denoted by ρ(A)

2. Rank of a matrix is unique.

3. Every matrix will have a rank.

4. If A is a matrix of order mxn,

Rank of A ≤ min(m,n)

5. If ρ(A) = r then every minor of A of order r+1, or more is zero.

6. Rank of the Identity matrix I_n is n.

7. If A is a matrix of order n and A is non-singular then ρ(A) = n

Important Note:

1. The rank of a matrix is ≤r if all minors of $(r+1)^{th}$ order are zero.

2. The rank of a matrix is ≥r, if there is at least one minor of order 'r' which is not equal to zero.

PROBLEMS

1. Find the rank of the given matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

Soln: Given matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

→ det A = 1(48-40)-2(36-28)+3(30-28)

= 8-16+6 = -2 ≠ 0

We have minor of order 3

ρ(A) =3

2. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$

Sol: Given the matrix is of order 3x4

Its Rank ≤ min(3,4) = 3

Highest order of the minor will be 3.

Let us consider the minor $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{bmatrix}$

Determinat of minor is 1(-49)-2(-56)+3(35-48)

= -49+112-39 = 24 ≠ 0.

Hence rank of the given matrix is '3'.

*** Elementary Transformations on a Matrix:**

i). Interchange of i^{th} row and j^{th} row is denoted by $R_i \leftrightarrow R_j$

(ii). If i^{th} row is multiplied with k then it is denoted by $R_i \to K R_i$

(iii). If all the elements of i^{th} row are multiplied with k and added to the corresponding elements of j^{th} row then it is denoted by $R_j \to R_j + KR_i$

<u>**Note:**</u> 1. The corresponding column transformations will be denoted by writing 'c'. i.e

$c_i \leftrightarrow c_j$, $c_i \to k\, c_j$ $c_j \to c_j + kc_i$

2. The elementary operations on a matrix do not change its rank.

Equivalence of Matrices: If B is obtained from A after a finite number of elementary transformations on A, then B is said to be equivalent to A.

It is denoted as B~A.

Note : 1. If A and B are two equivalent matrices, then rank A = rank B.

2. If A and B have the same size and the same rank, then the two matrices are equivalent.

Echelon form of a matrix:

A matrix is said to be in Echelon form, if

(i). Zero rows, if any exists, they should be below the non-zero row.

(ii). The first non-zero entry in each non-zero row is equal to '1'.

(iii). The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

<u>**Note:**</u> 1. The number of non-zero rows in echelon form of A is the rank of 'A'.

2. The rank of the transpose of a matrix is the same as that of original matrix.
3. The condition (ii) is optional.

Eg: 1. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a row echelon form.

2. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is a row echelon form.

PROBLEMS

1. Find the rank of the matrix A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ by reducing it to Echelon form.

sol: Given A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$

Applying row transformations on A.

A ~ $\begin{bmatrix} 1 & -3 & -1 \\ 3 & -2 & 4 \\ 2 & 3 & 7 \end{bmatrix}$ $R_1 \leftrightarrow R_3$

~ $\begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 9 & 9 \end{bmatrix}$ $R_2 \rightarrow R_2 - 3R_1$

$R_3 \rightarrow R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad R_2 \to R_2/7, R_3 \to R_3/9$$

$$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

This is the Echelon form of matrix A.

The rank of a matrix A.

= Number of non – zero rows =2

2. For what values of k the matrix $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$ **has rank '3'.**

Sol: The given matrix is of the order 4x4

If its rank is 3 ⇨ det A =0

$$A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$$

Applying $R_2 \to 4R_2 - R_1$, $R_3 \to 4R_3 - kR_1$, $R_4 \to 4R_4 - 9R_1$

We get A ~ $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 8-4k & 8+3k & 8-k \\ 0 & 0 & 4k+27 & 3 \end{bmatrix}$

Since Rank A = 3 ⇨ det A =0

⇨ 4 $\begin{vmatrix} 0 & -1 & -1 \\ 8-4k & 8+3k & 8-k \\ 0 & 4k+27 & 3 \end{vmatrix} = 0$

⇨ 1[(8-4k)3]-1(8-4k)(4k+27)] = 0

⇨ (8-4k) (3-4k-27) = 0

⇨ (8-4k)(-24-4k) =0

⇨ (2-k)(6+k)=0

⇨ k =2 or k = -6

Normal Form:

Every mxn matrix of rank r can be reduced to the form $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ (or) (I_r) (or) $\begin{pmatrix} I_r \\ 0 \end{pmatrix}$(or) $\begin{pmatrix} I_r & 0 \end{pmatrix}$ by a finite number of elementary transformations, where I_r is the r – rowed unit matrix.

Note: 1. If A is an mxn matrix of rank r, there exists non-singular matrices P and Q such that PAQ = $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

2.Normal form another name is “canonical form”

e.g: By reducing the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ into normal form, find its rank.

Sol: Given A = $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

A ~ $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & 5 \\ 0 & -6 & -4 & -22 \end{bmatrix}$ $R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 3R_1$

A ~ $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 3 & 2 & 11 \end{bmatrix}$ $R_3 \rightarrow R_3/-2$

A ~ $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ $R_3 \rightarrow R_3+R_2$

A ~ $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ $c_2 \rightarrow c_2 - 2c_1$, $c_3 \rightarrow c_3 - 3c_1$, $c_4 \rightarrow c_4 - 4c_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 18 \end{bmatrix} c_3 \to 3c_3 - 2c_2,\ c_4 \to 3c_4 - 5c_2$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} c_2 \to c_2/-3,\ c_4 \to c_4/18$$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} c_4 \leftrightarrow c_3$$

This is in normal form $[I_3\ 0]$

Hence Rank of A is '3'.

Gauss – Jordan method

- The inverse of a matrix by elementary Transformations: **Gauss – Jordan method)**

1. suppose A is a non-singular matrix of order 'n' then we write $A = I_n A$
2. Now we apply elementary row-operations only to the matrix A and the pre-factor I_n of the R.H.S
3. We will do this till we get $I_n = BA$ then obviously B is the inverse of A.

*Find the inverse of the matrix A using elementary operations where A= $\begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

Sol:

Given A = $\begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

We can write A = I_3 A

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \rightarrow 2R_3 - R_2$, we get

$$\begin{bmatrix} 1 & 6 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} A$$

Applying $R_1 \rightarrow R_1 - 3R_2$, we get

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} A$$

Applying $R_1 \rightarrow R_1 + 5R_3$, $R_2 \rightarrow R_2 - 3R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -8 & 10 \\ 0 & 4 & -6 \\ 0 & -1 & 2 \end{bmatrix} A$$

Applying $R_2 \rightarrow R_2/2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -8 & 10 \\ 0 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix} A \Rightarrow I_3 = BA$$

B is the inverse of A.

System of linear equations – Triangular systems:

Consider the system of n linear algebraic equations in n unknowns

$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n = b_2$

$a_{n1}x_1 + a_{n2}x_2 + \dots\dots\dots + a_{nn}x_n = bn$

The given system we can write Ax =B

i.e $$\begin{bmatrix} a_{11} & a_{12}\dots\dots a_{1n} \\ a_{21} & a_{22}\dots\dots a_{2n} \\ \dots\dots\dots\dots\dots\dots & \\ a_{n1} & a_{n2}\dots\dots\dots a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ .. \\ b_n \end{bmatrix}$$

Lower Triangular system:

Suppose the co-efficient matrix A is such that all the elements above the leading diagonal are zero. That is , A is a lower triangular matrix of the form.

$$A = \begin{bmatrix} a_{11} & 0 & 0......0 \\ a_{21} & a_{22}....0......0 & \\ .... & & \\ a_{n1} & a_{n2}..........a_{nn} & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ .. \\ b_n \end{bmatrix}$$

In this case the system will be of the form

$a_{11} x_1 = b_1$

$a_{21} x_1 + a_{22} x_2 = b_2$

$a_{n1} x_1 + a_{n2} x_2 + ... + a_{nn} = b_n$

from above equations, we get

$x_1 = b_1/a_{11}$

$$x_2 = \frac{b_2 - a_{21}x_1}{a_{22}} = \frac{1}{a_{22}}\left[b_2 - \frac{a_{21}}{a_{11}}b_1\right]$$

The method of constructing the exact solution is called method of forward substitution.

Upper triangular system:

Suppose the co-efficient matrix A is such that all the elements below the leading diagonal are all zero. i.e A is an upper triangular matrix of the form.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13}......a_{1n} \\ 0 & a_{22} & a_{23}....\ a_{2n} \\ .... & & \\ 0 & 0 & 0..........a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ .. \\ b_n \end{bmatrix}$$

Above system can be of the form

$a_{11}\,x_1+a_{12}x_2+\dots\dots\dots+a_{1n}\,x_n=b_1$

$a_{22}x_2+\dots\dots\dots+a_{2n}\,x_n=b_2$

$a_{nn}\,x_n=b_n$

from the above equations, we get

$$x_{n-1}=\frac{b_n}{a_{nn}}$$

$$x_{n-1}=\frac{b_{n-1}-a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$=\frac{1}{a_{n-1,n-1}}\left[b_{n-1}-\frac{a_{n-1,n}}{a_{nn}}b_n\right]$$ and so on.

The method of constructing the exact solution is called method of backward substitution

Solution of linear systems – Direct methods

Method of Factorization (Triangularisation):

Triangular Decomposition Method:

This method is based on the fact that a square matrix A can be factorized into the form LU, where L is the unit lower triangular matrix and U is the upper triangular matrix.

Note:

1. The principle minors of A must be non-singular
2. This factorization

Consider the linear system

$a_{11}\,x_1+a_{12}x_2+a_{13}x_3 = b_1$

$a_{21}\,x_1+a_{22}x_2+a_{23}x_3 = b_2$

$a_{31}\,x_1+a_{32}x_2+a_{33}x_3 = b3$

which can be written in the form Ax = B --------(1)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22}. & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = L.\,U \text{ -----(2) and } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\text{Where } L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Then (1) ➔ LUX = B.

If we put UX = Y where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Then (2) becomes LY = B

$$\left.\begin{aligned} &\text{The system is equals to } y_1 = b_1 \\ &l_{21}+y_2 = b_2 \\ &l_{31}y_1+l_{32}y_2+y_3 = b_3 \end{aligned}\right\} \quad (3)$$

here y_1,y_2,y_3 are solved by forward substitution using (3) we get

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$u_{11} x_1+u_{12}x_2+u_{13}x_3 =y_1$

$u_{22}x_2+u_{23}x_3 \quad =y_2$

$u_{33}x_3 \quad =y_3$

from these we can sole for x_1,x_2,x_3 by backward substitution .

The method of computing L and U is outlined below from (2) * we get

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22}. & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Equating corresponding elements, we get

$u_{11} = a_{11}$ $\quad l_{21} u_{11} = a_{21} \Rightarrow l_{21} = a_{21}/a_{11}$ and

$u_{12} = a_{12}$ $\quad l_{31} u_{11} =a_{31} \Rightarrow l_{31} = a_{31}/a_{11}$

$u_{13} = a_{13}$ $\quad l_{21} u_{12} + u_{22} = a_{22} \Rightarrow l_{21}a_{12} + u_{22} = a_{22} \Rightarrow u_{22} = a_{22}- a_{21}a_{12}/a_{11}$

$l_{31}u_{12}+l_{32}u_{22} = a_{32} \Rightarrow l_{32} = [a_{32}-l_{31} u_{12}] / u_{22}$ and

$l_{31}a_{13}+l_{32}a_{23}+u_{33} = a_{33}$ from which u_{33} can be calculated.

Ex : Solve the following system by the method of factorization x+3y+8z =4, x+4y+3z = -2, x+3y+4z =1

Soln: The given system can write AX = B;

A ➔ $\begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and B = $\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$

Let A = LU

Where L = $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and U = $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

➔ $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

➔ $u_{11} = 1$, $u_{12} = 3$, $u_{13} = 8$

$l_{21} u_{11} = 1$ ➔ $l_{21} = 1$ and $l_{31}u_{11} = 1$ ➔ $l_{31} = 1$

from the equations $l_{21}u_{12}+u_{22} = 4$ and

$l_{21}u_{13}+u_{23} = 3$, we get

$u_{22} = 4-l_{21}u_{12}$

$= 4-3 = 1$

$u_{23} = 3-l_{21}u_{13}$

$= 3-8 = -5$

By using $l_{31}u_{12}+l_{32}u_{22} = 3$ and $l_{31}u_{13}+l_{32}u_{23}+u_{33} = 4$ we get

$l_{33} = 4-l_{31}u_{13}-l_{32}u_{23}$

$= 4-1(8)-(0)5 = -4$

Thus L = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and U = $\begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & -4 \end{bmatrix}$

Let UX = Y where Y = $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ then LY = B

➔ $\begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ ---------(1)

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$ -----(2)

From (2) $y_1= 4$, $y_2= -2$ and $y_1+y_3 =1$

$y_3= -3$

from (1)

x = - 29/4 , y =7/4 and z= ¾

<u>Solution of Tridiagonal system:</u>

Consider the system of equations defined by

$b_1u_1+c_1u_2 =d_1$

$a_2u_1+b_2u_2+c_2u_3 = d_2$

$a_nu_{n-1}+b_nu_n =d_n$

The co-efficient matrix is

$$\begin{bmatrix} b_1 & c_1 & 0 & 0----0 & & \\ a_2 & b_2 & c_2 & 0----0 & & \\ 0 & a_3 & b_3 & c_3----0 & & \\ ------------ & & & & & \\ 0 & 0 & 0 & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & 0 & 0 & 0 & a_n & b_n \end{bmatrix}$$

This matrix is solved by using factorization method.

Ex : Solve the following tri-diagonal system of Equations

$$\begin{aligned} 2x_1+x_2 &= 2 \\ x_1+2x_2+x_3 &= 2 \\ x_2+2x_3+x_4 &= 2 \\ x_3+2x_4 &= 1 \end{aligned}$$

step1:

soln: The given system of equations in matrix notation can be coriten as Ax = B

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

Let A = LU

$$\Rightarrow \begin{bmatrix} w_1 & 0 & 0 & 0 \\ \beta_2 & w_2 & 0 & 0 \\ 0 & \beta_3 & w_3 & 0 \\ 0 & 0 & \beta_4 & w_4 \end{bmatrix} \begin{bmatrix} 1 & \alpha_1 & 0 & 0 \\ 0 & 1 & \alpha_2 & 0 \\ 0 & 0 & 1 & \alpha_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} w_1 & w_1\alpha_1 & 0 & 0 \\ \beta_2 & \alpha_1\beta_2 + w_2 & \alpha_2 w_2 & 0 \\ 0 & \beta_3 & \alpha_2\beta_3 + w_3 & \alpha_3 w_3 \\ 0 & 0 & \beta_4 & \alpha_3\beta_4 + w_4 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} 1$$

Equating the corresponding elements of first row ➔ $w_1 = 2$, $w_1\alpha_1 = 1$

➔ $\alpha_1 = ½$

Equating the corresponding elements of second row, we get $\beta_2 = 1$, $\alpha_1\beta_2 + w_2 = 2$ ➔

$w_2 = 3/2$ $\alpha_2 = 2/3$

$w_3 = 4/3$ $\alpha_3 = ¾$

equating the corresponding elements of fourth row, we get

$\beta_4 = 1$, $\alpha_3\beta_4 + w_4 = 2 \Rightarrow w_4 = 5/4$

substituting these values

L = $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & \frac{3}{2} & 0 & 0 \\ 0 & 1 & \frac{4}{3} & 0 \\ 0 & 0 & 1 & \frac{5}{4} \end{bmatrix}$ and U= $\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Step2: LUX = B

LY = B where UX = Y

$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & \frac{3}{2} & 0 & 0 \\ 0 & 1 & \frac{4}{3} & 0 \\ 0 & 0 & 1 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

$2y_1 = 2 \Rightarrow y_1 = 1$ $\quad$ $y_2 + 4/3 y_3 = 2 \Rightarrow y_3 = 1$

$y_1 + 3/2\, y_2 = 2 \Rightarrow y_2 = 2/3$ $\quad$ $y_3 + 5/4\, y_4 = 1 \Rightarrow y_4 = 0$

step3: UX = Y

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}$$

$x_1 + ½\, x_2 = 1$

$x_2 + 2/3 x_3 = 2/3$

$x_3 + 3/4 x_4 = 1$

$x_4 = 0$

solving the solution is given by

$x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$

UNIT-II
LINEAR TRANSFORMATION

Eigen Values & Eigen Vectors

Def: Characteristic vector of a matrix:

Let A= [a_{ij}] be an n x n matrix. A non-zero vector X is said to be a Characteristic Vector of A if there exists a scalar such that AX=λX.

Note: If AX=λX (X≠0), then we say 'λ' is the eigen value (or) characteristic root of 'A'.

Eg: Let A= $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ X = $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

AX = $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ = $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1.\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$= 1.X$

Here Characteristic vector of A is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and Characteristic root of A is "1".

Note: We notice that an eigen value of a square matrix A can be 0. But a zero vector cannot be an eigen vector of A.

Method of finding the Eigen vectors of a matrix.

Let A = [a_{ij}] be a nxn matrix. Let X be an eigen vector of A corresponding to the eigen value λ.

Then by definition AX = λX.

⇨ AX = λIX

⇨ AX –λIX = 0

⇨ (A-λI)X = 0 ------- (1)

This is a homogeneous system of n equations in n unknowns.

(1) Will have a non-zero solution X if and only |A-λI| = 0

- A-λI is called characteristic matrix of A
- |A-λI| is a polynomial in λ of degree n and is called the characteristic polynomial of A
- |A-λI|=0 is called the characteristic equation

Solving characteristic equation of A, we get the roots , $\lambda_1, \lambda_2, \lambda_3, \ldots \ldots . \lambda_n,$ These are called the characteristic roots or eigen values of the matrix.

- Corresponding to each one of these n eigen values, we can find the characteristic vectors.

Procedure to find eigen values and eigen vectors

Let A = $\begin{bmatrix} a_{11} & a_{12} \dots \dots & a_{1n} \\ a_{21} & a_{22} \dots . . & a_{2n} \\ \dots \dots . & \dots \dots . & \dots \dots . \\ a_{n1} & a_{n2} \dots \dots . & a_{nn} \end{bmatrix}$ $be\ a\ given\ matrix$

$Characteristic\ matrix\ of\ A\ is\ A - \lambda I$

$$i.e., A-\lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

Then the characterstic polynomial is $|A-\lambda I|$

$$say\phi(\lambda) = |A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix}$$

The characteristic equation is $|A-\lambda I| = 0$ we solve the $\emptyset(\lambda) = |A - \lambda I| = 0$, we get n roots, these are called eigen values or latent values or proper values.

Let each one of these eigen values say λ their eigen vector X corresponding the given value λ is obtained by solving Homogeneous system

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ 0 \end{bmatrix}$$

And determining the non-trivial solution.

PROBLEMS

1. **Find the eigen values and the corresponding eigen vectors of** $\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

$$sol: Let\ A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$$

$$Characteristic\ matrix = A - \lambda \mathrm{I}$$

$$= \begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix}$$

$Characteristic\ equation\ of\ \ A\ is |A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)(2-\lambda) + 8 = 0$$

$$\Rightarrow 16 + \lambda^2 - 10\lambda + 8 = 0$$

$$\Rightarrow \lambda^2 - 10\,\lambda + 24 = 0$$

$$\Rightarrow (\lambda - 6)(\lambda - 4) = 0$$

$$\Rightarrow \lambda = 6, 4 \text{ are eigen values of A}$$

$$Consider\ system\ \begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Eigen vector corresponding to $\lambda = 4$

Put $\lambda = 4$ in the above system, we get

$$\begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow\ 4x_1 - 4x_2 = 0 --- (1)$$

$$2x_1 - 2x_2 = 0 --- (2)$$

$$from\ (1) and\ (2) we\ have\ x_1 = x_2$$

Let $x_1 = \alpha$

$$Eigen\ vector\ is \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}\ is\ a\ Eigen\ vector\ of\ matrix\ A, corresponding\ eigen\ value\ \lambda = 4$$

Eigen Vector corresponding to $\lambda = 6$

$$put\ \lambda = 6 \text{ in the above system, we get}$$

$$\begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow\ 2x_1 - 4x_2 = 0 - - - (1)$$
$$2x_1 - 4x_2 = 0 - - - (2)$$

from (1) and (2) we have $x_1 = 2x_2$

Say $x_2 = \alpha \Rightarrow x_1 = 2\alpha$

$$Eigen\ vector = \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}\ is\ eigen\ vector\ of\ matrix\ A\ corresponding\ eigen\ value\ \lambda = 6$$

2. **Find the eigen values and the corresponding eigen vectors of matrix** $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Sol: Let A = $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

The characteristic equation is |A-λI|=0

i.e. |A-λI| = $\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$

$$\Rightarrow (2-\lambda)(2-\lambda)^2 - 0 + [-(2-\lambda)] = 0$$
$$\Rightarrow (2-\lambda)^3 — (\lambda - 2) = 0$$
$$\Rightarrow \lambda - 2\ [-(\lambda-2)^2 - 1] = 0$$
$$\Rightarrow \lambda - 2\ [-\lambda^2 + 4\lambda - 3] = 0$$
$$\Rightarrow (\lambda-2)(\lambda-3)(\lambda-1) = 0$$

⇨ λ=1,2,3

The eigen values of A is 1,2,3.

$$For\ finding\ eigen\ vector\ the\ system\ is\ (A - \lambda I)X = 0$$
$$\Rightarrow \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$Eigen\ vector\ corresponding\ to\ \lambda = 1$$

$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$x_1 + x_3 = 0$

$x_2 = 0$

$x_1 + x_3 = 0$

$x_1 = -x_3, x_2 = 0$

$say\ x_3 = \alpha$

$x_1 = -\alpha \quad x_2 = 0, \quad x_3 = \alpha$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-\alpha\\0\\\alpha\end{bmatrix}=\alpha\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$\begin{bmatrix}-1\\0\\1\end{bmatrix}$ *is Eigen vector*

Eigen vector corresponding to $\lambda = 2$

$$\begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Here $x_1 = 0$ *and* $x_3 = 0$ *and we can take any arbitary value* x_2 *i.e* $x_2 = \alpha$ *(say)*

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\\alpha\\0\end{bmatrix}=\alpha\begin{bmatrix}0\\1\\0\end{bmatrix}$$

Eigen vector is $\begin{bmatrix}0\\1\\0\end{bmatrix}$

Eigen vector corresponding to $\lambda = 3$

$$\begin{bmatrix}-1 & 0 & 1\\0 & -1 & 0\\1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$-x_1 + x_3 = 0$

$-x_2 = 0$

$x_1 - x_3 = 0$

here by solving we get $x_1 = x_3, x_2 = 0$ *say* $x_3 = \propto$

$x_1 = \propto \ , \quad x_2 = 0 \quad , x_3 = \propto$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}\alpha\\0\\\alpha\end{bmatrix}=\alpha\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$Eigen\ vector\ is\ \begin{bmatrix}1\\0\\1\end{bmatrix}$

Properties of Eigen Values:

Theorem 1: The sum of the eigen values of a square matrix is equal to its trace and product of the eigen values is equal to its determinant.

Proof: Characteristic equation of A is |A-λI|=0

i.e, $\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$ $expanding\ this\ we\ get$

$(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)-a_{12}$ (*a polynomial of degree n – 2*)

+ a_{13} (*a polynomial of degree n -2*) + … = 0

$$\Rightarrow (-1)^n\left[\lambda^n-(a_{11}+a_{22}+....+a_{nn})\lambda^{n-1}+a\ polynomial\ \ of\ \deg ree(n-2)\right]=0$$

$$(-1)^n\lambda^n+(-1)^{n+1}(Trace\,A)\lambda^{n-1}+a\ polynomial\ of\ \deg ree(n-2)\,in\,\lambda=0$$

$$If\ \lambda_1,\lambda_2\ .....\lambda_n\ are\ the\ roots\ of\ this\ equation$$

$sum\ of\ the\ roots=\frac{(-1)^{n+1}Tr(A)}{(-1)^n}=Tr(A)$s

$$Further\ |A-\lambda I|=(-1)^n\lambda^n+\cdots.+a_0$$

$$put\ \lambda=0\ then\ |A|=a_0$$

$$(-1)^n\lambda^n+a_{n-1}\lambda^{n-1}+a_{n-2}\lambda^{n-2}+\ ....+a_0=0$$

$$Product\ of\ the\ roots=\frac{(-1)^n\,a_0}{(-1)^n}=a_0$$

$$but\ a_0=\ |A|=\det A$$

$$Hence\ the\ result$$

Theorem 2: If λ is an eigen value of A corresponding to the eigen vector X, then λ^n is eigen value A^n corresponding to the eigen vector X.

Proof: Since λ is an eigen value of A corresponding to the eigen value X, we have

$AX = \lambda X$ ----------(1)

Pre multiply (1) by A, $A(AX) = A(\lambda X)$

$(AA)X = \lambda(AX)$

$A^2X = \lambda(\lambda X)$

$A^2X = \lambda^2 X$

λ^2 is eigen value of A^2 with X itself as the corresponding eigen vector. Thus the theorm is true for n=2

Let we assume it is true for n = k

i.e,, $A^K X = \lambda^K X$------------(2)

Premultiplying (2) by A, we get

$A(A^k X) = A(\lambda^K X)$

$(AA^K)X = \lambda^K(AX) = \lambda^K(\lambda X)$

$A^{K+1}X = \lambda^{K+1}X$

λ^{K+1} is eigen value of A^{K+1} with X itself as the corresponding eigen vector.

Thus, by Mathematical induction., λ^n is an eigen value of A^n

Theorem 3: A Square matrix A and its transpose A^T have the same eigen values.

Proof: We have $(A - \lambda I)^T = A^T - \lambda I^T$

$= A^T - \lambda I$

$|(A - \lambda I)^T| = |A^T - \lambda I|$ (or)

$|A - \lambda I| = |A^T - \lambda I|$ $\left[\because |A^T| = |A|\right]$

$|A - \lambda I| = 0$ if and only if $|A^T - \lambda I| = 0$

λ is eigen value of A if and only if λ is eigen value of A^T

Hence the theorm

Theorem 4: If A and B are n-rowed square matrices and If A is invertible show that $A^{-1}B$ and B A^{-1} have same eigen values.

Proof: Given A is invertble

i.e, A^{-1} exist

We know that if A and P are the square matrices of order n such that P is non-singular then A and

$P^{-1}AP$ have the same eigen values.

Taking A=B A^{-1} and P=A, we have

B A^{-1} and A^{-1}(B A^{-1})A have the same eigen values

ie.,B A^{-1} and (A^{-1}B)(A^{-1}A) have the same eigen values

ie.,B A^{-1} and (A^{-1}B)I have the same eigen values

ie.,B A^{-1} and A^{-1}B have the same eigen values

Theorem 5: If $\lambda_1, \lambda_2, \ldots\ldots\ldots\lambda_n$ are the eigen values of a matrix A then k λ_1, k λ_2, ….. k λ_n are the eigen value of the matrix KA, where K is a non-zero scalar.

Proof: Let A be a square matrix of order n. Then $|KA- \lambda KI| = |K(A- \lambda I)| = K^n\,|A- \lambda I|$

Since K≠0, therefore $|KA- \lambda KI| = 0$ if and only if $|A - \lambda I| = 0$

$i.e., K\lambda\, is\, an\, eigen\ value\, of\ KA \Leftrightarrow if\ \lambda\, is\, an\, eigen\ value\, of\ A$

Thus k λ_1, k λ_2 … k λ_n *are the eigen svalues of the matrix KA if*

λ_1, λ_2 … λ_n are the eigen values of the matrix A

Theorem 6: If λ is an eigen values of the matrix A then λ+K is an eigen value of the matrix A+KI

Proof: Let λ be an eigen value of A and X the corresponding eigen vector. Then by definition AX= λX

Now (A+KI)X

$= AX + IKX = \lambda X + KX$

$=(\lambda + K)\, X$

$\lambda + K$ is an eigen value of the matrix $A + KI$

Theorem 7: If $\lambda_1,\ \lambda_2 \ldots\ \lambda_n$ are the eigen values of A, then $\lambda_1 - K,\ \lambda_2 - K, \ldots\ \lambda_n - K,$

are the eigen values of the matrix $(A - KI)$, where K is a non – zero scalar

Proof: Since $\lambda_1, \lambda_2, \ldots . \lambda_n$ are the eigen values of A.

The characteristic polynomial of A is

$|A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda)$-----------------------1

Thus the characteristic polynomial of A-KI is

$|(A - KI) - \lambda I| = |A - (k + \lambda)I|$

$= [\lambda_1 - (\lambda + K)]\ [\lambda_2 - (\lambda + K)] \ldots . . [\lambda_n - (\lambda + K)]$
$= [(\lambda_1 - K) - \lambda][(\lambda_2 - K) - \lambda] \ldots . [(\lambda_n - K) - \lambda]$

Which shows that the eigen values of A-KI are $\lambda_1 - K, \lambda_2 - K, \ldots \ldots . . \lambda_n - K$

Theorem 8: If $\lambda_1, \lambda_2 \ldots\ \lambda_n$ are the eigen values of A, find the eigen values of the matrix $(A - \lambda I)^2$

Proof: First we will find the eigen values of the matrix A- λI

Since $\lambda_1, \lambda_2 \ldots\ \lambda_n$ are the eigen values of A

The characteristics polynomial is

$|A - \lambda I| = (\lambda_1 - K)\ (\lambda_2 - K) \ldots . . (\lambda_n - K)\ \ ------(1)\ where\ K\ is\ scalar$

The characteristic polynomial of the matrix (A- λI) is

$|A - \lambda I - KI| = |A - (\lambda + K)I|$

$= [\lambda_1 - (\lambda + K)]\ [\lambda_2 - (\lambda + K)] \dots [\ \lambda_n - (\ \lambda + K)]$

$= [(\lambda_1 - \lambda) - K)]\ [(\lambda_2 - \lambda) - K] \dots [\ (\lambda_n - \ \lambda) - K)]$

Which shows that eigen values of (A- λI) are $\lambda_1 - \lambda, (\lambda_2 - \lambda) \dots . \lambda_n - \lambda$

We know that if the eigen values of A are $\lambda_1, \lambda_2 \dots \ \lambda_n$ then the eigen values of A^2 are $\lambda_1^2, \lambda_2^2 \dots . . \lambda_n^2$

Thus eigen values of $(A - \lambda I)^2\ are (\lambda_1 - \lambda)^2, (\lambda_2 - \lambda)^2, \dots .. (\lambda_n - \lambda)^2$

Theorem 9: If λ is an eigen value of a non-singular matrix A corresponding to the eigen vector X, then λ^{-1} *is an eigen value of* A^{-1} *and corresponding eigen vector X itself.*

Proof: Since A is non-singular and product of the eigen values is equal to |A|, it follows that none of the eigen values of A is 0.

If λ is an eigen vector of the non-singular matrix A and X is the corresponding eigen vector $\lambda \neq 0$ and AX= λX. Premultiplying this with A^{-1}, we get $A^{-1}(AX) = A^{-1}(\ \lambda X)$

$\Rightarrow (A^{-1}A)X = \lambda A^{-1}X \Rightarrow IX = \lambda A^{-1}X$

$\therefore X = \lambda A^{-1}X \Rightarrow A^{-1}X = \lambda^{-1}X\ (\lambda \neq 0)$

Hence λ^{-1} is an eigen value of A^{-1}

Theorem 10: If

λ is an eigen value of a non − singular matrix A, then $\frac{|A|}{\lambda}$ is an eigen value of the matrix Adj A

Proof: Since λ is an eigen value of a non-singular matrix, therfore $\lambda \neq 0$

Also λ is an eigen value of A implies that there exists a non-zero vector X such that AX = λX -----(1)

$\Rightarrow (adj\ A)AX = (Adj\ A)(\lambda X)$

$\Rightarrow [(adj\ A)A]X = \ \lambda(\text{adj A})X$

$\Rightarrow |A|IX = \ \lambda\ (\text{adj A})X \left[\because (adjA)A = |A|I\right]$

$$\Rightarrow \frac{|A|}{\lambda} X = (adj\ A)\ X\ or\ (adj\ A)\ X = \frac{|A|}{\lambda} X$$

Since X is a non – zero vector, therefore the relation (1)

it is clear that $\frac{|A|}{\lambda}$ is an eigen value of the matrix Adj A

Theorem 11: If λ is an eigen value of an orthogonal matrix then $\frac{1}{\lambda}$ is also an eigen value

Proof: We know that if λ is an eigen value of a matrix A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1}

Since A is an orthogonal matrix, therefore $A^{-1} = A^{1}$

$\frac{1}{\lambda}$ is an eigen value of A^{1}

But the matrices A and A^1 have the same eigen values, since the determinants $|A-\lambda I|$ and $|A^1-\lambda I|$ are same.

Hence $\frac{1}{\lambda}$ is also an eigen value of A.

Theorem 12: If λ is eigen value of A then prove that the eigen value of $B = a_0A^2+a_1A+a_2I$ is $a_0\lambda^2+a_1\lambda+a_2$

Proof: If X be the eigen vector corresponding to the eigen value λ, then $AX = \lambda X$ --- (1)

Premultiplying by A on both sides

$$\Longrightarrow A(AX) = A(\lambda X)$$
$$\Longrightarrow A^2X = \lambda(AX) = \lambda(\lambda X) = \lambda^2 X$$

This shows that λ^2 is an eigen value of A^2

we have $B = a_0A^2+a_1A+a_2I$

$\therefore BX = (a_0A^2+a_1A+a_2I)X$

$= a_0A^2X+a_1AX+a_2X$

$= a_0\lambda^2X+a_1\lambda X+a_2X \quad = (a_0\lambda^2+a_1\lambda+a_2)X$

$(a_0\lambda^2+a_1\lambda+a_2)$ is an eigen value of B and the corresponding eigen vector of B is X.

Theorem 14: Suppose that A and P be square matrices of order n such that P is non singular. Then A and $P^{-1}AP$ have the same eigen values.

Proof: Consider the characteristic equation of $P^{-1}AP$

It is $|(P^{-1}AP)-\lambda I| = |P^{-1}AP-\lambda P^{-1}IP|$ $(\because I = P^{-1}P)$

$= |P^{-1}(A-\lambda I)P| = |P^{-1}|\ |A-\lambda I|\ |P|$

$= |A-\lambda I|$ since $|P^{-1}|\ |P| = 1$

Thus the characteristic polynomials of $P^{-1}AP$ and A are same. Hence the eigen values of $P^{-1}AP$ and A are same.

Corollary 1: If A and B are square matrices such that A is non-singular, then $A^{-1}B$ and BA^{-1} have the same eigen values.

Proof: In the previous theorem take BA^{-1} in place of A and A in place of B.

We deduce that $A^{-1}(BA^{-1})A$ and (BA^{-1}) have the same eigen values.

i.e, $(A^{-1}B)(A^{-1}A)$ and BA^{-1} have the same eigen values.

i.e, $(A^{-1}B)I$ and BA^{-1} have the same eigen values

i.e, $A^{-1}B$ and BA^{-1} have the same eigen values

Corollary 2: If A and B are non-singular matrices of the same order, then AB and BA have the same eigen values.

Proof: Notice that $AB=IAB = (B^{-1}B)(AB) = B^{-1}(BA)B$

Using the theorem above BA and $B^{-1}(BA)B$ have the same eigen values.

i.e, BA and AB have the same eigen values.

Theorem 15: The eigen values of a triangular matrix are just the diagonal elements of the matrix.

Proof: Let A = $\begin{vmatrix} a_{11} & a_{12}\dots\dots & a_{1n} \\ 0 & a_{22}\dots\dots & a_{2n} \\ \dots\dots & \dots\dots & \dots\dots \\ 0 & 0\dots\dots & a_{nn} \end{vmatrix}$ be a triangular matrix of order n

The characteristic equation of A is $|A-\lambda I|=0$

i.e., $\begin{vmatrix} a_{11-\lambda} & a_{12}\dots\dots & a_{1n} \\ 0 & a_{22-\lambda}\dots\dots & a_{2n} \\ \dots\dots & \dots\dots & \dots\dots \\ 0 & 0\dots\dots & a_{nn-\lambda} \end{vmatrix} = 0$

i.e, $(a_{11}-\lambda)(a_{22}-\lambda)\dots\dots(a_{nn}-\lambda)=0$

$\Longrightarrow \lambda = a_{11}, a_{22}, \dots. a_{nn}$

Hence the eigen values of A are $a_{11}, a_{22}, \dots. a_{nn}$ which are just the diagonal elements of A.

Note: Similarly we can show that the eigen values of a diagonal matrix are just the diagonal elements of the matrix.

Theorem 16: The eigen values of a real symmetric matrix are always real.

Proof: Let λ be an eigen value of a real symmetric matrix A and Let X be the corresponding eigen vector then AX= $\lambda X - - - -(1)$

Take the conjugate $\bar{A}\bar{X} = \bar{\lambda}\,\bar{X}$

Taking the transpose $\bar{X}^T(\bar{A})^T = \bar{\lambda}\,\bar{X}^T$

Since $\bar{A} = A \text{ and } A^T = A, \text{we have } \bar{X}^T A = \bar{\lambda}\,\bar{X}^T$

Post multiplying by X, we get $\bar{X}^T AX = \bar{\lambda}\,\bar{X}^T X$------- (2)

Premultiplying (1) with $\bar{X}^T$, we get $\bar{X}^T AX = \lambda \bar{X}^T X$ ------ (3)

$(2)-(3)$ gives $(\lambda-\bar{\lambda})\bar{X}^T X = 0$ but $\bar{X}^T X \neq 0 \Rightarrow \lambda - \bar{\lambda} = 0$

$\Rightarrow \lambda - \bar{\lambda} \Rightarrow \lambda$ is real. Hence the result follows

Theorem 17: For a real symmetric matrix, the eigen vectors corresponding to two distinct eigen values are orthogonal.

Proof: Let λ_1, λ_2 be eigen values of a symmetric matrix A and let X_1, X_2 be the corresponding eigen vectors.

Let $\lambda_1 \neq \lambda_2$. We want to show that X_1 is orthogonal to X2 (i.e., $X_1^T X_2 = 0$)

Sice X_1, X_2 are eigen values of A corresponding to the eigen values λ_1, λ_2 we have

$AX_1 = \lambda_1 X_1$ ----- (1) $AX_2 = \lambda_2 X_2$ ------- (2)

Premultiply (1) by X_2^T

$\Longrightarrow \; X_2^T A X_1 = \lambda_1 X_2^T X_1$

Taking transpose to above, we have

$\Rightarrow X_2^T A^T \left(X_2^T\right)^T = \lambda_1 X_1^T \left(X_2^T\right)^T$

$i.e, X_1^T A X_2 = \lambda_1 X_1^T X_2$-------- (3) s

Premultiplying (2) by $X_1^T, we\ get\ X_1^T A X_2 = \lambda_2 X_1^T X_2 - - - - - (4)$

Hence from (3) and (4) we get

$(\lambda_1 - \lambda_2) X_1^T X_2 = 0$

$\Longrightarrow \; X_1^T X_2 = 0$

$(\because \lambda_1 \neq \lambda_2)$

$X_1\ is\ orthogonal\ to\ X_2$

Note: If λ is an eigen value of A and f(A) is any polynomial in A, then the eigen value of f(A) is f(λ)

PROBLEMS

1. Find the eigen values and eigen vectors of the matrix A and its inverse, where

   $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

Sol: Given A = $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

The characteristic equation of A is given by |A-λI| = 0

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)] = 0$

$\Rightarrow \lambda = 1, 2, 3$

$Characteristic\ roots\ are\ 1, 2, 3$

$Characteristic\ vector\ for\ \lambda = 1$

$For\ \lambda = 1, becomes \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow 3x_2 + 4x_3 = 0$

$x_2 + 5x_3 = 0$

$2x_3 = 0$

$x_2 = 0, x_3 = 0$ and $x_1 = \alpha$

$X = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the solution where α is arbritary constant $\therefore X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Is the eigen vector corresponding to $\lambda = 1$

$Characteristic\ vector\ for\ \lambda = 2$

$For\ \lambda = 2, becomes \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -x_1 + 3x_2 + 4x_3 = 0$

$5x_3 = 0 \Rightarrow x_3 = 0$

$$-x_1+3x_2=0 \Rightarrow x_1=3x_2$$

$$Let\, x_2=k$$

$$x_1=3k$$

$$X=\begin{bmatrix}3k\\k\\0\end{bmatrix}=k\begin{bmatrix}3\\1\\0\end{bmatrix}$$

is the solution $\therefore X=\begin{bmatrix}3\\1\\0\end{bmatrix}$

Is the eigen vector corresponding to $\lambda=2$

Hence the characteristic vector is $\begin{bmatrix}1\\0\\0\end{bmatrix}$

Characteristic vector for $\lambda=3$

$$For\ \lambda=3, becomes\begin{bmatrix}-2&3&4\\0&-1&5\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Longrightarrow\ -2x_1+3x_2+4x_3=0$$

$$-x_2+5x_3=0$$

$$Say\ x_3=K\ \Longrightarrow\ x_2=5K$$

$$x_1=\frac{19}{2}K$$

X = $\begin{bmatrix}\frac{19}{2}K\\5K\\K\end{bmatrix}=\frac{K}{2}\begin{bmatrix}19\\10\\2\end{bmatrix}$

$\therefore X=\begin{bmatrix}19\\10\\2\end{bmatrix}$ is the eigen vector corresponding to $\lambda=3$

Eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$

$\Rightarrow$ *Eigen values of* A^{-1} *are* $1, \frac{1}{2}, \frac{1}{3}$

We know Eigen vectors of A^{-1} are same as eigen vectors of A.

2. *Find the Eigen values of* $3A^3 + 5A^2 - 6A + 2I$ *where* $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Sol: The characteristic equation of A is $|A - \lambda I| = 0$

i.e, $\begin{vmatrix} 1-\lambda & 2 & -3 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$

$\Longrightarrow [(1-\lambda)(3-\lambda)(-2-\lambda) - 0] = 0$

$\Longrightarrow (1-\lambda)(3-\lambda)(2+\lambda) = 0 \qquad \lambda = 1,3,-2$

Eigen values of A are $1,3,-2$

We know that if λ *is an eigen value of A and* $f(A)$ *is a polynomial in A.*

then the eigen value of $f(A)$ *is* $f(\lambda)$

Let f(A) = $3A^3 + 5A^2 - 6A + 2I$

Then eigen values of f(A) are f(1), f(3) and f(-2)

$f(1) = 3(1)^3+5(1)^2-6(1)+2(1) = 4$

$f(3) = 3(3)^3+5(3)^2-6(3)+2(1) = 110$

$f(-2) = 3(-2)^3+5(-2)^2-6(-2)+2(1) = 10$

Eigen values of $3A^3 + 5A^2 - 6A + 2I$ are 4,110,10

Diagonalization of a matrix:

Theorem: If a square matrix A of order n has n linearly independent eigen vectors $(X_1,X_2 \ldots X_n)$ corresponding to the n eigen values $\lambda_1,\lambda_2 \ldots \lambda_n$ respectively then a matrix P can be found such that

$P^{-1}AP$ is a diagonal matrix.

Proof: Given that $(X_1,X_2 \ldots X_n)$ be eigen vectors of A corresponding to the eigen values $\lambda_1,\lambda_2 \ldots \lambda_n$ respectively and these eigen vectors are linearly independent Define $P = (X_1,X_2 \ldots X_n)$ Since the n columns of P are linearly independent $|P| \neq 0$

Hence P^{-1} exists

Consider AP = A$[X_1, X_2 \ldots X_n]$

= $[AX_1, AX_2 \ldots\ldots AX_n]$

= $[\lambda X_1, \lambda_2 X_2 \ldots\ldots \lambda_n X_n]$

$$[X_1, X_2 \ldots X_n]\begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

$= PD$

Where D = diag $(\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\ldots\ldots \lambda_n)$

AP=PD

$P^{-1}(AP) = P^{-1}(PD) \Rightarrow P^{-1}AP = (P^{-1}P)D$

$\Rightarrow P^{-1}AP = (I)D$

$= D$

= diag $(\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\ldots\ldots \lambda_n)$

Hence the theorem is proved.

Modal and Spectral matrices:

The matrix P in the above result which diagonalize the square matrix A is called modal matrix of A and the resulting diagonal matrix D is known as spectral matrix.

Note 1: If $X_1, X_2 \ldots X_n$ are not linearly independent this result is not true.

2: Suppose A is a real symmetric matrix with n pair wise distinct eigen values $\lambda_1, \lambda_2 \cdots \lambda_n$ then the corresponding eigen vectors $X_1, X_2 \ldots X_n$ are pairwise orthogonal.

Hence if P = $(e_1, e_2 \ldots e_n)$

Where $e_1 = (X_1 / \|X_1\|)$, $e_2 = (X_2 / \|X_2\|) \ldots . e_n = (X_n)/ \|X_n\|$

then P will be an orthogonal matrix.

i.e, $P^TP=PP^T=I$

Hence $P^{-1} = P^T$

$P^{-1}AP = D \Longrightarrow P^TAP=D$

Calculation of powers of a matrix:

We can obtain the power of a matrx by using diagonalization

Let A be the square matrix then a non-singular matrix P can be found such that $D = P^{-1}AP$

$D^2=(P^{-1}AP)(P^{-1}AP)$

$= P^{-1}A(PP^{-1})AP$

$= P^{-1}A^2P$ (since $PP^{-1}=I$)

Simlarly $D^3 = P^{-1}A^3P$

In general $D^n = P^{-1}A^nP$……..(1)

To obtain A^n, Premultiply (1) by P and post multiply by P^{-1}

Then $PD^nP^{-1} = P(P^{-1}A^nP)P^{-1}$

$= (PP^{-1})A^n(PP^{-1}) = A^n \Rightarrow A^n = PD^nP^{-1}$

Hence $A^n = P\begin{bmatrix} \lambda_1^n & 0 & 0\cdots & 0 \\ 0 & \lambda_2^n & 0\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \lambda_n^n \end{bmatrix}P^{-1}$

PROBLEMS

1. Determine the modal matrix P of $= \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. Verify that $P^{-1}AP$ is a diagonal matrix.

Sol: The characteristic equation of A is $|A-\lambda I| = 0$

i.e, $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

$which\ gives\ (\lambda-5)(\lambda+3)^2 = 0$

Thus the eigen values are $\lambda=5$, $\lambda=-3$ and $\lambda=-3$

when λ=5 $\Longrightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By solving above we get $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Similarly, for the given eigen value λ=-3 we can have two linearly independent eigen vectors $X_2 =$ $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $and\ X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

$P = (X_1\ X_2\ \ X_3)$

$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = modal\ matrix\ of\ A$

$Now \det P = 1(-1) - 2(2) + 3(0-1) = -8$

$P^{-1} = \dfrac{adj\ P}{\det P} = -\dfrac{1}{8} \begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}$

$= -\frac{1}{8} \begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

$= -\dfrac{1}{8} \begin{bmatrix} -5 & -10 & 15 \\ 6 & -12 & -18 \\ 3 & 6 & 15 \end{bmatrix}$

$P^{-1}AP = -\frac{1}{8} \begin{bmatrix} -40 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$

$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = diag\ (5, -3, -3)$

$Hence\ P^{-1}AP$ is a diagonal matrix.

2. Find a matrix P which transform the matrix A = $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ $to\ diagonal\ form. Hence\ calculate\ A^4$

Sol: Characteristic equation of A is given by |A-λI| = 0 i.e, $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$

$\Longrightarrow (1-\lambda)[(2-\lambda)(3-\lambda)-2]-0-1[2-2(2-\lambda0\} = 0$

$\Longrightarrow (\lambda-1)(\lambda-2)(\lambda-3) = 0$

$\Longrightarrow \lambda = 1, \lambda = 2, \lambda = 3$

$Thus\ the\ eigen\ values\ of\ A\ are\ 1, 2, 3$

If x_1, x_2, x_3 be the components of an eigen vector corresponding to the eigen value λ, we have

$[A\text{-}\lambda I]X = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$For\ \lambda = 1, eigen\ vectors\ are\ given\ by$

$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ i.e, $0.x_1+0.x_2+0.x_3=0$ and $x_1+x_2+x_3=0$

$x_3=0$ and $x_1+x_2+x_3=0$

$x_3=0$, $x_1=-x_2$

$x_1=1$, $x_2=-1$, $x_3=0$

Eigen vector is $[1,-1,0]^T$

Also every non-zero multiple of this vector is an eigen vector corresponding to λ=1

For λ=2, λ=3 we can obtain eigen vector $[-2,1,2]^T$ and $[-1,1,2]^T$

$P = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

The Matrix P is called modal matrix of A

$$P^{-1} = -\frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

$$Now\, P^{-1}AP = \begin{bmatrix} 0 & -1 & \frac{1}{2} \\ -1 & -1 & 0 \\ 1 & 1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - D\ (say)$$

$$A^4 = PD^4P^{-1}$$

$$= \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}\begin{bmatrix} 0 & -1 & \frac{-1}{2} \\ -1 & 1 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$$

Cayley - Hamilton Theorem:

Every square matrix satisfies its own characterstic equation

PROBLEMS

1. Show that the matrix A = $\begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ satisfies its characteristic equation Hence find A^{-1}

Sol: Characteristic equation of A is det (A-λI) = 0

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 & 2 \\ 1 & -2-\lambda & 3 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0 \quad C2 \rightarrow C2+C3$$

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1-\lambda & 3 \\ 0 & 1-\lambda & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

By Cayley – Hamilton theorem, we have $A^3 - A^2 + A - I = 0$

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} \quad A^2 = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \quad A^3 = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$A^3 - A^2 + A - I = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Multiplying with A^{-1} we get $A^2 - A + I = A^{-1}$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

2. Using Cayley - Hamilton Theorem find the inverse and A^4 of the matrix A = $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

   Sol: Let A = $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

The characteristic equation is given by $|A-\lambda I|=0$

$$i.e., \begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 6 & 2 & -(1+\lambda) \end{vmatrix} = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley – Hamilton theorem we have $A^3-5A^2+7A-3I=0$.....(1)

Multiply with A^{-1} we get

$A^{-1} = \frac{1}{3}\,[A^2 - 5A + 7I]$

$$A^2 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} \quad A^3 = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

$Multiply\ (1) with\ A, we\ get$

$$A^4 - 5A^3 + 7A^2 - 3A = 0$$

$$= \begin{bmatrix} 395 & 130 & -130 \\ -390 & -125 & 130 \\ 390 & 130 & -125 \end{bmatrix} - \begin{bmatrix} 175 & 56 & -56 \\ -168 & -49 & 56 \\ 168 & 56 & -69 \end{bmatrix} + \begin{bmatrix} 21 & 6 & -6 \\ -18 & -3 & 6 \\ 18 & 6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 241 & 80 & -80 \\ -240 & -79 & 80 \\ 240 & 80 & -79 \end{bmatrix}$$

<u>Problems</u>

1. Diagonalize the matrix (i) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

3. Verify Cayley – Hamilton Theorem for A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$. Hence find A^{-1}.

Linear dependence and independence of Vectors :

1. Show that the vectors (1,2,3), (3,-2,1), (1,-6,-5) from a linearly dependent set.

Sol. The Given Vector $X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} X_2 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} X_3 = \begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix}$

The Vectors X_1, X_2, X_3 from a square matrix.

Let $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{bmatrix}$

Then $|A| = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{bmatrix}$

= 1(10+6)-2(15-1)+3(-18+2)

=16+32-48=0

The given vectors are linearly dependent $\because |A|=0$

2. Show that the Vector X_1=(2,2,1), X_2=(1,4,-1) and X_3=(4,6,-3) are linearly independent.

Sol. Given Vectors X_1=(2,-2,1) X_2=(1,4,-1) and X_3=(4,6,-3) The Vectors X_1, X_2, X_3 form a square matrix.

$A = \begin{bmatrix} 2 & 1 & 4 \\ -2 & 4 & 6 \\ 1 & -1 & -3 \end{bmatrix}$

Then $|A| = \begin{bmatrix} 2 & 1 & 4 \\ -2 & 4 & 6 \\ 1 & -1 & -3 \end{bmatrix}$

=2(-12+6)+2(-3+4)+1(6-16)

$=-20 \neq 0$

$\therefore$ The given vectors are linearly independent

$\therefore$ $|A| \neq 0$

Real and complex matrices

Conjugate of a matrix:

If the elements of a matrix A are replaced by their conjugates then the resulting matrix is defined as the conjugate of the given matrix. We denote it with $\bar{A}$

e.g If A=$\begin{bmatrix} 2+3i & 5 \\ 6-7i & -5+i \end{bmatrix}$ then $\bar{A}=\begin{bmatrix} 2-3i & 5 \\ 6+7i & -5-i \end{bmatrix}$

The transpose of the conjugate of a square matrix:

If A is a square matrix and its conjugate is $\bar{A}$, then the transpose of $\bar{A}$ is $\left(\bar{A}\right)^T$.

It can be easily seen that $\left(\bar{A}\right)^T = \overline{A^T}$

It is denoted by A^θ

$A^\theta = \left(\bar{A}\right)^T = \overline{A^T}$

Note: If A^θ and B^θ be the transposed conjugates of A and B respectively, then

i) $\left(A^\theta\right)^\theta = A$ ii) $\left(A \pm B\right)^\theta = A^\theta \pm B^\theta$ iii) $\left(KA\right)^\theta = \bar{K}A^\theta$ iv) $\left(AB\right)^\theta = B^\theta A^\theta$

Hermitian matrix:

A square matrix A such that $\bar{A} = A^T$ (or) $\left(\bar{A}\right)^T$ =A is called a hermitian matrix

e.g A=$\begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$ then $\bar{A}=\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ and A^θ=$\begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$

Here $\left(\bar{A}\right)^T$ =A, Hence A is called Hermitian

Note:

1) The element of the principal diagonal of a Hermitian matrix must be real
2) A hermitian matrix over the field of real numbers is nothing but a real symmetric.

Skew-Hermitian matrix

A square matrix A such that A^T =-$\bar{A}$ (or) $\left(\bar{A}\right)^T$ =-A is called a Skew-Hermitian matrix

e.g. Let A=$\begin{bmatrix} -3i & 2+i \\ -2+i & -i \end{bmatrix}$ then $\bar{A}=\begin{bmatrix} 3i & 2-i \\ -2-i & i \end{bmatrix}$ and $\left(\bar{A}\right)^T = \begin{bmatrix} 3i & -2-i \\ 2-i & i \end{bmatrix}$

$\therefore$ $\left(\bar{A}\right)^T$ =-A

A is skew-Hermitian matrix.

Note:
1) The elements of the leading diagonal must be zero (or) all are purely imaginary
2) A skew-Hermitian matrix over the field of real numbers is nothing but a real skew-symmetric matrix.

Unitary matrix:

A square matrix A such that $\left(\overline{A}\right)^T = A^{-1}$

i.e $\left(\overline{A}\right)^T$ A=A$\left(\overline{A}\right)^T$ =I

If A^θ A=I then A is called Unitary matrix

Theorem: The Eigen values of a Hermitian matrix are real.

Proof: Let A be Hermitian matrix. If X be the Eigen vector corresponding to the eigen value λ of A, then AX = λ X -------------------- (1)

Pre multiplying both sides of (1) by X^θ,we get

$X^\theta AX = \lambda X^\theta X$ ----------------------- (2)

Taking conjugate transpose of both sides of (2)

We get $\left(X^\theta AX\right)^\theta = \left(\lambda X^\theta X\right)^\theta$

i.e $X^\theta A^\theta \left(X^\theta\right)^\theta = \overline{\lambda} X^\theta \left(X^\theta\right)^\theta \left[\because (ABC)^\theta = C^\theta B^\theta A^\theta \text{ and } (KA)^\theta = \overline{K} A^\theta\right]$

(or) $X^\theta A^\theta X = \overline{\lambda} X^\theta X \left[\because \left(X^\theta\right)^\theta = X, \left(A^\theta\right)^\theta = A\right]$-------------------- (3)

From (2) and (3), we have

$\lambda X^\theta X = \overline{\lambda} X^\theta X$

i.e $\left(\lambda - \overline{\lambda}\right) X^\theta X = 0 \Rightarrow \lambda - \overline{\lambda} = 0$

$\Rightarrow \lambda = \overline{\lambda} \left(\because X^\theta X \neq 0\right)$

$\therefore$ Hence λ is real.

Note: The Eigen values of a real symmetric are all real

Corollary: The Eigen values of a skew-Hermitian matrix are either purely imaginary (or) Zero

Proof: Let A be the skew-Hermitian matrix

If X be the Eigen vector corresponding to the Eigen value λ of A, then

$AX = \lambda X (or) (iA) X = (i\lambda) X$

From this it follows that $i\lambda$ is an Eigen value of iA

Which is Hermitian (since A is skew-hermitian)

$\therefore A^{\theta} = -A$

Now $(iA)^{\theta} = \bar{i}A^{\theta} = -iA^{\theta} = -i(-A) = iA$

Hence $i\lambda$ is real. Therefore λ must be either

Zero or purely imaginary.

Hence the Eigen values of skew-Hermitian matrix are purely imaginary or zero

<u>**Theorem 3:**</u> The Eigen values of an unitary matrix have absolute value l.

Proof: Let A be a square unitary matrix whose Eigen value is λ with corresponding eigen vector X

$\Rightarrow \quad AX = \lambda X \rightarrow (1)$

$\Rightarrow \quad \overline{A}\overline{X} = \overline{\lambda}\overline{X} \;\Rightarrow \overline{X}^{T}\overline{A}^{T} = \overline{\lambda}X^{T} \rightarrow (2)$

Since A is unitary, we have $\left(\overline{A}\right)^{T} A = I \rightarrow (3)$

(1) and (2) given $\overline{X}^{T}\overline{A}^{T} \quad (AX) = \lambda\overline{\lambda}\overline{X}^{T}X$

i.e $\quad \overline{X}^{T}X = \lambda\overline{\lambda}\overline{X}^{T}X$ From (3)

$\Rightarrow \overline{X}^{T}X\left(1-\lambda\overline{\lambda}\right) = 0$

Since $\overline{X}^{T}X \neq 0$,we must have $1-\lambda\overline{\lambda} = 0$

$\Rightarrow \lambda\overline{\lambda} = 1$

Since $|\lambda| = \overline{|\lambda|}$

We must have $|\lambda|=1$

Note 1: From the above theorem, we have "The characteristic root of an orthogonal matrix is unit modulus".

2. The only real eigen values of unitary matrix and orthogonal matrix can be $\pm$ 1

Theorem 4: Prove that transpose of a unitary matrix is unitary.

Proof: Let A be a unitary matrix

Then $A.A^{\theta} = A^{\theta}.A = I$

Where A^{θ} is the transposed conjugate of A.

$$\therefore \left(AA^{\theta}\right)^{T} = \left(A^{\theta}A\right)^{T} = \left(I\right)^{T}$$

$$\therefore \left(AA^{\theta}\right)^{T} = \left(A^{\theta}A\right)^{T} = \left(I\right)^{T}$$

$$\Rightarrow \left(A^{\theta}\right)^{T} A^{T} = A^{T}\left(A^{\theta}\right)^{T} = I$$

$$\Rightarrow \left(A^{T}\right)^{\theta} A^{T} = A^{T}\left(A^{T}\right)^{\theta} = I$$

Hence A^{T} is a unitary matrix.

PROBLEMS

1) Find the eigen values of A= $\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

Sol: we have A= $\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

So $\overline{A} = \begin{bmatrix} -3i & 2-i \\ -2-i & i \end{bmatrix}$ and $A^{T} = \begin{bmatrix} 3i & -2+i \\ 2+i & -i \end{bmatrix}$

$\Rightarrow \overline{A} = -A^{T}$

Thus A is a skew-Hermitian matrix.

$\therefore$ The characteristic equation of A is $|A-\lambda I| = 0$

$$\Rightarrow A^{T} = \begin{vmatrix} 3i-\lambda & -2+i \\ -2+i & -i-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^{2} - 2i\lambda + 8 = 0$

$\Rightarrow \lambda = 4i, -2i$ are the Eigen values of A

2) Find the eigen values of $A = \begin{bmatrix} \frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i \end{bmatrix}$

$$Now\ \overline{A} = \begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix} and$$

$$\left(\overline{A}\right)^{T} = \begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix}$$

We can see that $\overline{A}^{T}.A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Thus A is a unitary matrix

$\therefore$ The characterstic equation is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} \frac{1}{2}i-\lambda & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i-\lambda \end{vmatrix} = 0$$

Which gives $\lambda = \frac{\sqrt{3}}{2} + i\frac{1}{2}\, and\, \frac{-\sqrt{3}}{2} + \frac{1}{2}i$ and

$$\lambda = 1/2\sqrt{3} + 1/2i$$

Hence above λ values are Eigen values of A.

3) **If A=**$\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ **then show that**

A is Hermitian and iA is skew-Hermitian.

Sol: Given A=$\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ then

$\overline{A} = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ -2+5i & 3+i & 4 \end{bmatrix}$ And $\left(\overline{A}\right)^T = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$

$\therefore A = \left(\overline{A}\right)^T$ Hence A is Hermitian matrix.

Let B= iA

i.e B=$\begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix}$ then

$$\overline{B} = \begin{bmatrix} -3i & 4-7i & -5+2i \\ -4-7i & 2i & -1-3i \\ 5+2i & 1-3i & -4i \end{bmatrix}$$

$$\left(\overline{B}\right)^T = \begin{bmatrix} -3i & -4-7i & 5+2i \\ 4-7i & 2i & 1-3i \\ -5+2i & -1-3i & -4i \end{bmatrix} = (-1)\begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix} = -B$$

$\therefore \left(\overline{B}\right)^T$ =-B

$\therefore$ B= iA is a skew Hermitian matrix.

4) If A and B are Hermitian matrices, prove that AB-BA is a skew-Hermitian matrix.

Sol: Given A and B are Hermitan matrices

$\therefore \left(\overline{A}\right)^T = A$ And $\left(\overline{B}\right)^T = B$ ------------- (1)

Now $\overline{(AB-BA)}^T = \left(\overline{AB}-\overline{BA}\right)^T$

$$= \left(\overline{A}\,\overline{B}-\overline{B}\,\overline{A}\right)^T$$

$$= \left(\overline{A}\,\overline{B}\right)^T - \left(\overline{B}\,\overline{A}\right)^T = \left(\overline{B}\right)^T\left(\overline{A}\right)^T - \left(\overline{A}\right)^T\left(\overline{B}\right)^T$$

$$= BA - AB \text{ (By (1))}$$

$$= -(AB-BA)$$

Hence AB-BA is a skew-Hemitian matrix.

5) Show that A=$\begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ is unitary if and only if $a^2+b^2+c^2+d^2=1$

Sol: Given A=$\begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$

Then $\overline{A} = \begin{bmatrix} a-ic & -b-id \\ b-id & a+ic \end{bmatrix}$

Hence $A^\theta = \left(\overline{A}\right)^T = \begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$

$$\therefore AA^\theta = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}\begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$$

$$= \begin{pmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{pmatrix}$$

$\therefore AA^\theta = I$ if and only if $a^2+b^2+c^2+d^2=1$

6) **Show that every square matrix is uniquely expressible as the sum of a Hermitian matrix and a skew- Hermitian matrix.**

Sol. Let A be any square matrix

Now $\left(A+A^{\theta}\right)^{\theta}=A^{\theta}+\left(A^{\theta}\right)^{\theta}$

$$=A^{\theta}+A$$

$\left(A+A^{\theta}\right)^{\theta}=A+A^{\theta}\Rightarrow A+A^{\theta}$ is a Hermitian matrix.

$\therefore \frac{1}{2}\left(A+A^{\theta}\right)$ is also a Hermitian matrix

Now $\left(A-A^{\theta}\right)^{\theta}=A^{\theta}-\left(A^{\theta}\right)^{\theta}$

$$=A^{\theta}-A=-\left(A-A\right)^{\theta}$$

Hence $A-A^{\theta}$ is a skew-Hermitian matrix

$\therefore \frac{1}{2}\left(A-A^{\theta}\right)$ is also a skew –Hermitian matrix.

Uniqueness:

Let A =R+S be another such representation of A

Where R is Hermitian and

S is skew-Hermitian

Then $A^{\theta}=\left(R+S\right)^{\theta}$

$$=R^{\theta}+S^{\theta}$$

$$=R-S \quad \left(\because R^{\theta}=R, S^{\theta}=-S\right)$$

$\therefore R=\frac{1}{2}\left(A+A^{\theta}\right)=P \text{ and } S=\frac{1}{2}\left(A-A^{\theta}\right)=Q$

Hence P=R and Q=S

Thus the representation is unique.

7) Given that A=$\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, show that $(I-A)(I+A)^{-1}$ is a unitary matrix.

Sol: we have $I-A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}-\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$

$$=\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \text{ And}$$

$$I+A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$\therefore (I+A)^{-1}=\frac{1}{1-\left(4i^2-1\right)}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$=\frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

Let $B=(I-A)(I+A)^{-1}$

$$B=\frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}=\frac{1}{6}\begin{bmatrix} 1+(1-2i)(-1-2i) & -1-2i-1-2i \\ 1-2i+1-2i & (-1-2i)(1-2i)+1 \end{bmatrix}$$

$$B=\frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$$

Now $\overline{B}=\frac{1}{6}\begin{bmatrix} -4 & -2+4i \\ 2+4i & -4 \end{bmatrix}$ and $\overline{(B)}^T=\frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$

$$B\left(\overline{B}\right)^T=\frac{1}{36}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$$

$$=\frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$$

$$\left(\overline{B}\right)^T=B^{-1}$$

i.e., B is unitary matrix.

$\therefore (I-A)(I+A)^{-1}$ is a unitary matrix.

8) Show that the inverse of a unitary matrix is unitary.

Sol: Let A be a unitary matrix. Then $AA^{\theta} = I$

i.e $\left(AA^{\theta}\right)^{-1} = I^{-1}$

$\Rightarrow \left(A^{\theta}\right)^{-1} A^{-1} = I$

$\Rightarrow \left(A^{-1}\right)^{\theta} A^{-1} = I$

Thus A^{-1} is unitary.

UNIT-III
DIFFERENTIAL EQUATIONS OF FIRST ORDER AND THEIR APPLICATIONS

ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER

& FIRST DEGREE

Definition: An equation which involves differentials is called a Differential equation.

Ordinary differential equation: An equation is said to be ordinary if the derivatives have reference to only one independent variable.

Ex . (1) $\frac{dy}{dx} + 7xy = x^2$ (2) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^x$

(1) **Partial Differential equation:** A Differential equation is said to be partial if the derivatives in the equation have reference to two or more independent variables.

E.g: 1. $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4z$

2. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$

Order of a Differential equation: A Differential equation is said to be of order 'n' if the n^{th} derivative is the highest derivative in that equation.

E.g : (1). $(x^2+1) . \frac{dy}{dx} + 2xy = 4x^2$

Order of this Differential equation is 1.

(2) $x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = e^x$

Order of this Differential equation is 2.

(3). $\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^2 + 2y = 0$.

Order=2 , degree=1.

(4). $\frac{\partial^2 u}{\partial x^2}$ + $\frac{\partial^2 u}{\partial y^2}$ =0. Order is 2.

Degree of a Differential equation: Degree of a differential Equation is the highest degree of the highest derivative in the equation, after the equation is made free from radicals and fractions in its derivations.

E.g : 1) $y = x \cdot \frac{dy}{dx} + \sqrt{1 + (\frac{dy}{dx})^2}$ on solving we get

$(1-x^2)\,(\frac{dy}{dx})^2 + 2xy \cdot \frac{dy}{dx} + (1 - y^2) = 0$. Degree = 2

2) $a. \frac{d^2 y}{dx^2} = [1 + (\frac{dy}{dx})^2]^{3/2}$ on solving . we get

$a^2 . (\frac{d^2 y}{dx^2})^2 = [1 + (\frac{dy}{dx})^2]^3$. Degree = 2

Formation of Differential Equation : In general an O.D Equation is Obtained by eliminating the arbitrary constants c_1,c_2,c_3--------c_n from a relation like $\Phi(x, y, c_1, c_2, ...c_n) = 0$. ------(1).

Where c_1,c_2,c_3,---------c_n are arbitrary constants.

Differentiating (1) successively w.r.t x, n- times and eliminating the n-arbitrary constants c_1,c_2,----c_n from the above (n+1) equations, we obtain the differential equation F(x , y, y_1, y_2,-----) =0.

PROBLEMS

1.Obtain the Differential Equationy= Ae^{-2x} + B e^{5x} **by Eliminating the arbitrary Constants:**

Sol. y= Ae^{-2x} + B e^{5x} ---------------------(1).

$y_1 = A(-2)e^{-2x}$ + B(5) e^{5x} ----------(2).

y_2 = A (4) . e^{-2x} + B(25) e^{5x} ----------(3).

Eliminating A and B from (1), (2) & (3).

$$\Rightarrow \begin{vmatrix} e^{-2x} & e^{5x} & -y \\ (-2)e^{-2x} & 5e^{5x} & -y_1 \\ (4).e^{-2x} & 25e^{5x} & -y_2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & y \\ (-2) & 5 & y_1 \\ 4 & 25 & y_2 \end{vmatrix} = 0$$

⇨ $y_2 - 3y_1$-10y =0.

The required D. Equation obtained by eliminating A & B is

y_2- $3y_1$ -10y = 0

2). Log $\left(y/x\right) = cx$

Sol: Log $\left(y/x\right) = cx$ -------------(1).

=> log y –log x= cx

=>$\frac{1dy}{ydx}$ $-\frac{1}{x}$ =c ---------------(2).

(2) in (1) => Log $\left(y/x\right)$ =x[$\frac{1dy}{ydx}$ $-\frac{1}{x}$].

3) $\sin^{-1} x + \sin^{-1} y$ = c.

Sol: Given equation) $\sin^{-1} x + \sin^{-1} y$ = c

$$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

⇨ $$\frac{dy}{dx} = \frac{-\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

4) y = e^x[Acosx +B sinx]

Sol: Given equation is y = e^x[Acosx +B sinx]

$\frac{dy}{dx}$ = e^x[Acosx +B sinx] + e^x[-Asinx +B cosx]

=> $\frac{dy}{dx}$ = y + e^x(-Asinx +B cosx).

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-A\sin x + B\cos x) + e^x(-A\cos x - B\sin x)$$

$$\frac{dy}{dx} + \frac{dy}{dx} - y - y$$

$$= \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0 \text{ is required equation}$$

5) y= a $\tan^{-1} x$ + b.

Sol: $\frac{dy}{dx} = \frac{a}{1+x^2}$

=> $(1+x^2)\,.\,\frac{d^2y}{dx^2}$ + 2x.$\frac{dy}{dx}$ =0

=> $(1+x^2)\,.\,\frac{d^2y}{dx^2}$ + 2x.$\frac{dy}{dx}$ =0 is the required equation.

6) y=a e^x + be^{-2x}

Sol: $\frac{d^2y}{dx^2} + \frac{dy}{dx}$ -2y =0

7) Find the differential equation of all the circle of radius

Sol. The equation of circles of radius a is $(x-h)^2 + (y-k)^2 = a^2$ where (h ,k) are the co-ordinates of the centre of circle and h,k are arbitrary constants.

Sol: $[1 + (\frac{dy}{dx})^2\ ]^3 = a^2\,.\frac{d^2y}{dx^2}$

8) Find the differential equation of the family of circle passing through the origin and having their centre on x-axis.

Ans: Let the general equation of the circle is $x^2+y^2+2gx+2fy+c=0$.

Since the circle passes through origin, so c=0 also the centre (-g,-f) lies on x-axis. So the y-coordinate of the centre i.e, f=0. Hence the system of circle passing through the origin and having their centres on x-axis is $x^2+y^2+2gx=0$.

Ans. 2xy . $\frac{dy}{dx} + x^2 - y^2$ =0.

9) $\sin^{-1}(xy) + 4x$ =c .

Ans: x . $\frac{dy}{dx}$ +y+ 4. $\sqrt{1-x^2}y^2$ =0

10) y= $\frac{a+x}{x^2+1}$

Sol: (x^2+1) . $\frac{dy}{dx}$ + 2xy -1=0.

11) r=a(1+cosθ)

Sol: r=a(1+cosθ) $-----------(1)$

$\frac{dr}{d\theta}$ = - asinθ $----------(2)$

Put a value from (1) in (2).

$$\frac{dr}{d\theta} = \frac{-r}{1+cos\theta} . sin\theta$$

$$\frac{dr}{d\theta} = \frac{-r . 2\sin^{\theta}/_{2} . cos^{\theta}/_{2}}{2cos^2 \, ^{\theta}/_{2}}$$

$$= -r \tan^{\theta}/_{2}$$

Hence $\frac{dr}{d\theta} + r\tan^{\theta}/_{2} = 0.$

Differential Equations of first order and first degree:

The general form of first order ,first degree differential equation is $\frac{dy}{dx} =$ f(x,y) or [Mdx + Ndy =0 Where M and N are functions of x and y]. There is no general method to solve any first order differential equation The equation which belong to one of the following types can be easily solved.

In general the first order differential equation can be classified as:

(1). Variable separable type

(2). (a) Homogeneous equation and

(b)Non-Homogeneous equations which to exact equations.

(3) (a) exact equations and

(b)equations reducible to exact equations.

4) (a) Linear equation &

(b) Bernoulli's equation.

Type –I : VARIABLE SEPARABLE:

If the differential equation $\frac{dy}{dx}$=f(x,y) can be expressed of the form $\frac{dy}{dx}=\frac{f(x)}{g(y)}$ or f(x) dx –g(y)dy =0 where f and g are continuous functions of a single variable, then it is said to be of the form variable separable.

General solution of variable separable is $\int f(x)dx - \int g(y)dy$= c

Where c is any arbitrary constant.

PROBLEMS:

1) tan y $\frac{dy}{dx}$ = sin(x+y) + sin(x-y).

Sol: Given that sin(x+y) + sin(x-y) = tan y $\frac{dy}{dx}$

⇨ 2sinx.cosx = tan y $\frac{dy}{dx}$ [**Note**: sinC+sinD =2sin($\frac{C+D}{2}$).cos($\frac{C-D}{2}$)]

⇨ 2sinx = tany secy $\frac{dy}{dx}$

General solution is $2\int \sin x\, dx = \int secy\, .tany\, .dy$

=> -2cosx = secy +c

$\Rightarrow$ sec y + 2 cos x +c =0 .//

2) Solve $(x^2+1)\cdot\frac{dy}{dx}+(y^2+1)=0$, y(0) =1.

Sol: Given $(x^2+1)\cdot\frac{dy}{dx}+(y^2+1)=0$

$$\Rightarrow \quad \frac{dx}{x^2+1}+\frac{dy}{y^2+1}=0$$

On Integrations

$$\Rightarrow \int \frac{1}{(1+x^2)}dx+\int \frac{1}{(1+y^2)}dy=0$$

$\Rightarrow \tan^{-1}x+\tan^{-1}y=c$ ---------------(1)

Given y(0)=1 => At x=0 ,y=1 ---------(2)

(2) in (1) $\Rightarrow \tan^{-1}0+\tan^{-1}1=c$.

$\Rightarrow 0+\frac{\pi}{4}=c$

$\Rightarrow c=\frac{\pi}{4}$.

Hence the required solution is $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}$

Exact Differential Equations:

Def: Let M(x,y)dx +N(x,y) dy =0 be a first order and first degree Differential Equation where M & N are real valued functions of x,y . Then the equation Mdx + Ndy =0 is said to be an exact Differential equation if ∃ a function f ∋.

$$d[f(x,y)] = \frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

Condition for Exactness: If M(x,y) & N (x,y) are two real functions which have continuous partial derivatives then the necessary and sufficient condition for the Differential equation Mdx+ Ndy =0 is to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence solution of the exact equation M(x,y)dx +N(x,y) dy =0. Is

$$\int M\,dx + \int N\,dy = c.$$

(y constant) (terms free from x).

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PROBLEMS

1) Solve $\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$

Sol: Hence M = $1+e^{\frac{x}{y}}$ & N = $e^{\frac{x}{y}}(1-\frac{x}{y})$

$\frac{\partial M}{\partial y}$ = $e^{\frac{x}{y}}$ $(\frac{-x}{y^2})$&$\frac{\partial N}{\partial x}$ = $e^{\frac{x}{y}}\left(\frac{-1}{y}\right)$ $+(1-\frac{x}{y})e^{\frac{x}{y}}(\frac{1}{y})$

$\frac{\partial M}{\partial y}$ = $e^{\frac{x}{y}}$ $(\frac{-x}{y^2})$&$\frac{\partial N}{\partial x}$ = $e^{\frac{x}{y}}$ $(\frac{-x}{y^2})$

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ equation is exact

General solution is

$\int M dx$ + $\int N dy$ = c.

(y constant) (terms free from x)

$\int(1+e^{\frac{x}{y}})\,dx$ + $\int\ 0\ dy$ = c.

=> $x+\frac{e^{\frac{x}{y}}}{\frac{1}{y}}$ = c

$=> x+y\,e^{\frac{x}{y}}$= C

2. Solve (e^y+1) .cosx dx + $e^y\ sinx\ dy$ =0.

Ans: (e^y+1) . sinx =c $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ = e^xcosx

3. Solve (r+sinθ $-$casθ) dr + r (sin$\theta+cos\theta$) d$\theta=0$.

Ans: $r^2+2r(\sin\theta-\cos\theta)=2c$

$\frac{\partial M}{\partial r}=\frac{\partial N}{\partial \theta}$ = sin$\theta+cos\theta$.

4. Solve $[y(1+\frac{1}{x})+\cos y]\,dx+[x+\log x-x\sin y]dy=0$.

Sol: hence $M = y(1+\frac{1}{x})+\cos y$, $N = x+\log x-x\sin y$.

$\frac{\partial M}{\partial y} = 1+\frac{1}{x}-\sin y$ $\qquad$ $\frac{\partial N}{\partial x} = 1+\frac{1}{x}-\sin y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so the equation is exact

General sol $\int M\,dx + \int N\,dy = c.$

(y constant) (terms free from x)

$$\int [y+\frac{y}{x}+\cos y]dx + \int o\,.dy = c.$$

$\Rightarrow y(x+\log x)+x\cos y = c.$

5. Solve $y\sin 2x\,dx-(y^2+\cos x)\,.dy=0$.

6. Solve $(\cos x-x\cos y)dy-(\sin y+(y\sin x))dx=0$

Sol: $N = \cos x-x\cos y$ & $M = -\sin y-y\sin x$

$\frac{\partial N}{\partial x} = -\sin x-\cos y$ $\qquad$ $\frac{\partial M}{\partial y} = -\cos y-\sin x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ the equation is exact.

General sol $\int M\,dx + \int N\,dy = c.$

(y constant) (terms free from x)

$$=> \int(-\sin y-y\sin x).\,dx + \int o\,.dy = c$$

$=> -x\sin y+y\cos x = c$

$=> y\cos x-x\sin y = c.$

7. Solve $(\sin x\,.\sin y-x\,e^y)\,dy = (e^y+\cos x-\cos y)\,dx$

Ans: $xe^y+\sin x.\cos y = c.$

8. Solve $(x^2+y^2-a^2)\,x\,dx+(x^2-y^2-b^2)\,.\,y\,.dy=0$

Ans: $x^4+2x^2y^2-2a^2x^2-2b^2y^2 = c$.

REDUCTION OF NON-EXACT DIFFERENTIAL EQUATIONS TO EXACT USING INTEGRATING FACTORS

Definition: If the Differential Equation M(x,y) dx + N (x,y) dy = 0 be not an exact differential equation. It Mdx+Ndy=0 can be made exact by multiplying with a suitable function u (x,y) $\neq$ 0. Then this function is called an Integrating factor(I.F).

Note: There may exits several integrating factors.

<u>Some methods to find an I.F to a non-exact Differential Equation Mdx+N dy =0</u>

Case -1: Integrating factor by inspection/ (Grouping of terms).

Some useful exact differentials

1 . d (xy) = xdy +y dx

2. $d\left(\frac{x}{y}\right) = \frac{ydx-xdy}{y^2}$

3. $d\left(\frac{y}{x}\right) = \frac{xdy-ydx}{x^2}$

4. $d\left(\frac{x^2+y^2}{2}\right) = x\,dx + y\,dy$

5. $d\left(\log\left(\frac{y}{x}\right)\right) = \frac{xdy-ydx}{xy}$

6. $d\left(\log\left(\frac{x}{y}\right)\right) = \frac{ydx-xdy}{xy}$

7. $d\left(tan^{-1}\left(\frac{x}{y}\right)\right) = \frac{ydx-xdy}{x^2+y^2}$

8. $d\left(tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{xdy-ydx}{x^2+y^2}$

9. $d(\log(xy)) = \frac{xdy+ydx}{xy}$

10. $d(\log(x^2+y^2)) = \frac{2(xdx+ydy)}{x^2+y^2}$

11. $d\left(\frac{e^x}{y}\right) = \frac{ye^x\,dx-e^x\,dy}{y^2}$

PROBLEMS:

1 . Solve $xdx + y\,dy + \frac{xdy-ydx}{x^2+y^2} = 0$.

Sol: Given equation $x\,dx + y\,dy + \frac{xdy-ydx}{x^2+y^2} = 0$

$$d\left(\frac{x^2+y^2}{2}\right) + d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = 0$$

on Integrating

$$\frac{x^2+y^2}{2} + \tan^{-1}\left(\frac{y}{x}\right) = c.$$

2 . Solve $y(x^3 . e^{xy} - y)\,dx + x\,(y + x^3 . e^{xy})\,dy = 0$.

Sol: Given equation is on Regrouping

We get $yx^3e^{xy}\,dx - y^2dx + xy\,dy + x^4e^{xy}\,dy = 0$.

$$x^3e^{xy}(ydx + xdy) + y\,(x\,dy - ydx) = 0$$

Dividing by x^3

$$e^{xy}\,(ydx + xdy) + \left(\frac{y}{x}\right) . \left(\frac{xdy-ydx}{x^2}\right) = 0$$

$$d\,(e^{xy}) + \left(\frac{y}{x}\right) .d + \left(\frac{y}{x}\right) = 0$$

on Integrating

$$e^{xy} + ½\left(\frac{y}{x}\right)^2 = C \text{ is required G.S.}$$

3. Solve $(1+xy)\,x\,dy + (1- yx)\,y\,dx = 0$

Sol: Given equation is $(1+xy)\,x\,dy + (1-yx)\,y\,dx = 0$.

$$(xdy + y\,dx) + xy\,(xdy - y\,dx) = 0.$$

Divided by x^2y^2 => $\left(\frac{xdy+ydx}{x^2y^2}\right) + \left(\frac{xdy-ydx}{xy}\right) = 0$

$$\Rightarrow \left(\frac{d(xy)}{x^2y^2}\right) + \frac{1}{y}\,dy - \frac{1}{x}\,dx = 0.$$

On integrating => $\frac{-1}{xy} + \log y - \log x = \log c$

$$-\frac{1}{xy} - \log x + \log y = \log c.$$

4. Solve $ydx - x\,dy = a\,(x^2 + y^2)\,dx$

Sol: Given equation is $ydx - x\,dy = a\,(x^2 + y^2)\,dx$

$$\Rightarrow \frac{ydx - x\,dy}{(x^2+y^2)} = a\,dx$$

$$\Rightarrow d\left(\tan^{-1}\frac{x}{y} = a\,dx\right)$$

Integrating on $\tan^{-1}\frac{x}{y}$ = ax +c where c is an arbitrary constant.

Method -2: If M(x,y) dx + N (x,y) dy =0 is a homogeneous differential equation and Mx +Ny $\neq$ 0 then $\frac{1}{Mx+Ny}$ is an integrating factor of Mdx+ Ndy =0.

1 . Solve x^2y dx – (x^3+ y^3) dy = 0

Sol : Given equation is x^2y dx – (x^3+ y^3) dy = 0-----------------(1)

Where M = x^2y & N = ($-x^3$- y^3)

Consider $\frac{\partial M}{\partial y}$ = x^2 & $\frac{\partial N}{\partial x}$ = $-3x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ equation is not exact .

But given equation(1) is homogeneous differential equation then

So Mx+ Ny = $x(x^2y) - y(x^3+y^3) = -y^4 \neq 0$.

$$\text{I.F} = \frac{1}{Mx+Ny} = \frac{-1}{y^4}$$

Multiplying equation (1) by $\frac{-1}{y4}$

$= > \frac{x^2y}{-y^4}$ dx - $\frac{x^3+y^3}{-y^4}$ dy = 0---------------------(2)

$= > -\frac{x2}{y3}$ dx - $\frac{x3+y3}{-y4}\,dy = 0$

This is of the form $M_1dx + N_1dy = 0$

For $M_1 = \frac{-x2}{y3}$ & $N_1 = \frac{x^3+y^3}{y^4}$

$= > \frac{\partial M1}{\partial y} = \frac{3x^2}{y4}$ & $\frac{\partial N1}{\partial x} = \frac{3x^2}{y^4}$

$= > \frac{\partial M1}{\partial y} = \frac{\partial N1}{\partial x}$ equation (2) is an exact D.equation.

General sol $\int M_1 dx + \int N_1 dy = c$

(y constant) (terms free from x in N_1)

$$\Rightarrow \int \frac{-x^2}{y^3} dx + \int \frac{1}{y} dy = c.$$

$$\Rightarrow \frac{-x^3}{3y^3} + \log|y| = c$$

2. Solve $y^2\,dx + (x^2 - xy - y^2)dy = 0$

Ans: $(x-y) \cdot y^2 = c1^2(x+y)$.

3. Solve $y(y^2 - 2x^2)\,dx + x(2y^2 - x^2)\,dy = 0$

Sol: Given equation is $y(y^2 - 2x^2)\,dx + x(2y^2 - x^2)\,dy = 0$ -----------(1)

It is the form Mdx +Ndy =0

Where $M = y(y^2 - 2x^2)$, $N = x(2y^2 - x^2)$

Consider $\frac{\partial M}{\partial y} = 3y^2 - 2x^2$ & $\frac{\partial N}{\partial x} = 2y^2 - 3x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ equation is not exact .

Since equation(1) is Homogeneous differential equation then

Consider $Mx + Ny = x[y(y^2 - 2x^2)] + y[x(2y^2 - x^2)]$

$= 3xy(y^2 - x^2) \neq 0.$

$\Rightarrow$ I.F. $= \frac{1}{3xy(y^2 - x^2)}$

Multiplying equation (1) by $\frac{1}{3xy(y^2 - x^2)}$ we get

$$\Rightarrow \frac{y(y^2 - 2x^2)}{3xy(y^2 - x^2)} dx + \frac{x(2y^2 - x^2)}{3xy(y^2 - x^2)} dy = 0$$

Now it is exact

$$\frac{(y^2 - x^2) - x^2}{3x(y^2 - x^2)} dx + \frac{y^2 + (y^2 - x^2)}{3y(y^2 - x^2)} dy = 0$$

$\frac{dx}{x} - \frac{xdx}{y^2-x^2} + \frac{ydy}{y^2-x^2} + \frac{dy}{y}$ =0.

$$\left(\frac{dx}{x}+\frac{dy}{y}\right)+\frac{2ydy}{2\left(y^2-x^2\right)}-\frac{2xdx}{2\left(y^2-x^2\right)}=0$$

log x +log y +$\frac{1}{2}$ log (y^2-x^2) - $\frac{1}{2}$ log (y^2-x^2)=logc $\Rightarrow$ xy = c

4. Solve r (θ^2+r^2) d $\theta - \theta$ (θ^2+2r^2) dr =0

Ans: $\frac{\theta^2}{2r^2}$ + logθ - log r^2 =c.

Method- 3: If the equation Mdx + N dy =0 is of the form y. f(x, y) .dx + x . g (x, y) dy = 0 & Mx- Ny $\neq 0$ then $\frac{1}{Mx-Ny}$ is an integrating factor of Mdx+ Ndy =0.

Problems:

1 . Solve (xy sinxy +cosxy) ydx + (xy sinxy –cosxy)x dy =0.

Sol: Given equation (xy sinxy +cosxy) ydx + (xy sinxy –cosxy)x dy =0 -------(1).

Equation (1) is of the form y. f(xy) .dx + x . g (xy) dy =0.

Where M =(xy sinxy + cos xy) y

N= (xy sinxy- cos xy) x

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ equation (1) is not an exact

Now consider Mx-Ny

Here M =(xy sinxy + cos xy) y

N= (xy sinxy- cos xy) x

Consider Mx-Ny =2xycosxy

Integrating factor = $\frac{1}{2xycosxy}$

So equation (1) x I.F

$$\Rightarrow \frac{(xy\sin xy+\cos xy)y}{2xy\cos xy}dx+\frac{(xy\sin xy-\cos xy)x}{2xy\cos xy}dy=0$$

$\Rightarrow$ (y tan xy + $\frac{1}{x}$) dx + (y tan xy -$\frac{1}{y}$) dy =0

$\Rightarrow M_1$ dx + N_1 dx =0

Now the equation is exact.

General sol $\int$ M_1 dx + $\int$ N_1 dy = c.

(y constant) (terms free from x in N_1)

$$=>\int(ytanxy+\frac{1}{x})dx + \int\frac{-1}{y}dy \quad =c.$$

$$=>\frac{y.\log|secxy|}{y} +\log x + (-\log y) =\log c$$

=> log|sec(xy)| +log$\frac{x}{y}$ =log c.

=>$\frac{x}{y}$. secxy =c.

2. Solve (1+xy) y dx + (1-xy) x dy =0

Sol : I.F =$\frac{1}{2x^2y^2}$

$$=>\int \frac{1}{2x^2y} + \frac{1}{2x})dx + \int\frac{-1}{2y}dy \quad =c$$

$$=>\frac{-1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y =c.$$

=>$\frac{-1}{xy}$ +log($\frac{x}{y}$) =c^1 where c^1 = 2c.

3. Solve (2xy+1) y dx + (1+ 2xy-x^3y^3) x dy =0

Ans: log y + $\frac{1}{x^2y^2}$ + $\frac{1}{3x^3y^3}$ =c.

4. solve (x^2y^2 +xy +1) ydx +(x^2y^2- xy+1) xdy =0

Ans: xy -$\frac{1}{xy}$ + log($\frac{x}{y}$) =c .

Method -4: If there exists a continuous single variable function $f(x)$ such that $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}$ =f(x),then I.F. of Mdx + N dy =0 is $e^{\int f(x)dx}$

PROBLEMS

1 . Solve (3xy – 2ay^2) dx + ($x^2 - 2axy$) dy =0

Sol: Given equation is (3xy – 2ay^2) dx + ($x^2 - 2axy$) dy =0

This is of the form Mdx+ Ndy = 0

$$\Rightarrow \quad M = 3xy - 2ay^2 \;\&\; N = x^2 - 2axy$$

$$\frac{\partial M}{\partial y} = 3x-4ay \quad \& \frac{\partial N}{\partial x} = 2x-2ay$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \text{equation not exact .}$$

Now consider $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{(3x-4ay)-(2x-2ay)}{x(x-2ay)}$

$$\Rightarrow \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{1}{x} = f(x).$$

=> $e^{\int \frac{1}{x}dx} = x$ is an Integrating factor of (1)

equation (1) Multiplying with I.F then

$$\Rightarrow \frac{(3xy-2ay^2)}{1} \; x\,dx + \frac{(x^2-2axy)}{1} \; x\,dy = 0$$

=> ($3x^2y$ -$2ay^2x$) dx + (x^3-$2ax^2y$) dy =0

It is the form M_1dx + N_1dy =0

$M_1 = 3x^2y - 2ay^2x, N_1 = x^3 - 2ax^2y$

$\frac{\partial M_1}{\partial y} = 3x^2 - 4axy, \frac{\partial N_1}{\partial x} = 3x^2 - 4axy$

$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

$\therefore$ equation is an exact

General sol $\int M_1 dx + \int N_1 dy = c.$

(y constant) (terms free from x in N_1)

$\Rightarrow \int (3x^2y - 2ay^2x) dx + \int 0 dy = c$

=> $x^3y - ax^2y^2 = c$.

2 . Solve ydx-xdy+(1+x^2)$dx + x^2 \sin y \, dy = 0$

Sol : Given equation is (y+1+x^2) $dx + (x^2 \sin y - x)$ dy =0.

M= y+1+x^2& N =$x^2 \sin y - x$

$\frac{\partial M}{\partial y} = 1$ $\frac{\partial N}{\partial x} =$ 2x sin y -1

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ = > the equation is not exact.

So consider $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{(1 - 2x\sin y + 1)}{x^2 \sin y - x} = \frac{-2x\sin y + 2}{x^2 \sin y - x} = \frac{-2(x\sin y - 1)}{x(x\sin y - 1)} = \frac{-2}{x}$

I.F = $e^{\int f(x)dx} = e^{-2\int \frac{1}{x}dx} = e^{-2\log x} = \frac{1}{x^2}$

Equation (1) X I.F =>$\frac{y+1+x^2}{x^2} \, dx + \frac{x^2 \sin y - x}{x^2}$ dy =0

It is the form of $M_1dx+ N_1\, dy =0$.

Gen soln $\quad \Rightarrow \int (\frac{y}{x^2} + \frac{1}{x^2} + 1)dx + \int sinydy = 0$

$\Rightarrow \frac{-y}{x} - \frac{1}{x} + x - \cos y = c.$

$\Rightarrow x^2 - y - 1 - x\cos y = cx.$

----

3. Solve $2xy\, dy - (x^2+y^2+1)dx = 0$

Ans: $-x + \frac{y^2}{x} + \frac{1}{x} = c$.

4. Solve $(x^2+y^2)\, dx - 2xy\, dy = 0$

Ans: $x^2 - y^2 = cx$.

Method -5: For the equation $Mdx + N\,dy = 0$ if $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y)$ (is a function of y alone) then $e^{\int g(y)dy}$ is an integrating factor of $M\,dx + N\,dy = 0$.

Problems:

1 . Solve $(3x^2y^4+2xy)dx + (2x^3y^3-x^2)\, dy = 0$

Sol: Given equation $(3x^2y^4+2xy)dx + (2x^3y^3-x^2)\, dy = 0$ -----------------(1).

Equation of the form $M\,dx + N\,dy = 0$.

Where $M = 3x^2y^4+2xy$ $\quad$ & $N = 2x^3y^3-x^2$

$$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x, \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ equation (1) not exact.

So consider $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-2}{y} = g(y)$

I.F $= e^{\int g(y)dy} = e^{-2\int \frac{1}{y}dy} = e^{-2\log y} = \frac{1}{y^2}$.

Equation (1) x I.F $\Rightarrow$ $\Rightarrow \left(\dfrac{3x^2y^4+2xy}{y^2}\right)dx+\left(\dfrac{2x^3y^3-x^2}{y^2}\right)dy=0$

$$\Rightarrow\left(3x^2y^2+\frac{2x}{y}\right)dx+\left(2x^3y-\frac{x^2}{y^2}\right)dy=0$$

It is the form $M_1dx + N_1\, dy =0$

General sol $\int M_1dx + \int N_1dy = c$

(y constant) (terms free from x in N_1)

$$\Rightarrow \int\left(3x^2y^2+\frac{2x}{y}\right)dx + \int o\ dy = c.$$

$$\Rightarrow \frac{3x^3y^2}{3}+\frac{2x^2}{2y} = c.$$

$$\Rightarrow x^3y^2+\frac{x^2}{y} = c.$$

2 . Solve $(xy^3+y)\,dx + 2(x^2y^2+x+y^4)\,dy =0$

Sol: $\dfrac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}{M} = \dfrac{(4xy^2+2)-(3xy^2+1)}{xy^3+y} = \dfrac{1}{y} = g(y).$

I.F $= e^{\int g(y)dy} = e^{\int \frac{1}{y}dy} = y.$

Gen sol: $\int\left(xy^4+y^2\right)dx+\int\left(2y^5\right)dy=c$

$\dfrac{x^2y^4}{2} + y^2x + \dfrac{2y^6}{6} = c$.

3 . solve $(y^4+2y)dx + (xy^3 +2y^4 - 4x)\,dy =0$

Sol: $\dfrac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}{M} = \dfrac{(y^3-4)-(4y^3+2)}{y^4+2y} = \dfrac{-3}{y} = g(y).$

I.F $= e^{\int g(y)dy} = e^{-3\int \frac{1}{y}dy} = \dfrac{1}{y^3}$

Gen soln : $\int\left(y+\dfrac{2}{y^2}\right)dx+\int 2y\,dy = c$.

$$\left(y+\frac{2}{y^2}\right)x+y^2=c.$$

4. Solve $(y+y^2)dx + xy\,dy = 0$

Ans: $x + xy = c$.

5. Solve $(xy^3+y)\,dx + 2(x^2y^2+x+y^4)dy = 0$.

Ans: $(x^2+y^4-1)\ e^{x^2} = c$.

LINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER:

Def: An equation of the form $\frac{dy}{dx}+P(x).y=Q(x)$ is called a linear differential equation of first order in y.

Working Rule: To solve the liner equation $\frac{dy}{dx}+P(x).y=Q(x)$

first find the integrating factor I.F $=e^{\int p(x)dx}$

General solution is y x I.F = $\int Q(x)\times I.F.dx+c$

Note: An equation of the form $\frac{dx}{dy}+p(y).x=$ Q(y) called a linear Differential equation of first order in x.

Then integrating factor $=e^{\int p(y)dy}$

General solutionis = x X I.F = $\int Q(y)\times I.F.dy+c$

PROBLEMS:

1 . Solve $(1+y^2)\,dx=(\tan^{-1}y-x)\,dy$

Sol: Given equation is $(1+y^2)\frac{dx}{dy}=(\tan^{-1}y-x)$

$$\frac{dx}{dy}+\left(\frac{1}{1+y^2}\right).\ x=\frac{\tan^{-1}y}{1+y^2}$$

It is the form of $\frac{dx}{dy}+p(y).x=Q(y)$

$$\text{I.F}=e^{\int p(y)dy}=e^{\int\frac{1}{1+y^2}dy}\ =\ e^{\tan^{-1}y}$$

=> General solution is x. $e^{tan^{-1}y} = \int \frac{\tan^{-1} y}{1+y^2} . e^{\tan^{-1} y} dy + c$

$=>$ x. $e^{tan^{-1}y} = \int t \,.\, e^t \,dt + c$

[put $tan^{-1} y = t$

$\Rightarrow \frac{1}{1+y^2} dy = dt$]

$\Rightarrow$x. $e^{tan^{-1}y}$ = t . e^t -e^t +c

$=>$ x. $e^{tan^{-1}y} = tan^{-1} y . e^{tan^{-1}y}$ -$e^{tan^{-1}y}$ +c

$=>$ x $= tan^{-1} y - 1$ + c/$e^{tan^{-1}y}$ is the required solution

2. Solve (x+y+1) $\frac{dy}{dx}$ **= 1.**

Sol: Given equation is (x+y+1) $\frac{dy}{dx}$ = 1.

$=>\frac{dx}{dy} - x$ = y+1.

It is of the form $\frac{dx}{dy}$ + p(y).x = Q(y)

Where p(y) = -1 ; Q(y) = 1+y

$=>$ I.F =$e^{\int p(y) dy}$ = $e^{-\int dy}$ = e^{-y}

General solution is x X I.F = $\int Q(y) \times I.F.dy + c$

$=>$ x . e^{-y} = $\int (1+y)\, e^{-y} dy + c$

$=>$ x . e^{-y} = $\int e^{-y} dy + \int y e^{-y} dy + c$

$=>$ xe^{-y} = - $e^{-y} -$ y xe^{-y} - e^{-y} +c

$=>$ xe^{-y} = - $e^{-y}(2+y)$ +c .//

3. Solve $y^1 + y = e^{e^x}$

Sol: Given equation is $y^1 + y = e^{e^x}$

It is of the form $\frac{dy}{dx} + p(x).y = \emptyset(x)$

Where p(x) =1 Q(x) = e^{e^x}

=> I.F = $e^{\int p(x)dx}$ $= e^{\int dx}$ $=e^x$

General solution is y x I.F = $\int Q(x)\times I.F.dx + c$

=> y. $e^x = \int e^{e^x} e^x dx + c$

=> y. $e^x = \int e^t dt + c$ $\left\{ \begin{array}{l} \text{put } e^x = t \\ e^x dx = dt \end{array} \right\}$

=> y. e^x = e^t+c

=> y. e^x = e^{e^x}+c

4. Solve x. $\frac{dy}{dx}$ + y =log x

Sol : Given equation is x. $\frac{dy}{dx}$ + y =log x

It is of the form $\frac{dy}{dx}$ +p(x)y = Q(x)

Where p(x) = $\frac{1}{x}$& Q(x)=$\frac{\log x}{x}$

i.e , $\frac{dy}{dx} + \frac{1}{x} . y = \frac{\log x}{x}$

=> I.F = $e^{\int p(x)dx}$ $= e^{\int \frac{1}{x}dx}$ $=e^{\log x}$ =x.

General solution is y x I.F = $\int Q(x)\times I.F.dx + c$

=> y.x = $\int \frac{\log x}{x}\, x\, dx + c$

=> y .x = x (logx-1) +c.

5 . Solve $(1+y^2) + (x- e^{tan^{-1}y})\frac{dy}{dx} = 0$.

Sol : Given equation is $\frac{dx}{dy} + \frac{x}{1+y2} = \frac{e^{tan^{-1}y}}{1+y2}$

It is of the form $\frac{dx}{dy}$ + p(y) .x = Q(y)

Where p (y) = $\frac{1}{1+y2}$, Q(x)=$\frac{e^{tan^{-1}y}}{1+y2}$.

I.F $= e^{\int p(y)dy} = e^{\int\frac{1}{1+y2}dy} = e^{tan^{-1}y}$.

General solution is x x I.F = $\int Q(y) \times I.F.dy + c$.

$=>$ x . $e^{tan^{-1}y}$. $= \int \frac{e^{tan^{-1}y}}{1+y2} e^{tan^{-1}y}$. dy +c

$=>$ x . $e^{tan^{-1}y} = \int e^t e^t$. dt +c

[**Note: put** $tan^{-1}y$ = t

$=> \frac{1}{1+y2}\ dy$ = dt]

$=>$ x . $e^{tan^{-1}y} = \int e^{2t}$. dt +c

$=>$ x . $e^{tan^{-1}y} = \frac{e^{2t}}{2}$+ c

$=>$ x . $e^{tan^{-1}y} = \frac{e^{2\,tan^{-1}y}}{2}$+ c

6. solve $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{sin2x}{logx}$

Ans: ylogx = $\frac{-cos\,2x}{2}$ + c.

7 . $\frac{dy}{dx}$ + (y-1). Cosx = $e^{-sinx}cos^2x$

Ans: y.e^{sinx} = $\frac{x}{2} + \frac{sin2x}{4}$ + e^{sinx} +c

8. $\frac{dy}{dx} + \frac{2x}{1+x^2}$. y= $\frac{1}{(1+x^2)^2}$ given y = 0 , when x= 1.

Ans : y(1+x^2) = $tan^{-1}x - \frac{\pi}{4}$

9. Solve $\frac{dy}{dx} - \frac{tany}{1+x}$ = (1+x) e^x . sec y

Sol : The above equation can be written as

Divided by sec y => cos y $\frac{dy}{dx}$ - $\frac{siny}{1+x}$ = (1+x) e^x ------------(1)

Put sin y = u

$=>$ cos y $\frac{dy}{dx} = \frac{du}{dx}$

Differential Equation (1) is $\frac{du}{dx} - \frac{1}{1+x}$. u = (1+x) e^x

this is of the form $\frac{du}{dx} + p(x).u = \emptyset($ Q(x)

Where p(x) =$\frac{-1}{1+x}$ Q(x) =(1+x) e^x

=> I.F = $e^{\int p(x)dx}$ $= e^{\int \frac{-1}{1+x}dx}$ $=e^{-\log(1+x)}$ $=\frac{1}{1+x}$

General solution is u x I.F = $\int Q(y) \times I.F.dy + c$

=> u. $\frac{1}{1+x} = \int (1+x)\ e^x \frac{1}{1+x} dx\ + c$

=> u. $\frac{1}{1+x} = \int\ e^x dx\ + c$

=> (sin y) $\frac{1}{1+x}$ = e^x +c

(Or)

= > sin y = (1+x) e^x +c . (1+x) is required solution.

10. Solve $\frac{dy}{dx}$ - ytanx $=\frac{sinx\ .\ cos^2x}{y^2}$

Ans : $y^3\ cos^3x\ =\ \frac{-\ cos^6x}{2}$ +c .

11 .Solve $\frac{dy}{dx}$ – yx = $y^2\ e^{\frac{x^2}{2}}\ .sinx$

Ans: $\frac{1}{y}\ e^{\frac{-x^2}{2}}$ = cosx + c.

12. e^{x} . $\frac{dy}{dx}$ = $2xy^2$ + y e^{x}

Ans : $\frac{1}{y}\ e^{x}$ = x^2 + c.

13. $\frac{dy}{dx}$ + ycos x = y^3 sin x

Ans : $\frac{1}{y^2}$ = (1+ 2 sinx) + c e^{2sinx} (or)

$\frac{-1}{y^2}\ e^{-2sinx}$= -(1+ 2 sinx) e^{-2sinx}+ c .

14. $\frac{dy}{dx}$ + y cot x = $y^2\ sin^2x\ cos^2\ x$

Ans: ysinx (c + $cos^3\ x)$ =3.

BERNOULLI'S EQUATION :

(EQUATIONS REDUCIBLE TO LINEAR EQUATION)

Def: An equation of the form $\frac{dy}{dx}$ **+** p(x) .y =$\emptyset(x,$ Q(x) **--- --------(1)**

is called Bernoulli's Equation, where P&Q are function of x and n is a real constant.

Working Rule:

Case -1 : If n=1 then the above equation becomes $\frac{dy}{dx}$ + p. y = Q.

=> General solution of $\frac{dy}{dx}+(P-Q)y=0$ is

$\int\frac{dy}{y}+(P-Q)dx=c$ by variable separation method.

Case -2: If $n\neq 1$ then divide the given equation (1) by y^n

$\Rightarrow y^{-n}\,.\,\frac{dy}{dx}$ **+** p(x) . y^{1-n} **=**Q ---------(2)

Then take y^{1-n} = u

(1-n) $y^{-n}.\frac{dy}{dx}=\frac{du}{dx}$

$\Rightarrow y^{-n}.\frac{dy}{dx}=\frac{1}{1-n}\frac{du}{dx}$

Then equation (2) becomes

$\frac{1}{1-n}\frac{du}{dx}$ + p(x) . u =Q

$\frac{du}{dx}$ + (1-n) p.u = (1-n)Q which is linear and hence we can solve it.

Problems:

1 . Solve x $\frac{dy}{dx}$ + y = $x^3 y^6$

Sol: Given equation is x $\frac{dy}{dx}$ + y = $x^3 y^6$

Given equation can be written as $\frac{dy}{dx} + \left(\frac{1}{x}\right) y = x^2y^6$

Which is of the form $\frac{dy}{dx}$ + p(x).y =Qy^n

Where p(x) =$\frac{1}{x}$Q(x) = x^2& n=6

Divided by y^6 =>$\frac{1}{y^6} . \frac{dy}{dx} + \frac{1}{x}\frac{1}{y^5} = x^2$ ---------------(2)

Take $\frac{1}{y^5}$ = u

$\Rightarrow \frac{-5}{y^6}\frac{dy}{dx} = \frac{du}{dx}$ }----------------(3)

$\Rightarrow \frac{1}{y^6}\frac{dy}{dx} = \frac{-1}{5}\frac{du}{dx}$ }----------------(3)

(3) in (2) =>$\frac{du}{dx} - \frac{5}{x}u = -5x^2$

Which is a Linear differential equation in u

I.F = $e^{\int p(x)dx} = e^{-5\int \frac{1}{x}dx} = e^{-5 \log x} = \frac{1}{x^5}$

General solution is u .I.F = $\int Q(x) \times I.F.dx + c$

$$u.\frac{1}{x^5} = \int -5x^2.\frac{1}{x^5}\,dx + c$$

$\frac{1}{y^5x^5} = \frac{5}{2x^2}$ +c (or) $\frac{1}{y^5} = \frac{5\,x^3}{2} + cx^5$

2. Solve $\frac{dy}{dx}$ $(x^2y^3 + xy) = 1$

Sol: Given equation is $\frac{dy}{dx}$ $(x^2y^3 + xy) = 1$

This can be written as $\frac{dx}{dy}$ -x .y= x^2y^3 => $\frac{1}{x^2} \cdot \frac{dx}{dy}$ - $\frac{1}{x}$.y = y^3 --------------------(1)

Put $\frac{1}{x}$ =u

$\Rightarrow \frac{-1}{x^2} \cdot \frac{dx}{dy} = \frac{du}{dx}$ ----------------(2).

(2) in (1) $\Rightarrow -\frac{du}{dx} -$ u .y = y^3

(Or) $\frac{du}{dx}$ + u .y = -y^3.

This is a Linear Differential Equation in ' u'

I.F = $e^{\int P(y)dy} = e^{\int y dy} = e^{-\frac{y^2}{2}}$

General solution $\Rightarrow$ u .I.F = $\int Q(y) \times I.F.dy + c$

$\Rightarrow$ u . $e^{-\frac{y^2}{2}} = \int y^3 . e^{-\frac{y^2}{2}} dy$ +c

$\Rightarrow \frac{e^{-\frac{y^2}{2}}}{x}$ = -2$(\frac{y^2}{2} - 1)$. $e^{-\frac{y^2}{2}}$ +c

(or)

x(2-y^2)+ cx$e^{-\frac{y^2}{2}}$ =1.

3. Solve $\frac{dy}{dx}$+y tanx = y^2 sec x

Ans: I.F = $e^{-\int tanx dx} = e^{\int \log\cos x} = \cos x$

General solution $\frac{1}{y}$ cos x= -x +c .

4. (1-x^2) $\frac{dy}{dx}$ +xy = $y^3 sin^{-1}$x

Sol: Given equation can be written as

$\frac{dy}{dx} + \frac{x}{1-x^2}$ y = $\frac{y^3}{1-x^2} sin^{-1}$x

Which is a Bernoulli's equation in ' y '

Divided by $y^3 \Rightarrow$ $\frac{1}{y^3}$. $\frac{dy}{dx} + \frac{1}{y^2}\frac{x}{1-x^2}$ = $\frac{sin^{-1}x}{1-x^2}$ -----------(1).

Let $\frac{1}{y^2}$ = u

$\Rightarrow \frac{-2}{y^3}\frac{dy}{dx} = \frac{du}{dx}$ $=> \frac{1}{y^3}\frac{dy}{dx} = -\frac{1}{2}\frac{du}{dx}$-------------(2)

(2) in (1) $\Rightarrow -\frac{1}{2}\frac{du}{dx} + \frac{x}{1-x^2}.u = \frac{sin^{-1}x}{1-x^2} \Rightarrow \frac{du}{dx} - \frac{2x}{1-x^2}.u = \frac{-2\sin^{-1}x}{1-x^2}$

Which is a Linear differential equation in u

$\Rightarrow$ I.F = $e^{\int p(x)dx}$ $= e^{-\int\frac{2x}{1-x^2}dx}$ $= e^{\log(1-x^2)}$ $= (1-x^2)$

General solution $\Rightarrow$ u .I.F = $\int Q(x)\times I.F.dx + c$

$\Rightarrow \frac{1}{y^2}(1-x^2)$ = $-\int\frac{2sin^{-1}x}{1-x^2}(1-x^2)dx + c$

$=> \frac{(1-x^2)}{y^2}$ = -2 [$x sin^{-1}x + \sqrt{1-x^2}$] +c

5 . $e^x\frac{dy}{dx}$ = $2xy^2$ + y . e^x

Ans: $\frac{e^x}{y}$ = $-x^2 + c$.

APPLICATION OF DIFFERENTIAL EQUATIONS OF FIRST ORDER

ORTHOGONAL TRAJECTORIES (O.T)

Def: A curve which cuts every member of a given family of curves at a right angle is an orthogonal trajectory of the given family.

Orthogonal trajectories in Cartesian co-ordinates:

Working rule:<u>**To find the family of O.T in Cartesian form**</u> . Let f(x,y,c) =0 ……(1)

be the given equation of family of curves in Cartesian form.

Step: (1) Differentiate with respect to ' x ' and obtain F(x, y, y^1) = 0 ----------(2)

of the given family of curves.

(2) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ is (2)

Then the Differential Equation of family of O.T is

F(x, y, $-\frac{dx}{dy}$) =0 ----------------(3).

(3) Solve equation(3) to get the equation of family of O.T's of equation(1).

PROBLEMS:

1 . Find the O.T's of family of semi-cubical parabolas $ay^2=x^3$ where a is a parameters.

Sol : The given family of semi-cubical parabola is $ay^2=x^3$ ---------------(1)

Differentiating with respect to 'x ' => a 2y$\frac{dy}{dx}$ = 3 x^2-------------(2)

Eliminating 'a ' =>$\frac{x^3}{y^2}$ .2y . $\frac{dy}{dx}$ =3 x^2

$$=> \frac{2x^3}{y}\frac{dy}{dx} = 3x^2$$

Replace $\frac{dy}{dx}$ by -$\frac{dx}{dy}$ $=> \frac{2x^3}{y}\left(-\frac{dx}{dy}\right) = 3x^2$

$$=> \frac{-2}{3}x\frac{dx}{dy} = y$$

$$=> \int \frac{-2}{3}xdx - \int ydy = -c$$

$$=> \frac{-x^2}{3} - \frac{y^2}{2} = c$$

$$=> \frac{x^2}{3c} + \frac{y^2}{2c} = 1$$

2. Find the O.T of the family of circles $x^2+y^2+2gx+c=0$, Where g is the parameter

Sol: $x^2+y^2+2gx+c=0$. ---------(1) is represents a system of co- axial circles with g as parameter

Differentiating with respect to 'x ' => $2x+2y\frac{dy}{dx}+2g=0$--------------(2)

Substituting equation from (2) in (1)

$$=> x^2+y^2 -\left(2x+2y\frac{dy}{dx}\right)x+c=0.$$

$$=> y^2-x^2-2xy\frac{dy}{dx}+c=0$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$=> y^2-x^2-2xy\left(-\frac{dx}{dy}\right)+c=0$$

$$=> y^2-x^2+2xy\left(\frac{dx}{dy}\right)+c=0$$

This can be written as

$$2x\cdot\frac{dx}{dy} - \frac{1}{y}x^2 = \frac{-(c+y^2)}{y}$$

This is a Bernoulli's equation in x

So put $x^2 = u \Rightarrow 2x\,.\frac{dx}{dy} = \frac{du}{dy}$

$$\Rightarrow \frac{du}{dy} - \frac{1}{y}u = \frac{-(c+y^2)}{y}$$

Which is a linear equation in 'u '

$$\Rightarrow \text{I.F} = e^{\int -\frac{1}{y}dy} = e^{-\log y} = \frac{1}{y}$$

General solution is u .I .F = $\int Q(y).I.F.dy + K$

$\Rightarrow$ $x^2 \frac{1}{y}$ = $\int \frac{-(c+y^2)}{y}\frac{1}{y}dy + k$

$= -c\left(\frac{-1}{y}\right) - y + k$

$\Rightarrow \frac{x^2}{y}$ = $\frac{c}{y}$ -y + k

3. Find the O.T's of the family of parabolas through origin and foci on y –axis.

Sol : The equation of the family of parabolas through the origin and foci on y-axis is x^2=4ay where a is parameter

$\Rightarrow$ 2x =4a . $\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}$ = $\frac{2y}{x}$

Ans: $\frac{x^2}{2c} + \frac{y^2}{c} = 1$

4. Find the O.T of the one parameter family of curves e^x+ e^{-y}=c.

Sol: Given equation is e^x+ e^{-y}=c.

Differentiating with respective 'x' $\Rightarrow e^x$+ $e^{-y}(\frac{dy}{dx})$=c

O.T $\Rightarrow e^x$+ $e^{-y}(\frac{-dx}{dy})$=c

Ans: $e^y - e^{-x} = k$.

5. Find the O.T of the family of circle passing through origin and centre on x-axis.
   Hint : Given family of circles is x^2+y^2+2gx=0 .

   Ans: $\frac{x2}{y}$= - y +c.

6. Prove that the system of parabolas $y^2=4a(x+a)$ is self orthogonal

ORTHOGONAL TRAJECTORIES IN POLAR FORM

Working Rule: To find the O.T of a given family of curves in polar-co ordinates.

Let $f(r, \theta, c) = 0$ -----(1) be the given family of curves in polar form.

1 .) Differentiating with respect to θ and obtain $F\left[r, \theta, \frac{dr}{d\theta}\right]=0$ by eliminating the parameter c.

2.) Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ then the Differential Equation of family of O.T

$$F\left[r, \theta, -r^2 \frac{d\theta}{dr}\right]=0$$

3.) Solve the above equation to get the equation of O.T of (1)

Problems:

1 . Find the O.T of family of

a) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ where a is a parameter.

b) $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ λ is a parameter, is self –orthogonal

c) $r\sin 2\theta = \lambda$, λ is a parameter

Ans: $r^4 \cos 2\theta = C^4$.

2 . **Find the O.T** of family of curves $r^n = a^n \cos n\theta$

Ans: $r^n = c \sin n\theta$

3. Find the O.T of family of curves $r=2a(\cos\theta + \sin\theta)$

ans : $r = (\cos\theta - \sin\theta) . c$

4. Find the O.T of family of curves $r^n \sin n\theta = a^n$

Ans: $r^n = c^n \sec n\theta$

5. Find the O.T of the co focal and coaxial parabolas $r = \frac{2a}{1+\cos\theta}$

Ans: $r = \frac{c}{1-\cos\theta}$

NEWTON'S LAW OF COOLING

STATEMENT: The rate of change of the temperature of a body is proportional to the difference of the temperature of the body and that of the surrounding medium.

Let $'\theta'$ be the temperature of the body at time 't' and θo be the temperature of its surrounding medium(usually air). By the Newton's law of cooling , we have

$\frac{d\theta}{dt} \alpha\ (\theta - \theta o) \Rightarrow -\frac{d\theta}{dt}$- $k(\theta - \theta o)$ k is +ve constant

$$\Rightarrow \int \frac{d\theta}{(\theta-\theta o)} . = -k \int dt$$

$$\Rightarrow \log(\theta - \theta o) = -kt + c.$$

If initially $\theta = \theta_1$ is the temperature of the body at time t=0 then

$$c = \log(\theta_1 - \theta_0) \Rightarrow \log(\theta - \theta o) = -kt + \log(\theta_1 - \theta_0)$$

$$\Rightarrow \log \frac{(\theta - \theta_0)}{(\theta_1 - \theta_0)} (= -kt.$$

$$\Rightarrow \frac{(\theta - \theta_0)}{(\theta_1 - \theta_0)} = e^{-kt}$$

$$\theta = \theta o + (\theta_1 - \theta_0) . e^{-kt}$$

Which gives the temperature of the body at time 't' .

Problems:

1 A body is originally at 80^0C and cools down to 60^0C in 20 min . If the temperature of the air is 40^0 Cfind the temperature of body after 40 min.

Sol: By Newton's law of cooling we have

$\frac{d\theta}{dt}$ = -k$(\theta - \theta o)$ where θo is the temperature of the air.

$$\Rightarrow \int \frac{d\theta}{(\theta-\theta_0)} = -k\int dt \Rightarrow \log(\theta-\theta_0) = -kt + \log c$$

Here $\theta o = 40^0$ c

$\Rightarrow$ log($\theta - 40)$ = -kt + log c

$\Rightarrow$ log$(\frac{\theta - 40}{c})$ =-kt

$\Rightarrow \frac{\theta - 40}{c} = e^{-kt}$

$\Rightarrow \theta = 40 + c\ e^{-kt}$ --------------(1)

When t=0 , $\theta = 80^0$C$\Rightarrow$ 80 = 40 +c $\Rightarrow c = 40$------------(2).

When t=20 , $\theta = 60^0$C$\Rightarrow$60 = 40 +ce^{-20k} ------------(3).

Solving (2) & (3) $\Rightarrow$c$e^{-20k} = 20$

$\Rightarrow$ 40e^{-2k} =20

=> k = $\frac{1}{20}$ log2

When t= 40^0C => equation (1) is $\theta = 40 + 40\ e^{-(\frac{1}{20}\log 2)40}$

= 40 +40 $e^{-2\log 2}$

= 40 + (40 x $\frac{1}{4}$)

$\Rightarrow \theta = 50^0$C

2 . An object whose temperature is 75^0C cools in an atmosphere of constant temperature 25^0C, at the rate of k θ ,θ being the excess temperature of the body over that of the temperature. If after 10min , the temperature of the object falls to 65^0C , find its temperature after 20 min. Also find the time required to cool down to 55^0C.

Sol : We will take one minute as unit of time.

It is given that $\frac{d\theta}{dt}$ = - kθ

$\Rightarrow \theta = c\, e^{-kt}$ ------------(1).

Initially when t=0 $\Rightarrow \theta = 75^0 - 25^0 = 50^0$

$\Rightarrow$ c= 50^0

Hence $C = 50 \Rightarrow \theta = 50.e^{-kt}$ -----------------(2)

When t= 10 min $\Rightarrow \theta = 65^0 - 25^0 = 40^0$

$\Rightarrow$ 40= 50 e^{-10k}

$\Rightarrow e^{-10k} = \frac{4}{5}$ ---------------------(3).

The value of θ when t=20 $\Rightarrow \theta = c\, e^{-kt}$

$\theta = 50e^{-20k}$

$\theta = 50(e^{-10k})^2$

$\theta = 50(\frac{4}{5})^2$

When t=20 $\Rightarrow \theta = 32^0$C.

Hence the temperature after 20min $=32^0+25^0=57^0$C

When the temperature of the object = 55^0C

$\theta = 55^0 - 25^0 = 30^0 C$

Let t, be the corresponding time from equ. (2)

$30 = 50.e^{-kt_1}$ -----------------(4)

From equation (3) $\left(e^{-k}\right)^{10} = \frac{4}{5}\, i.e., e^{-k} = \left(\frac{4}{5}\right)^{\frac{1}{10}}$

From Equ(4) we get $30 = 50\left(\frac{4}{5}\right)^{\frac{t_1}{10}} \Rightarrow \frac{t_1}{10}\log\frac{4}{5} = \log\frac{3}{5}$

$$\Rightarrow t_1 = 10\left[\frac{\log\left(3/5\right)}{\log\left(4/5\right)}\right] = 22.9\,\text{min}$$

3. A body kept in air with temperature 25^0C cools from 140^0C to 80^0C in 20 min. Find when the body cools down in 35^0C.

Sol : By Newton's law of cooling $\frac{d\theta}{dt} = -k(\theta - \theta_0) \Rightarrow \frac{d\theta}{\theta - \theta_0} = -kdt$

$\Rightarrow \log(\theta - \theta_0) = -kt + c$ Here θo $=25^0 c$

$\Rightarrow$ log $(\theta - 25)$ = -kt +c -------------(1).

When t=0 , θ $=140^0$ c $\Rightarrow$ log (115) =c

$\Rightarrow$ c =log (115).

$\Rightarrow$ kt = - log $(\theta - 25)$+ log 115--------(2)

When t=20 , θ $=80^0$ c

$\Rightarrow$ log (80-25)= -20k + log 115

$\Rightarrow$ 20 k =log (115) - log(55) ---------(3)

(2)/ (3) => $\frac{kt}{20k} = \frac{\log 115 - \log(\theta - 25)}{\log 115 - \log 55}$

$\frac{t}{20} = \frac{\log 115 - \log(\theta - 25)}{\log 115 - \log 55}$

When θ $=35^0$ C $\Rightarrow \frac{t}{20} = \frac{\log 115 - \log(10)}{\log 115 - \log 55}$

$\Rightarrow \frac{t}{20} = \frac{\log(11.5)}{\log(\frac{23}{11})}$ = 3. 31

$\Rightarrow$temperature = 20 $\times$ 3.31 = 66.2

The temp will be 35^0C after 66.2 min.

4 . If the temperature of the air is 20^0C and the temperature of the body drops from 100^0C to 80^0C in 10 min. What will be its temperature after 20min. When will be the temperature 40^0C .

Sol: log $(\theta - 20)$ = -kt + log c

c $=80^0$ C and $e^{-10k} = \frac{3}{4}$.

$$t = \frac{10\log(\frac{1}{4})}{\log(\frac{5}{4})} = 4.82\text{min}$$

5. The temperature of the body drops from 100^0C to 75^0C in 10 min. When the surrounding air is at 20^0 C temperature. What will be its temp after half an hour.When will the temperature be 25^0C .

Sol : $\frac{d\theta}{dt} = -k(\theta - \theta o)$

$\log(\theta - 20) = -kt + \log c$

when t=0 , $\theta = 100^0$ => c=80

when t=10 , $\theta = 75^0$ => $e^{-10k} = \frac{11}{16}$.

when t =30min => $\theta = 20 + 80\left(\frac{1331}{4096}\right) = 46^O C$

when $\theta = 25^0$c => $t = 10\frac{\log 5 - \log 80}{(\log 11 - \log 16)} = 74.86\,\text{min}$

LAW OF NATURAL GROWTH OR DECAY

Statement : Let x(t) or x be the amount of a substance at time ' t' and let the substance be getting converted chemically . A law of chemical conversion states that the rate of change of amount x(t) of a chemically changed substance is proportional to the amount of the substance available at that time

$\frac{dx}{dt} \alpha \quad x$ (or) $\frac{dx}{dt} = -kx$; (k >0)

Where k is a constant of proportionality

Note: Incase of Natural growth we take

$$\frac{dx}{dt} = k.x \quad (k > 0)$$

PROBLEMS

1 The number N of bacteria in a culture grew at a rate proportional to N . The value of N was initially 100 and increased to 332 in one hour. What was the value of N after $1\frac{1}{2} hrs$

Sol: The differentialequation to be solved is $\frac{dN}{dt}$ = kN

$\Rightarrow \frac{dN}{N}$ **=** k dt

$\Rightarrow \int \frac{dN}{N} . = \int kdt$

$\Rightarrow$ log N = kt + log c

$\Rightarrow$ N = c e^{kt} ------------(1).

When t= 0sec , N =100 $\Rightarrow$ 100 =c $\Rightarrow$ c =100

When t =3600 sec , N =332 $\Rightarrow$ 332 =100 e^{3600k}

$\Rightarrow e^{3600k} = \frac{332}{100}$

Now when t =$\frac{3}{2}$ hors = 5400 sec then N=?

$\Rightarrow$ N =100 e^{5400k}

$\Rightarrow$ N =100$[e^{3600k}]^{\frac{3}{2}}$

$\Rightarrow$ N = 100 $[\frac{332}{100}]^{\frac{3}{2}}$ = 605.

$\Rightarrow$ N = 605.

2 . In a chemical reaction a given substance is being converted into another at a rate proportional to the amount of substance unconverted. If $\left(\frac{1}{5}\right)^{th}$ of the original amount has been transformed in 4 min, how much time will be required to transform one half.

Ans: t= 13 mins.

3. The temperature of a cup of coffee is 92^0C, when freshly poured the room temperature being 24^0C. In one min it was cooled to 80^0C. How long a period must elapse, before the temperature of the cup becomes 65^0C.

Sol: : By Newton's Law of cooling,

$\frac{d\theta}{dt}$ = -k($\theta - \theta o$) ; k>0

θo = 24^0C$\Rightarrow$ log ($\theta - 24$) = -kt + log c--------------(1).

When t=0 ; θ =92 $\Rightarrow$ c =68

When t =1 ; $\theta = 80^0$C$\Rightarrow e^{-k} = \frac{68}{56}$

$\Rightarrow$ k = log $\frac{56}{68}$.

When $\theta = 65^0$C , t =?

Ans: t = $\frac{65\times 41}{68^2} = 0.576\,\text{min}$

RATE OF DECAY OR RADIO ACTIVE MATERIALS

Statement : The disintegration at any instant is proportional to the amount of material present in it.

If u is the amount of the material at any time 't' , then $\frac{du}{dt}$ = - ku , where k is any constant (k >0).

Problems:

1) If 30% of a radioactive substance disappears in 10days,how long will it take for 90% of it to disappear.

Ans: 64.5 days

2) The radioactive material disintegrator at a rate proportional to its mass. When mass is 10 mgm , the rate of disintegration is 0.051 mgm per day . how long will it take for the mass to be reduced from 10 mgm to 5 mgm.

Ans: 136 days.

3. Uranium disintegrates at a rate proportional to the amount present at any instant. If M_1 and M_2 are grams of uranium that are present at times T_1 and T_2 respectively, find the half-life of uranium.

Ans: $T = \frac{(T2-T1)\,log2}{\log(\frac{M_1}{M_2})}$.

4. The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number double in 2 hrs, in how many hours will it be triple.

Ans: $\frac{2log3}{log2}$ hrs.

5. a) If the air is maintained at 30^0C and the temperature of the body cools from 80^0C to 60^0C in 12 min, find the temperature of the body after 24 min.

Ans: 48^0C

b) If the air is maintained at 150^0C and the temperature of the body cools from 70^0C to 40^0C in 10 min, find the temperature after 30 min.

UNIT - IV
HIGHER ORDER DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS

LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER

Definition: An equation of the form $\frac{d^n y}{dx^n}$ + $P_1(x)$. $\frac{d^{n-1} y}{dx^{n-1}}$ + $P_2(x)$. $\frac{d^{n-2} y}{dx^{n-2}}$ + --------+ $P_n(x)$ .y = Q(x) Where $P_1(x)$, $P_2(x)$, $P_3(x)$… …..$P_n(x)$ and Q(x) (functions of x) continuous is called a linear differential equation of order n.

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Def: An equation of the form $\frac{d^n y}{dx^n}$ + P_1 . $\frac{d^{n-1} y}{dx^{n-1}}$ + P_2 . $\frac{d^{n-2} y}{dx^{n-2}}$ + --------+ P_n .y = Q(x) where $P_1, P_2, P_3, \ldots P_n$, are real constants and Q(x) is a continuous function of x is called an linear differential equation of order ' n' with constant coefficients.

Note:

1. Operator D =$\frac{d}{dx}$; D^2 =$\frac{d^2}{dx^2}$; …………………… D^n =$\frac{d^n}{dx^n}$

   Dy =$\frac{dy}{dx}$; D^2 y=$\frac{d^2 y}{dx^2}$; …………………… D^n y=$\frac{d^n y}{dx^n}$

2. Operator $\frac{1}{D}$Q = $\int Q$ i e D^{-1}Q is called the integral of Q.

To find the general solution of f(D).y = 0 :

Where f(D) = $D^n + P_1 D^{n-1} + P_2 D^{n-2}$ +-----------+P_n is a polynomial in D.

Now consider the auxiliary equation : f(m) = 0

i.e f(m) = $m^n + P_1 m^{n-1} + P_2 m^{n-2}$ +-----------+P_n = 0

where p_1,p_2,p_3 ……………p_n are real constants.

Let the roots of f(m) =0 be $m_1, m_2, m_3, \ldots m_n$.

Depending on the nature of the roots we write the complementary function as follows:

Consider the following table

S.No	Roots of A.E f(m) =0	Complementary function(C.F)
1.	$m_1, m_2, ..m_n$ are real and distinct.	$y_c = c_1e^{m_1x} + c_2e^{m_2x} + \ldots + c_ne^{m_nx}$

2.	$m_1, m_2, ..m_n$ are and two roots are equal i.e., m_1, m_2 are equal and real(i.e repeated twice) &the rest are real and different.	$y_c = (c_1+c_2x)e^{m_1x} + c_3e^{m_3x} + \ldots + c_ne^{m_nx}$
3.	$m_1, m_2, ..m_n$ are real and three roots are equal i.e., m_1, m_2, m_3 are equal and real(i.e repeated thrice) &the rest are real and different.	$y_c = (c_1+c_2x+c_3x^2)e^{m_1x} + c_4e^{m_4x} + \ldots + c_ne^{m_nx}$
4.	Two roots of A.E are complex say $\alpha+i\beta\ \alpha - i\beta$ and rest are real and distinct.	$y_c = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) + c_3e^{m_3x} + \ldots + c_ne^{m_nx}$
5.	If $\alpha \pm i\beta$ are repeated twice & rest are real and distinct	$y_c = e^{\alpha x}[(c_1+c_2x)\cos\beta x + (c_3+c_4x)\sin\beta x)] + c_5e^{m_5x} + \ldots + c_ne^{m_nx}$
6.	If $\alpha \pm i\beta$ are repeated thrice & rest are real and distinct	$y_c = e^{\alpha x}[(c_1+c_2x+c_3x^2)\cos\beta x + (c_4+c_5x+c_6x^2)\sin\beta x)] + c_7e^{m_7x} + \ldots\ldots + c_ne^{m_nx}$
7.	If roots of A.E. irrational say $\alpha \pm \sqrt{\beta}$ and rest are real and distinct.	$y_c = e^{\alpha x}\left[c_1\cosh\sqrt{\beta}x + c_2\sinh\sqrt{\beta}x\right] + c_3e^{m_3x} + \ldots\ldots + c_ne^{m_nx}$

Solve the following Differential equations :

1. Solve $\frac{d^3y}{dx^3}$ **- 3**$\frac{dy}{dx}$ **+ 2y = 0**

Sol: Given equation is of the form f(D).y = 0

Where f(D) = $(D^3 - 3D + 2)\, y = 0$

Now consider the auxiliary equation f(m) = 0

$f(m) = m^3 - 3m + 2 = 0 \Rightarrow (m-1)(m-1)(m+2) = 0$

$\Rightarrow m = 1, 1, -2$

Since m_1 and m_2 are equal and m_3 is -2

We have $y_c = (c_1+c_2x)e^x + c_3e^{-2x}$

2. **Solve $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$**

Sol: Given $f(D) = (D^4 - 2D^3 - 3D^2 + 4D + 4)\, y = 0$

⇨ A.equation $f(m) = (m^4 - 2m^3 - 3m^2 + 4m + 4) = 0$

⇨ $(m + 1)^2 (m - 2)^2 = 0$

⇨ $m = -1, -1, 2, 2$

⇨ $y_c = (c_1+c_2x)e^{-x} + (c_3+c_4x)e^{2x}$

3. Solve $(D^4 + 8D^2 + 16)\, y = 0$

Sol: Given $f(D) = (D^4 + 8D^2 + 16)\, y = 0$

Auxiliary equation $f(m) = (m^4 + 8m^2 + 16) = 0$

⇨ $(m^2 + 4)^2 = 0$

⇨ $(m+2i)^2 (m+2i)^2 = 0$

⇨ $m = 2i, 2i, -2i, -2i$

$Y_c = e^{0x} [(c_1+c_2x)\cos 2x + (c_3+c_4x)\sin 2x)]$

4. Solve $y^{11}+6y^1+9y = 0$; $y(0) = -4$, $y^1(0) = 14$

Sol: Given equation is $y^{11}+6y^1+9y = 0$

Auxiliary equationf(D) y = 0 $\Rightarrow$ $(D^2 + 6D + 9)\, y = 0$

A.equation $f(m) = 0 \Rightarrow (m^2 + 6m + 9) = 0$

$\Rightarrow m = -3, -3$

$y_c = (c_1+c_2x)e^{-3x}$ -------------------> (1)

Differentiate of (1) w.r.to x $\Rightarrow$ $y^1 = (c_1+c_2x)(-3e^{-3x}) + c_2(e^{-3x})$

Given $y_1(0) = 14$ $\Rightarrow$ $c_1 = -4$ & $c_2 = 2$

Hence we get $y = (-4 + 2x)(e^{-3x})$

5. Solve $4y^{111} + 4y^{11} + y^1 = 0$

Sol: Given equation is $4y^{111} + 4y^{11} + y^1 = 0$

That is $(4D^3+4D^2+D)y=0$

Auxiliary equation $f(m) = 0$

$4m^3 + 4m^2 + m = 0$

$m(4m^2 + 4m + 1) = 0$

$m(2m + 1)^2 = 0$

$m = 0, -1/2, -1/2$

$y = c_1 + (c_2 + c_3x)\, e^{-x/2}$

6. Solve $(D^2 - 3D + 4)\, y = 0$

Sol: Given equation $(D^2 - 3D + 4)\, y = 0$
A.E. $f(m) = 0$

$m^2 - 3m + 4 = 0$

$m = \frac{3 \pm \sqrt{9-16}}{2} = \frac{3 \pm i\sqrt{7}}{2}$

$\alpha \pm i\beta = \frac{3 \pm i\sqrt{7}}{2} = \frac{3}{2} \pm i\frac{\sqrt{7}}{2}$

$y = e^{\frac{3}{2}x}\left(c_1 \cos\frac{\sqrt{7}}{2}x + c_2 \sin\frac{\sqrt{7}}{2}x\right)$

General solution of f(D) y = Q(x)

Is given by $y = y_c + y_p$

i.e. y = C.F+P.I

Where the P.I consists of no arbitrary constants and P.I of f (D) y = Q(x)

Is evaluated as P.I = $\frac{1}{f(D)}$. Q(x)

Depending on the type of function of Q(x).

P.I is evaluated as follows:

1. P.I of f (D) y = Q(x) where Q(x) =e^{ax} for (a) ≠ 0

Case1: P.I = $\frac{1}{f(D)} . Q(x) = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$

Provided f(a) ≠ 0

Case 2: If f(a) = 0 then the above method fails. Then

if $f(D) = (D-a)^k \phi(D)$

(i.e ‘ a’ is a repeated root k times).

Then P.I = $\frac{1}{\phi(a)} e^{ax} . \frac{1}{k!} x^k$ provided $\phi(a) \neq 0$

2. P.I of f(D) y =Q(x) where Q(x) = sin ax or Q(x) = cos ax where ‘ a ‘ is constant then

P.I = $\frac{1}{f(D)}$. Q(x).

Case 1: In f(D) put $D^2 = -a^2 \ni f(-a^2) \neq 0$ then P.I = $\frac{\sin ax}{f(-a^2)}$

Case 2: If $f(-a^2) = 0$ then $D^2 + a^2$ is a factor of $\phi(D^2)$ and hence it is a factor of f(D).

Then let $f(D) = (D^2 + a^2) .\Phi(D^2)$.

Then $\frac{\sin ax}{f(D)} = \frac{\sin ax}{(D^2+a^2)\Phi(D^2)} = \frac{1}{\Phi(-a^2)}\frac{\sin ax}{D^2+a^2} = \frac{1}{\Phi(-a^2)}\frac{-x\cos ax}{2a}$

$$\frac{\cos ax}{f(D)} = \frac{\cos ax}{(D^2+a^2)\Phi(D^2)} = \frac{1}{\Phi(-a^2)}\frac{\cos ax}{D^2+a^2} = \frac{1}{\Phi\left(-a^2\right)}\frac{x\sin ax}{2a}$$

3. **P.I for f(D) y = Q(x) where Q(x) = x^k where k is a positive integer f(D) can be express as f(D) =[1± $\emptyset(D)$]**

   Express $\frac{1}{f(D)} = \frac{1}{1\pm\emptyset(D)}$ = [1± $\emptyset(D)$] $^{-1}$

   Hence P.I = $\frac{1}{1\pm\emptyset(D)}$ Q(x).

   = [1± $\emptyset(D)$] $^{-1}$ .x^k

4. **P.I of f(D) y = Q(x) when Q(x) = e^{ax} V where 'a' is a constant and V is function of x. where V =sin ax or cos ax or x^k**

   Then P.I =$\frac{1}{f(D)}$ Q(x)

   =$\frac{1}{f(D)}$ e^{ax} V

   = e^{ax} [$\frac{1}{f(D+a)}$(V)]

   &$\frac{1}{f(D+a)}$ V is evaluated depending on V.

5. **P.I of f(D) y = Q(x) when Q(x) = x V where V is a function of x.**

   Then P.I = $\frac{1}{f(D)}$ Q(x)

   = $\frac{1}{f(D)}$ x V

   = [x - $\frac{1}{f(D)}$ f^1(D)] $\frac{1}{f(D)}$ V

6. **i. P.I. of f(D)y=Q(x) where Q(x)=x^mv where v is a function of x.**

   Then P.I. $= \frac{1}{f(D)} \times Q(x) = \frac{1}{f(D)} x^m v = I.P.of\ \frac{1}{f(D)} x^m(\cos ax + i\sin ax)$

   $= I.P.of\ \frac{1}{f(D)} x^m e^{iax}$

   ii. P.I. $= \frac{1}{f(D)} x^m \cos ax = R.P.of\ \frac{1}{f(D)} x^m e^{iax}$

Formulae

1. $\frac{1}{1-D} = (1-D)^{-1} = 1 + D + D^2 + D^3 +$ ------------------
2. $\frac{1}{1+D} = (1+D)^{-1} = 1 - D + D^2 - D^3 +$ ------------------
3. $\frac{1}{(1-D)^2} = (1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 +$ ------------------
4. $\frac{1}{(1+D)^2} = (1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 +$ ------------------
5. $\frac{1}{(1-D)^3} = (1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 +$ ------------------
6. $\frac{1}{(1+D)^3} = (1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 +$ ------------------

I. HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS:

1. Find the Particular integral of f(D) y = e^{ax} when f(a) ≠0
2. Solve the D.E $(D^2 + 5D + 6)\, y = e^x$
3. Solve $y^{11}+4y^1+4y = 4e^{3x}$; $y(0) = -1$, $y^1(0) = 3$
4. Solve $y^{11} + 4y^1 + 4y = 4\cos x + 3\sin x$, $y(0) = 1$, $y^1(0) = 0$
5. Solve $(D^2+9)\, y = \cos 3x$
6. Solve $y^{111} + 2y^{11} - y^1 - 2y = 1-4x^3$
7. Solve the D.E $(D^3 - 7D^2 + 14D - 8)\, y = e^x \cos 2x$
8. Solve the D.E $(D^3 - 4D^2 - D + 4)\, y = e^{3x} \cos 2x$
9. Solve $(D^2 - 4D + 4)\, y = x^2 \sin x + e^{2x} + 3$
10. Apply the method of variation parameters to solve $\frac{d^2y}{dx^2} + y = \text{cosec}x$
11. Solve $\frac{dx}{dt} = 3x + 2y$, $\frac{dy}{dt} + 5x + 3y = 0$
12. Solve $(D^2 + D - 3)\, y = x^2 e^{-3x}$
13. Solve $(D^2 - D - 2)\, y = 3e^{2x}$, $y(0) = 0$, $y^1(0) = -2$

SOLUTIONS:

1) Particular integral of f(D) y = e^{ax} when f(a) ≠0

Working rule:

Case (i):

In f(D), put D=a and Particular integral will be calculated.

Particular integral= $\frac{1}{f(D)}\ e^{ax} = \frac{1}{f(a)}\ e^{ax}$ provided f(a) $\neq 0$

Case (ii) :

If f(a)= 0 , then above method fails. Now proceed as below.

If f(D)= $(D-a)^K \phi(D)$

i.e. 'a' is a repeated root k times, then

Particular integral= $\frac{e^{ax}}{\phi(a)} \cdot \frac{x^k}{k!}$ provided $\phi(a) \neq 0$

2. Solve the Differential equation$(D^2+5D+6)y=e^x$

Sol : Given equation is $(D^2+5D+6)y=e^x$

Here $Q(x) = e^x$

Auxiliary equation is $f(m) = m^2+5m+6=0$

$m^2+3m+2m+6=0$

$m(m+3)+2(m+3)=0$

m=-2 or m=-3

The roots are real and distinct

C.F = $y_c = c_1 e^{-2x} + c_2 e^{-3x}$

Particular Integral = $y_p = \frac{1}{f(D)} . Q(x)$

$= \frac{1}{D2+5D+6} e^x \quad = \frac{1}{(D+2)(D+3)} e^x$

Put D = 1 in f(D)

P.I. = $\frac{1}{(3)(4)} e^x$

Particular Integral = $y_p = \frac{1}{12} . e^x$

General solution is $y=y_c+y_p$

$y=c_1e^{-2x}+c_2\, e^{-3x} + \frac{e^x}{12}$

3) Solve $y^{11}-4y^1+3y=4e^{3x}$, $y(0) = -1$, $y^1(0) = 3$

Sol : Given equation is $y^{11}-4y^1+3y=4e^{3x}$

i.e. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 4e^{3x}$

it can be expressed as

$D^2y-4Dy+3y=4e^{3x}$

$(D^2-4D+3)y=4e^{3x}$

Here $Q(x)=4e^{3x}$; $f(D)= D^2-4D+3$

Auxiliary equation is $f(m)=m^2-4m+3 = 0$

$m^2-3m-m+3 = 0$

$m(m-3) -1(m-3)=0 \Rightarrow m=3$ or 1

The roots are real and distinct.

C.F= $y_c=c_1e^{3x}+c_2e^x$ ----→ (2)

P.I.= $y_p= \frac{1}{f(D)} . Q(x)$

$= y_p= \frac{1}{D^2-4D+3} . 4e^{3x}$

$= y_p= \frac{1}{(D-1)(D-3)} . 4e^{3x}$

Put D=3

$$y_p = \frac{4e^{3x}}{(3-1)(D-3)} = \frac{4}{2}\frac{e^{3x}}{(D-3)} = 2\frac{x^1}{1!}e^{3x} = 2xe^{3x}$$

General solution is $y=y_c+y_p$

$y=c_1e^{3x}+c_2\, e^x+2xe^{3x}$ ------------------→ (3)

Equation (3) differentiating with respect to ‘x’

$y^1=3c_1e^{3x}+c_2e^x+2e^{3x}+6xe^{3x}$ -----------→ (4)

By data, $y(0) = -1$, $y^1(0)=3$

From (3), $-1=c_1+c_2$ -------------------→ (5)

From (4), $3=3c_1+c_2+2$

$3c_1+c_2=1$ -------------------→ (6)

Solving (5) and (6) we get $c_1=1$ and $c_2 = -2$

$y=-2e^{x} +(1+2x)e^{3x}$

(4). Solve $y^{11}+4y^1+4y= 4\cos x + 3\sin x$, $y(0) = 0$, $y^1(0) = 0$

Sol: Given differential equation in operator form

$(D^2 + 4D + 4)y= 4\cos x +3\sin x$

A.E is $m^2+4m+4 = 0$

$(m+2)^2=0$ then $m=-2, -2$

∴ C.F is $y_c= (c_1 + c_2x)e^{-2x}$

P.I is $= y_p= \frac{4\cos x+3\sin x}{(D^2+4D+4)}$ put $D^2 = -1$

$y_p= \frac{4\cos x+3\sin x}{(4D+3)} = \frac{(4D-3)(4\cos x+3\sin x)}{(4D-3)(4D+3)}$

$= \frac{(4D-3)(4\cos x+3\sin x)}{16D^2-9}$

Put $D^2 = -1$

$\therefore y_p= \frac{(4D-3)(4\cos x+3\sin x)}{-16-9}$

$= \frac{-16\sin x+12\cos x-12\cos x-9\sin x)}{-25} = \frac{-25\sin x}{-25} = \sin x$

∴General equation is $y = y_c+ y_p$

$y = (c_1 + c_2x)e^{-2x} + \sin x$ ------------ (1)

By given data, $y(0) = 0$ ∴ $c_1 = 0$ and

Diff (1) w.r.. t. $y^1 = (c_1 + c_2x)(-2)e^{-2x} + e^{-2x}(c_2)$ +cosx ------------ (2)

given $y^1(0) = 0$

(2) $\Rightarrow -2c_1 + c_2+1=0$ $\therefore c_2 = -1$

$\therefore$Required solution is $y = -xe^{-2x}$+sinx

5. Solve $(D^2+9)y = \cos 3x$

Sol:Given equation is $(D^2+9)y = \cos 3x$

A.E is $m^2+9 = 0$

$\therefore$ $m = \pm 3i$

y_c = C.F = $c_1 \cos 3x + c_2 \sin 3x$

y_c =P.I = $\frac{\cos 3x}{D^2+9} = \frac{\cos 3x}{D^2+3^2}$

$= \frac{x}{2(3)} \sin 3x = \frac{x}{6} \sin 3x$

General equation is $y = y_c + y_p$

$y = c_1 \cos 3x + c_2 \cos 3x + \frac{x}{6} \sin 3x$

6. Solve $y^{111}+2y^{11} - y^1-2y= 1-4x^3$

Sol:Given equation can be written as

$(D^3 + 2D^2 - \mathrm{D} - 2)\mathrm{y} = 1-4x^3$

A.E is $(m^3 + 2m^2 - \mathrm{m} - 2) = 0$

$(m^2 - 1)(m+2) = 0$

$m^2 = 1$ or $m=- 2$

m = 1, -1, -2

C.F $= c_1 e^{x} + c_2 e^{-x} + c_3 e^{-2x}$

P.I $= \dfrac{1}{(D^3+2D^2-D-2)}(1-4x^3)$

$= \dfrac{-1}{2\left[1-\dfrac{(D^3+2D^2-D)}{2}\right]}\ (1-4x^3)$

$= \dfrac{-1}{2}\left[1-\dfrac{(D^3+2D^2-D)}{2}\right]^{-1}(1-4x^3)$

$= \dfrac{-1}{2}\left[1+\dfrac{(D^3+2D^2-D)}{2}+\dfrac{(D^3+2D^2-D)^2}{4}+\dfrac{(D^3+2D^2-D)^3}{8}+\ldots\right](1-4x^3)$

$= \dfrac{-1}{2}\left[1+\dfrac{1}{2}(D^3+2D^2-D)+\dfrac{1}{4}(D^2-4D^3)+\dfrac{1}{8}(-D^3)\right](1-4x^3)$

$= \dfrac{-1}{2}\left[1-\dfrac{5}{8}(D^3)+\dfrac{5}{4}(D^2)-\dfrac{1}{2}D\right](1-4x^3)$

$= \dfrac{-1}{2}\left[(1-4x^3)-\dfrac{5}{8}(-24)+\dfrac{5}{4}(-24x)-\dfrac{1}{2}(-12x^2)\right]$

$= \dfrac{-1}{2}[-4x^3+6x^2-30x+16] =$

$= [2x^3-3x^2+15x-8]$

The general solution is

y= C.F + P.I

$y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{-2x} + [2x^3-3x^2+15x-8]$

7. Solve $(D^3-7D^2+14D-8)y = e^{x}\cos 2x$

Given equation is

$(D^3-7D^2+14D-8)y = e^{x}\cos 2x$

A.E is $(m^3-7m^2+14m-8) = 0$

(m-1) (m-2)(m-4) = 0

Then m = 1,2,4

C.F = $c_1 e^{x} + c_2 e^{2x} + c_3 e^{4x}$

P.I = $\frac{e^{x} cos2x}{(D^3-7D^2+14D-8)}$

$$= e^{x} \cdot \frac{1}{(D+1)^3-7(D+1)^2+14(D+1)-8} \cdot \text{Cos}2x$$

$$\left[\because P.I = \frac{1}{f(D)} e^{ax} v = e^{ax} \frac{1}{f(D+a)} v\right]$$

$$= e^{x} \cdot \frac{1}{(D^3-4D^2+3D)} .\cos 2x$$

$$= e^{x} \cdot \frac{1}{(-4D+3D+16)} .\cos 2x$$ (Replacing D^2 with -2^2)

$$= e^{x} \cdot \frac{1}{(16-D)} .\cos 2x$$

$$= e^{x} \cdot \frac{16+D}{(16-D)(16+D)} .\cos 2x$$

$$= e^{x} \cdot \frac{16+D}{256-D^2} .\cos 2x$$

$$= e^{x} \cdot \frac{16+D}{256-(-4)} .\cos 2x$$

$$= \frac{e^{x}}{260} (16\cos 2x - 2\sin 2x)$$

$$= \frac{2e^{x}}{260} (8\cos 2x - \sin 2x)$$

$$= \frac{e^{x}}{130} (8\cos 2x - \sin 2x)$$

General solution is $y = y_c + y_p$

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{4x} + \frac{e^x}{130}(8\cos 2x - \sin 2x)$$

8. Solve $(D^2 - 4D + 4)y = x^2 sinx + e^{2x} + 3$

Sol:Given $(D^2 - 4D + 4)y = x^2\ sinx + e^{2x} + 3$

A.E is $(m^2 - 4m + 4) = 0$

$(m - 2)^2 = 0$ then m=2,2

C.F. = $(c_1 + c_2x)e^{2x}$

P.I = $\frac{x^2 sinx + e^{2x} + 3}{(D-2)^2} = \frac{1}{(D-2)^2}(x^2 sinx) + \frac{1}{(D-2)^2} e^{2x} + \frac{1}{(D-2)^2}(3)$

Now $\frac{1}{(D-2)^2}(x^2 sinx) = \frac{1}{(D-2)^2}(x^2)$ (I.P of e^{ix})

$= \text{I.P of } \frac{1}{(D-2)^2}(x^2)(e^{ix})$

$= \text{I.P of } (e^{ix}).\frac{1}{(D+i-2)^2}(x^2)$

On simplification, we get

$\frac{1}{(D+i-2)^2}(x^2\ sinx) = \frac{1}{625}$ [(220x+244)cosx+(40x+33)sinx]

and $\frac{1}{(D-2)^2}(e^{2x}) = \frac{x^2}{2}(e^{2x})$,

$\frac{1}{(D-2)^2}(3) = \frac{3}{4}$

P.I = $\frac{1}{625}$ [(220x+244)cosx+(40x+33)sinx] + $\frac{x^2}{2}(e^{2x}) + \frac{3}{4}$

$y = y_c + y_p$

$y = (c_1 + c_2x)e^{2x} + \frac{1}{625}$ [(220x+244)cosx+(40x+33)sinx] + $\frac{x^2}{2}(e^{2x}) + \frac{3}{4}$

Variation of Parameters :

Working Rule :

1. Reduce the given equation of the form $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R$
2. Find C.F.
3. Take P.I. y_p=Au+Bv where A= $-\int\frac{vRdx}{uv^1-vu^1}$ $and\ B=\int\frac{uRdx}{uv^1-vu^1}$
4. Write the G.S. of the given equation $y=y_c+y_p$

9. Apply the method of variation of parameters to solve $\frac{d^2y}{d\,x^2}$ + y = cosecx

Sol: Given equation in the operator form is $(D^2+1)y = cosecx$----------------(1)

A.E is (m^2+1) = 0

$\therefore m=\pm i$

The roots are complex conjugate numbers.

∴ C.F. is $y_c=c_1\cos x + c_2\sin x$

Let y_p = Acosx + Bsinx be P.I. of (1)

$u\frac{dv}{dx}$ - $v\frac{du}{dx}$ = $\cos^2x+\ \sin^2x$ =1

A and B are given by

A= $-\int\frac{vRdx}{uv^1-vu^1}$ = - $\int\frac{\sin x\ cosec\ x}{1}$ dx = - $\int dx$ = - x

B = $\int\frac{uRdx}{uv^1-vu^1}$ = $\int \cos x.\,cosecx\,dx = \int \cot x\,dx$ = log(sinx)

∴y_p= -xcosx +sinx. log(sinx)

∴ General solution is y = y_c+ y_p.

y = $c_1$cosx + $c_2$sinx-xcosx +sinx. log(sinx)

10. Solve ($4D^2-4$D +1)y = 100

Sol:A.E is $(4m^2-4\text{m}+1)$ = 0

$(2m-1)^2 = 0$ then m = $\frac{1}{2}, \frac{1}{2}$

C.F = $(c_1+c_2x)\, e^{\frac{x}{2}}$

P.I = $\frac{100}{(4D^2-4D+1)} = \frac{100\, e^{0.x}}{(2D-1)^2} = \frac{100}{(0-1)^2}$ = 100

Hence the general solution is y = C.F +P.I

y= $(c_1+c_2x)\, e^{\frac{x}{2}}$ + 100

Applications of Differential Equations:

11. The differential equation satisfying a beam uniformly loaded (w kg/meter) with one end fixed and the second end subjected to tensile force p is given by

$$EI \frac{d^2y}{dx^2} = py - \frac{1}{2} wx^2$$

Show that the elastic curve for the beam with conditions y=0= $\frac{dy}{dx}$ at x=0 is given by y = $\frac{w}{n^2 p}$ (1-coshnx) + $\frac{wx^2}{2p}$ where $n^2 = \frac{p}{EI}$

Sol:The given differential equation can be written as

$\frac{d^2y}{dx^2} - \frac{p}{EI} y = \frac{-1}{2EI} wx^2$ (or)

$\frac{d^2y}{dx^2} - n^2 y = \frac{-w}{2EI} x^2$ (or)

$(D^2 - n^2)y = \frac{-w}{2EI} x^2$ -----------(1)

The auxiliary equation is $(m^2 - n^2) = 0$ => m = n and m= -n

$\therefore$ C.F = $y_c = c_1 e^{nx} + c_2 e^{-nx}$

P.I = $\frac{1}{(D^2-n^2)}$ $(\frac{-w}{2EI}x^2)$

$= \frac{w}{2EI}(\frac{1}{(n^2-D^2)}x^2)$

$= \frac{w}{2EI}(\frac{1}{\left(n^2(1-\frac{D^2}{n^2})\right)}x^2)$

$= \frac{w}{2EI.n^2}(1-\frac{D^2}{n^2})^{-1}.x^2$

$= \frac{w}{2EI.n^2}(1+\frac{D^2}{n^2}+ - - - - - -).x^2$

$= \frac{w}{2EI.n^2}(x^2+\frac{2}{n^2})$

∴ The general solution of equation (1) is given by y= C.F + P.I

y= $c_1 e^{nx} + c_2 e^{-nx} + \frac{w}{2EI.n^2}(x^2+\frac{2}{n^2})$

12. A condenser of capacity 'C' discharged through an inductance L and resistance R in series and the charge q at time t satisfies the equation $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$. Given that L=0.25H, R = 250ohms, c=2 * 10 $^{-6}$farads, and that when t =0, change q is 0.002 coulombs and the current $\frac{dq}{dt} = 0$, obtain the value of 'q' in terms of t.

Sol:

Given differential equation is

$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$ or $\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0$ -------------(1)

Substituting the given values in (1), we get

$\frac{d^2q}{dt^2} + \frac{250}{0.25}\frac{dq}{dt} + \frac{q}{0.25*2*10^{-6}} = 0$ or

$\frac{d^2q}{dt^2}$ + 1000$\frac{dq}{dt}$ +$2 * 10^6$q =0 or

$(D^2 + 1000D + 2 * 10^6)$q = 0

Its A.E is $m^2 + 1000m + 2 * 10^6$ =0

$\therefore$ m=$\frac{-1000 \pm \sqrt{10^6 - 8\times 10^6}}{2} = \frac{-1000 \pm 1000\sqrt{7i}}{2}$

= -500 $\pm$ 1323i

Thus the solution is q= e^{-500t}($c_1$cos1323t+$c_2$sin1323t)

When t=0, q=0.002 since c_1= 0.002

Now $\frac{dq}{dt} = -500e^{-500t}(c_1\cos 1323t + c_2 \sin 1323t) + e^{-500t} \times 1323(-c_1 \sin 1323t + c_2 \cos 1323t)$

When $t = 0, \frac{dq}{dt} = 0$

There fore c_2=0.0008

Hence the required solution is $q = e^{-500t}(0.002\cos 1323t + 0.0008\sin 1323t)$

13. A particle is executing S.H.M, with amplitude 5 meters and time 4 seconds. Find the time required by the particle in passing between points which are at distances 4 and 2 meters from the Centre of force and are on the same side of it.

Sol: The equation of S.H.M is $\frac{d^2x}{dt^2}$ = - $\mu^2 x$-------------(1)

Give time period = $\frac{2\pi}{\mu}$ =4

μ = $\frac{\pi}{2}$

We have the solution of (1) is x=acos μt a =5, $\mu = \frac{\pi}{2}$

x = 5cos $\frac{\pi}{2}t$-------------(2)

Let the times when the particle is at distances of 4 meters and 2 meters from the centre of motion respectively be t_1 sec and t_2 sec

$\therefore$ $t_1 = \frac{2}{\pi} cos^{-1}(\frac{4}{5})$ since $[4= 5\cos(\frac{\pi}{2}t_1)]$

and $t_2 = \frac{2}{\pi} cos^{-1}(\frac{2}{5})$ since $[2= 5\cos(\frac{\pi}{2}t_2)]$

time required in passing through these points

$$t_2 - t_1 = \frac{2}{\pi}\left[cos^{-1}\left(\frac{2}{5}\right) - cos^{-1}\left(\frac{4}{5}\right)\right] = 0.33\text{sec}$$

differentiating (2) w.r.to 't'

$$\frac{dx}{dt} = \frac{-5\pi}{2}\sin\frac{\pi}{2}t$$

$$= \frac{-5\pi}{2}\sqrt{1 - \frac{x^2}{25}}$$

$$\frac{dx}{dt} = \frac{-\pi}{2}\sqrt{25 - x^2}$$

When x=4 meters v = $\frac{\pi}{2}\sqrt{5^2 - 4^2}$ = 4.71 m/sec

When x=2 meters v=$\frac{\pi}{2}\sqrt{21}$ m/sec

14. A body weighing 10kgs is hung from a spring. A pull of 20kgs will stretch the spring to 10cms. The body is pulled down to 20cms below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time t seconds the maximum velocity and the period of oscillation.

Sol:Let 0 be the fixed end and A be the other end of the spring. Since load of 20kg attached to A stretches the spring by 0.1m.

Let e(AB) be the elongation produced by the mass 'm' hanging in equilibrium.

If 'k' be the restoring force per unit stretch of the spring due to elasticity, then for the equilibrium at B

Mg = T =ke

20 = T_0 = k * 0.1

K = 200kg/m

Let B be the equilibrium position when 10kg weight is

10 = T_B= k * AB => AB = $\frac{10}{200}$ = 0.05m

Now the weight is pulled down to c, where BC=0.2. After any time t of its release from c, let the weight be at p, where BP=x.

Then the tension T = k *AP

= 200(0.05+x) = 10 + 200x

∴ The equation of motion of the body is

$\frac{w}{g}\frac{d^2x}{dt^2}$ = w –T where g =9.8m/sec^2

= $\frac{10}{9.8}\frac{d^2x}{dt^2}$

= 10 – (10+200x)

⇨ $\frac{d^2x}{dt^2} = -\mu^2 x$ where $\mu = 14$

This shows that the motion of the body in simple harmonic about B as centre and the period of oscillation = $\frac{2\pi}{\mu}$= 0.45sec

Also the amplitude of motion being B C=0.2m, the displacement of the body from B at time t is given by x = 0.2cosect

$X = 0.2\cos \omega t = 0.2\cos 14t$ m.

Maximum velocity = μ (amplitude) = 14 * 0.2 = 2.8m/sec

UNIT - V

FUNCTIONS OF SINGLE AND SEVERAL VARIABLES

MEAN VALUE THEOREMS

I Rolle's Theorem:

Let f(x) be a function such that

(i). It is continuous in closed interval [a,b]

(ii). It is differentiable in open interval (a,b) and

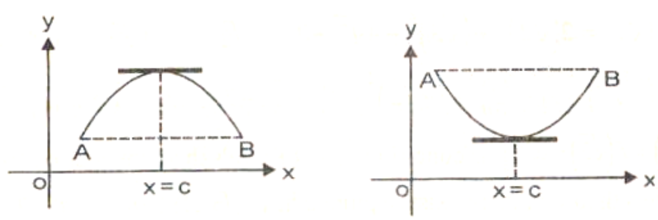
(iii). f(a) = f(b).

Then there exists at least one point 'c' in (a,b) such that

$f^1(c) = 0$.

Geometrical Interpretation of Rolle's Theorem :

Let $f:[a,b] \to R$ be a function satisfying the three conditions of Rolle's theorem. Then the graph.



1. y=f(x) in a continuous curve in [a,b].
2. There exist a unique tangent line at every point x=c, where a<c<b
3. The ordinates f(a), f(b) at the end points A,B are equal so that the points A and B are equidistant from the X-axis.
4. By Rolle's Theorem, There is at least one point x=c between A and B on the curve at which the tangent line is parallel to the x-axis and also it is parallel to chord of the curve.

1. Verify Rolle's theorem for the function f(x) = sinx/e^x or e^{-x} sinx in [0,π]

Sol: i) Since sinx and e^x are both continuous functions in [0, π].

Therefore, sinx/e^x is also continuous in [0,π].

ii) Since sinx and e^x be derivable in (0,π), then f is also derivable in (0,π).

iii) f(0) = sin0/e^0 = 0 and f(π)= sin π/e^{π} =0

∴ f(0) = f(π)

Thus all three conditions of Rolle's theorem are satisfied.

∴ There exists c є(0, π) such that $f^1(c)=0$

Now $f^1(x) = \frac{e^x \cos x - \sin x\, e^x}{(e^x)^2} = \frac{\cos x - \sin x}{e^x}$

$f^1(c)= 0$ => $\frac{\cos c - \sin c}{e^c} = 0$

cos c = sin c => tan c = 1

c = π/4 є(0,π)

Hence Rolle's theorem is verified.

2. Verify Rolle's theorem for the functions $\log\left(\frac{x^2+ab}{x(a+b)}\right)$ **in[a,b] , a>0, b>0,**

Sol: Let f(x)= $\log\left(\frac{x^2+ab}{x(a+b)}\right)$

= log(x^2+ab) – log x –log(a+b)

(i). Since f(x) is a composite function of continuous functions in [a,b], it is continuous in [a,b].

(ii). $f^1(x)$ = $\frac{1}{x^2+ab}.2x - \frac{1}{x} = \frac{x^2-ab}{x(x^2+ab)}$

$f^1(x)$ exists for all xє (a,b)

(iii). f(a) = $\log\left[\frac{a^2+ab}{a^2+ab}\right] = \log 1 = 0$

f(b) = $\log\left[\frac{b^2+ab}{b^2+ab}\right] = \log 1 = 0$

f(a) = f(b)

Thus f(x) satisfies all the three conditions of Rolle's theorem.

So, $\exists$ c $\in$ (a, b) $\Rightarrow f^1(c) = 0$,

$f^1(c) = 0$, $\Rightarrow \frac{c^2-ab}{c(c^2+ab)} = 0 \Rightarrow c^2 = ab$

$\Rightarrow c = \sqrt{ab} \in (a,b)$

Hence Rolle's theorem verified.

3. Verify whether Rolle 's Theorem can be applied to the following functions in the intervals.

i) f(x) = tan x in[0 , π] and ii) $f(x) = 1/x^2$ in [-1,1]

(i) f(x) is discontinuous at x = π/2 as it is not defined there. Thus condition (i) of Rolle 's Theorem is not satisfied. Hence we cannot apply Rolle 's Theorem here.

$\therefore$ Rolle's theorem cannot be applicable to f(x) = tan x in [0,π].

(ii). $f(x) = 1/x^2$ in [-1,1]

f(x) is discontinuous at x= 0.

Hence Rolle 's Theorem cannot be applied.

4. Verify Rolle's theorem for the function $f(x) = (x-a)^m(x-b)^n$ where m,n are positive integers in [a,b].

Sol: (i). Since every polynomial is continuous for all values, f(x) is also continuous in[a,b].

(ii) $f(x) = (x-a)^m(x-b)^n$

$f^1(x) = m(x-a)^{m-1}(x-b)^n+(x-a)^m.n(x-b)^{n-1}$

$= (x-a)^{m-1}(x-b)^{n-1}[m(x-b)+n(x-a)]$

$=(x-a)^{m-1}(x-b)^{n-1}[(m+n)x-(mb+na)]$

Which exists.

Thus f(x) is derivable in (a,b)

(iii) f(a) = 0 and f(b) = 0

$\therefore$ f(a) =f(b)

Thus three conditions of Rolle's theorem are satisfied.

$\therefore$ There exists cϵ(a,b) such that $f^1(c)=0$

$(c-a)^{m-1}(c-b)^{n-1}[(m+n)c-(mb+na)]=0$

$\Rightarrow$ (m+n)c-(mb+na)=0 => (m+n)c = mb+na

$\Rightarrow c = \frac{mb+na}{m+n} \; \epsilon(a,b)$

$\therefore$ Rolle's theorem verified.

5. Using Rolle 's Theorem, show that g(x) = $8x^3-6x^2-2x+1$ has a zero between

0 and 1.

Sol: g(x) = $8x^3-6x^2-2x+1$ being a polynomial, it is continuous on [0,1] and differentiable on (0,1)

Now g(0) = 1 and g(1)= 8-6-2+1 = 1

Also g(0)=g(1)

Hence, all the conditions of Rolle's theorem are satisfied on [0,1].

Therefore, there exists a number cϵ(0,1) such that $g^1(c)=0$.

Now $g^1(x) = 24x^2-12x-2$

$\therefore g^1(c)= 0 \Rightarrow 24c^2-12c-2 =0$

$\Rightarrow c= \frac{3 \pm \sqrt{21}}{12}$ *ie* c= 0.63 or -0.132

only the value c = 0.63 lies in (0,1)

Thus there exists at least one root between 0 and 1.

6. Verify Rolle's theorem for $f(x) = x^{2/3} - 2x^{1/3}$ in the interval (0,8).

Sol: Given $f(x) = x^{2/3} - 2x^{1/3}$

f(x) is continuous in [0,8]

$f^1(x) = 2/3 \,.\, 1/x^{1/3} - 2/3 \,.\, 1/x^{2/3} = 2/3(1/x^{1/3} - 1/x^{2/3})$

Which exists for all x in the interval (0,8)

$\therefore$ f is derivable (0,8).

Now $f(0) = 0$ and $f(8) = (8)^{2/3} - 2(8)^{1/3} = 4-4 = 0$

i.e., f(0) = f(8)

Thus all the three conditions of Rolle's Theorem are satisfied.

$\therefore$ There exists at least one value of c in(0,8) such that $f^1(c)=0$

ie. $\frac{1}{c^{\frac{1}{3}}} - \frac{1}{c^{\frac{2}{3}}} = 0 \Rightarrow c = 1 \in (0,8)$

Hence Rolle's Theorem is verified.

7. Verify Rolle's theorem for $f(x) = x(x+3)e^{-x/2}$ in [-3,0].

Sol: - (i). Since x(x+3) being a polynomial is continuous for all values of x and $e^{-x/2}$ is also continuous for all x, their product $x(x+3)e^{-x/2} = f(x)$ is also continuous for every value of x and in particular f(x) is continuous in the [-3,0].

(ii). we have $f^1(x) = x(x+3)(-1/2 \; e^{-x/2}) + (2x+3)e^{-x/2}$

$= e^{-x/2}\,[2x+3-\frac{x^2+3x}{2}]$

$= e^{-x/2}[6+x-x^2/2]$

Since $f^1(x)$ doesnot become infinite or indeterminate at any point of the interval(-3,0).

f(x) is derivable in (-3,0)

(iii) Also we have f(-3) = 0 and f(0) =0

$\therefore$ f (-3)=f(0)

Thus f(x) satisfies all the three conditions of Rolle's theorem in the interval [-3,0].

Hence there exist at least one value c of x in the interval (-3,0) such that $f^1(c)=0$

i.e., $\frac{1}{2} e^{-c/2}(6+c-c^2)=0 \Rightarrow 6+c-c^2=0$ ($e^{-c/2} \neq 0$ for any c)

$\Rightarrow c^2+c-6 = 0 \Rightarrow (c-3)(c+2)=0$

c=3,-2

Clearly, the value c= -2 lies within the (-3,0) which verifies Rolle's theorem.

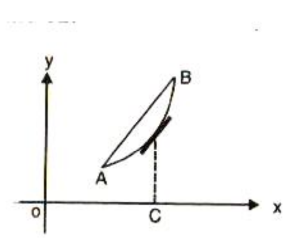
II. Lagrange's mean value Theorem

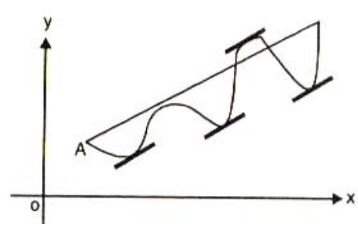
Let f(x) be a function such that (i) it is continuous in closed interval [a,b] & (ii) differentiable in (a,b).
Then $\exists$ at least one point c in (a,b) such that

$$f^1(c) = \frac{f(b)-f(a)}{b-a}$$

Geometrical Interpretation of Lagrange's Mean Value theorem:

Let $f:[a,b] \rightarrow R$ be a function satisfying the two conditions of Lagrange's theorem. Then the graph.





1. y=f(x) is continuous curve in [a,b]

2. At every point x=c, when a<c<b, on the curve y=f(x), there is unique tangent to the curve. By Lagrange's theorem there exists at least one point $c \in (a,b) \ni f^1(c) = \frac{f(b)-f(a)}{b-a}$

Geometrically there exist at least one point c on the curve between A and B such that the tangent line is parallel to the chord $\overleftrightarrow{AB}$

1. Verify Lagrange's Mean value theorem for f(x)= x^3-x^2-5x+3 in [0,4]

Sol: Let f(x)= x^3-x^2-5x+3 is a polynomial in x.

$\therefore$ It is continuous & derivable for every value of x.

In particular, f(x) is continuous [0,4] & derivable in (0,4)

Hence by Lagrange's Mean value theorem $\exists$ c$\in$ (0,4) $\ni$

$f^1(c) = \frac{f(4)-f(0)}{4-0}$

i.e., $3c^2-2c-5 = \frac{f(4)-f(0)}{4}$ ……………………(1)

Now $f(4) = 4^3-4^2-5.4+3 = 64-16-20-3 = 67-36 = 31$ & f(0)=3

$$\frac{f(4)-f(0)}{4} = \frac{(31-3)}{4} = 7$$

From equation (1), we have

$3c^2-2c-5 = 7 \Rightarrow 3c^2-2c-12 = 0$

$$c = \frac{2 \pm \sqrt{4+144}}{6} = \frac{2 \pm \sqrt{148}}{6} = \frac{1 \pm \sqrt{37}}{3}$$

We see that $\frac{1+\sqrt{37}}{3}$ lies in open interval (0,4) & thus Lagrange's Mean value theorem is verified.

2. Verify Lagrange's Mean value theorem for f(x) = $\log_e x$ in [1,e]

Sol: - f(x) = $\log_e x$

This function is continuous in closed interval [1,e] & derivable in (1,e). Hence L.M.V.T is applicable here. By this theorem, $\exists$ a point c in open interval (1,e) such that

$$f^1(c) = \frac{f(e)-f(1)}{e-1} = \frac{1-0}{e-1} = \frac{1}{e-1}$$

But $f^1(c) = \frac{1}{e-1} \implies \frac{1}{c} = \frac{1}{e-1}$

$\therefore$ c = e - 1

Note that (e-1) is in the interval (1,e).

Hence Lagrange's mean value theorem is verified.

3. Give an example of a function that is continuous on [-1, 1] and for which mean value theorem does not hold with explanations.

Sol:- The function f(x) = $|x|$ is continuous on [-1,1]

But Lagrange Mean value theorem is not applicable for the function f(x) as its derivative does not exist in (-1,1) at x=0.

4. If a<b, P.T $\frac{b-a}{1+b^2} < Tan^{-1}b - Tan^{-1}a < \frac{b-a}{1+a^2}$ **using Lagrange's Mean value theorem. Deduce the following.**

i). $\frac{\pi}{4} + \frac{3}{25} < Tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

ii). $\frac{5\pi+4}{20} < Tan^{-1}2 < \frac{\pi+2}{4}$

Sol: consider f(x) = Tan^{-1} x in [a,b] for 0<a<b<1

Since f(x) is continuous in closed interval [a,b] & derivable in open interval (a,b).

We can apply Lagrange's Mean value theorem here.

Hence there exists a point c in (a,b)∍

$f^1(c) = \dfrac{f(b)-f(a)}{b-a}$

Here $f^1(x) = \dfrac{1}{1+x^2}$ & hence $f^1(c) = \dfrac{1}{1+c^2}$

Thus ∃ c, a<c<b ∍

$$\frac{1}{1+c^2} = \frac{Tan^{-1}b - Tan^{-1}a}{b-a} \text{ ------- (1)}$$

We have $1+a^2<1+c^2<1+b^2$

$$\therefore \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2} \qquad \text{.......... (2)}$$

From (1) and (2), we have

$$\frac{1}{1+a^2} > \frac{Tan^{-1}b - Tan^{-1}a}{b-a} > \frac{1}{1+b^2}$$

or

$$\frac{b-a}{1+a^2} < Tan^{-1}b - Tan^{-1}a < \frac{b-a}{1+b^2} \qquad \text{.................(3)}$$

Hence the result

Deductions: -

(i) We have $\dfrac{b-a}{1+b^2} < Tan^{-1}b - Tan^{-1}a < \dfrac{b-a}{1+a^2}$

Take $b = \dfrac{4}{3}$ & a=1, we get

$$\frac{\frac{4}{3}-1}{1+\frac{16}{9}} < Tan^{-1}(\frac{4}{3}) - Tan^{-1}(1) < \frac{\frac{4}{3}-1}{1+1^2} \implies \frac{\frac{4-3}{3}}{\frac{25}{9}} < Tan^{-1}(\frac{4}{3}) - \frac{\pi}{4} < \frac{4-3}{\frac{3}{2}}$$

$$\Rightarrow \frac{3}{25} + \frac{\pi}{4} < Tan^{-1}(\frac{4}{3}) < \frac{\pi}{4} + \frac{1}{6}$$

(ii) Taking b=2 and a=1, we get

$$\frac{2-1}{1+2^2} < Tan^{-1}2 - Tan^{-1}1 < \frac{2-1}{1+1^2} \Rightarrow \frac{1}{5} < Tan^{-1}2 - \frac{\pi}{4} < \frac{1}{2}$$

$$\Rightarrow \frac{1}{5} + \frac{\pi}{4} < Tan^{-1}2 < \frac{2+\pi}{4}$$

$$\Rightarrow \frac{4+5\pi}{20} + < Tan^{-1}2 < \frac{2+\pi}{4}$$

5. Show that for any x > 0, 1 + x < e^x < 1 + xe^x.

Sol: - Let f(x) = e^x defined on [0,x]. Then f(x) is continuous on [0,x] & derivable

on (0,x).

By Lagrange's Mean value theorem ∃ a real number c ϵ(0,x) such that

$$\frac{f(x)-f(0)}{x-0} = f^1(c)$$

$$\Rightarrow \frac{e^x - e^0}{x-0} = e^c \Rightarrow \frac{e^x - 1}{x} = e^c \quad \text{............(1)}$$

Note that 0<c<x => $e^0<e^c<e^x$ (e^x is an increasing function)

=> $1 < \frac{e^x-1}{x} < e^x$ From (1)

=> $x<e^x-1<xe^x$

=> $1+x<e^x<1+xe^x$.

6. Calculate approximately $\sqrt[5]{245}$ by using L.M.V.T.

Sol:- Let f(x) = $\sqrt[5]{x}$ =$x^{1/5}$ & a=243 , b=245

Then $f^1(x) = 1/5\, x^{-4/5}$ & $f^1(c) = 1/5c^{-4/5}$

By L.M.V.T, we have

$$\frac{f(b)-f(a)}{b-a} = f^1(c)$$

=> $$\frac{f(245)-f(243)}{245-243} = \frac{1}{5}c^{\frac{-4}{5}}$$

=> f (245) =f(243)+2/5$c^{-4/5}$

=> c lies b/w 243 & 245 take c= 243

=> $\sqrt[5]{245}$ = $(243)^{1/5}$ +2/5$(243)^{-4/5}$ $=(3^5)^{\frac{1}{5}}+\frac{2}{5}(3^5)^{\frac{-4}{5}}$

= 3+ (2/5)(1/81) = 3+2/405 = 3.0049

7. Find the region in which f(x) = 1-4x-x^2 is increasing & the region in which it is decreasing using M.V.T.

Sol: - Given f(x) = 1-4x-x^2

f(x) being a polynomial function is continuous on [a,b] & differentiable on (a,b) $\forall$ a,b $\in$R

$\therefore$ f satisfies the conditions of L.M.V.T on every interval on the real line.

$f^1(x)$= - 4-2x= -2(2+x)$\forall$ x$\in$R

$f^1(x)$= 0 if x = -2

for x<-2, $f^1(x)$ >0 & for x>-2 , $f^1(x)$<0

Hence f(x) is strictly increasing on (-∞, -2) & strictly decreasing on (-2,∞)

8. Using Mean value theorem prove that Tan x > x in 0<x<π/2

Sol:- Consider f(x) = Tan x in $[\xi, x]$ where 0<ξ<x<π/2

Apply L.M.V.T to f(x)

$\exists$ a points c such that $0<\xi<c<x<\pi/2$ such that

$$\frac{Tan\, x - Tan\,\xi}{x-\xi} = \sec^2 c \implies$$

$$\text{Tan x - Tan } \xi = (\text{x} - \xi)\sec^2 \text{c}$$

$$Take\, \xi \to 0+0\, then\, Tan\, x = x\sec^2 x$$

But $\sec^2 c>1$.

Hence Tan x > x

9. If $f^1(x) = 0$ Through out an interval [a,b], prove using M.V.T f(x) is a constant in that interval.

Sol:- Let f(x) be function defined in [a,b] & let $f^1(x) = 0\ \forall$ x in [a,b].

Then $f^1(t)$ is defined & continuous in [a,x] where $a\le x\le b$.

& f(t) exist in open interval (a,x).

By L.M.V.T $\exists$ a point c in open interval (a,x) ∍

$$\frac{f(x)-f(a)}{x-a} = f^1(c)$$

But it is given that $f^1(c) = 0$

$$\therefore f(x) - f(a) = 0$$

$$\therefore f(x) = f(a)\ \forall\ x$$

Hence f(x) is constant.

10 Using mean value theorem

S.T i) $x > \log(1+x) > \frac{x}{1+x}$ x > 0

ii) $\pi/6 + (\sqrt{3}/15) < \sin^{-1}(0.6) < \pi/6 + (1/6)$

i) $1+x < e^x < 1+xe^x \quad \forall\, x > 0$

ii) $\frac{v-u}{1+v^2} < \tan^{(-1)}v - \tan^{(-1)}u < \frac{v-u}{1+u^2}$ where $0 < u < v$ hence deduce

a) $\pi/4 + (3/25) < \tan^{(-1)}(4/3) < \pi/4 + (1/6)$

III. Cauchy's Mean Value Theorem

If f: $[a,b] \to R$, $g:[a,b] \to R \ni$ (i) f,g are continuous on [a,b] (ii) f,g are differentiable on (a,b)

$$(iii)\, g^1(x) \neq 0 \forall x \in (a,b),\ then$$

$$\exists a\ po\text{int}\ c \in (a,b) \ni \frac{f^1(c)}{g^1(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

1. **Find c of Cauchy's mean value theorem for**

$$f(x) = \sqrt{x} \quad \&\ g(x) = \frac{1}{\sqrt{x}}$$ in [a,b] where 0<a<b

Sol: - Clearly f, g are continuous on $[a,b] \subseteq R^+$

We have $f^1(x) = \frac{1}{2\sqrt{x}}$ **a**nd $g^1(x) = \frac{-1}{2x\sqrt{x}}$ which exits on (a,b)

$\therefore$ f, g are differentiable on $(a, b) \subseteq R^+$

Also $g^1(x) \neq 0$, $\forall\ x \in (a,b) \subseteq R^+$

Conditions of Cauchy's Mean value theorem are satisfied on (a,b) so $\exists c \in (a,b) \ni$

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f^1(c)}{g^1(c)}$$

$$\frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}} = \frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c\sqrt{c}}} ==> \frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}} = \frac{-2c\sqrt{c}}{2\sqrt{c}} => \sqrt{ab} = c$$

Since a,b >0 , $\sqrt{ab}$ is their geometric mean and we have $a<\sqrt{ab}<b$

$c \in (a,b)$ which verifies Cauchy's mean value theorem.

2. Verify Cauchy's Mean value theorem for $f(x) = e^x$ & $g(x) = e^{-x}$ in [3,7] &

find the value of c.

Sol: We are given $f(x) = e^x$ & $g(x) = e^{-x}$

f(x) & g(x) are continuous and derivable for all values of x.

=>f & g are continuous in [3,7]

=> f & g are derivable on (3,7)

Also $g^1(x) = e^{-x} \neq 0 \ \forall \ x \in (3,7)$

Thus f & g satisfies the conditions of Cauchy's mean value theorem.

Consequently, $\exists$ a point $c \in (3,7)$ such that

$$\frac{f(7)-f(3)}{g(7)-g(3)} = \frac{f^1(c)}{g^1(c)} \implies \frac{e^7 - e^3}{e^{-7} - e^{-3}} = \frac{e^c}{-e^{-c}} \implies \frac{e^7 - e^3}{\frac{1}{e^7} - \frac{1}{e^3}} = -e^{2c}$$

=> $-e^{7+3} = -e^{2c}$

=> 2c = 10

=> $c = 5 \in (3,7)$

Hence C.M.T. is verified

FUNCTIONS OF SEVERAL VARIABLES

Jacobian (J): Let u = u (x , y) , v = v(x , y) are two functions of the independent variables x , y.

The jacobian of (u , v) w.r.t (x , y) is given by

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Note: $J = \dfrac{\partial(u,v)}{\partial(x,y)}$ *and* $J^1 = \dfrac{\partial(x,y)}{\partial(u,v)}$ *then* $JJ^1 = 1$

Similarly of u = u(x, y, z) , v = v (x, y , z) , w = w(x, y , z)

Then the Jacobian of u , v , w w.r.to x , y , z is given by

$$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Solved Problems:

1. **If $x + y^2 = u$, $y + z^2 = v$, $z + x^2 = w$ find $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$**

Sol : Given $x + y^2 = u$, $y + z^2 = v$, $z + x^2 = w$

We have $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 1 & 2y & 0 \\ 0 & 1 & 2z \\ 2x & 0 & 1 \end{vmatrix}$

= 1(1-0) – 2y(0 – 4xz) + 0

= 1 – 2y(-4xz)

= 1 + 8xyz

$$\Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\left[\frac{\partial(u,v,w)}{\partial(x,y,z)}\right]} = \frac{1}{1+8xyz}$$

2. **S.T the functions $u = x + y + z$, $v = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$ and $w = x^3 + y^3 + z^3 - 3xyz$ are functionally related. ('07 S-1)**

Sol: Given $u = x + y + z$

$v = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$

$w = x^3 + y^3 + z^3 - 3xyz$

we have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x-2y-2z & 2y-2x-2z & 2z-2y-2x \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$=6 \begin{vmatrix} 1 & 1 & 1 \\ x-y-z & y-x-z & z-y-x \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix}$$

$c_1 \to c_1 - c_2$

$c_2 \to c_2 - c_3$

$$=6 \begin{vmatrix} 0 & 0 & 1 \\ 2x-2y & 2y-2z & z-y-x \\ x^2-yz-y^2+xz & y^2-xz-z^2+xy & z^2-xy \end{vmatrix}$$

$=6[2(x - y)(y^2 + xy - xz - z^2) - 2(y - z)(x^2 + xz - yz - y^2)]$

$=6[2(x - y)(y - z)(x + y + z) - 2(y - z)(x - y)(x + y + z)]$

$=0$

Hence there is a relation between u,v,w.

3. If x + y + z = u , y + z = uv , z = uvw then evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ ('06 S-1)

Sol: $x + y + z = u$

$y + z = uv$

$z = uvw$

$y = uv - uvw = uv(1 - w)$

$x = u - uv = u(1 - v)$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$R_2 \to R_2 + R_3$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix}$$

$= uv[u - uv + uv]$

$= u^2v$

4. If $u = x^2 - y^2$, $v = 2xy$ where $x = r\cos\theta$, $y = r\sin\theta$ S.T $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$ ('07 S-2)

Sol: Given $u = x^2 - y^2$, $v = 2xy$

$= r^2\cos^2\theta - r^2\sin^2\theta$ $\qquad = 2r\cos\theta \, r\sin\theta$

$= r^2(\cos^2\theta - \sin^2\theta)$ $\qquad = r^2 \sin 2\theta$

$= r^2 \cos 2\theta$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} u_r & u_\theta \\ v_r & v_\theta \end{vmatrix} = \begin{vmatrix} 2r\cos 2\theta & r^2(-\sin 2\theta)2 \\ 2r\sin 2\theta & r^2(\cos 2\theta)2 \end{vmatrix}$$

$$= (2r)(2r)\begin{vmatrix} \cos 2\theta & -r\sin 2\theta \\ \sin 2\theta & r(\cos 2\theta) \end{vmatrix}$$

$= 4r^2 [r\cos^2 2\theta + r\sin^2 2\theta]$

$= 4r^2(r)[\cos^2 2\theta + \sin^2 2\theta]$

$= 4r^3$

5. If $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ ('08 S-4)

Sol: Given $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$

We have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$u_x = yz(-1/x^2) = \frac{-yz}{x^2}$, $u_y = \frac{z}{x}$, $u_z = \frac{y}{x}$

$v_x = \frac{z}{y}$, $v_y = xz(-1/y^2) = \frac{-xz}{y^2}$, $v_z = \frac{x}{y}$

$w_x = \frac{y}{z}$, $w_y = \frac{x}{z}$, $w_z = xy(-1/z^2) = \frac{-xy}{z^2}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{x^2}\cdot\frac{1}{y^2}\cdot\frac{1}{z^2}\begin{vmatrix} -yz & xz & xy \\ yz & -xz & xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{(yz)(xz)(xy)}{x^2y^2z^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= 1[-1(1-1) -1(-1-1) + (1+1)]$

$= 0 - 1(-2) + (2)$

$= 2 + 2$

$= 4$

Assignment

Calculate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ if $x = \sqrt{vw}$, $y = \sqrt{wu}$, $z = \sqrt{uv}$ and $u = r\sin\theta\cos\emptyset$, $v = r\sin\theta\sin\emptyset$, $w = r\cos\theta$

6. If $x = e^r\sec\theta$, $y = e^r\tan\theta$ P.T $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

Sol: Given $x = e^r\sec\theta$, $y = e^r\tan\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}, \quad \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix}$$

$x_r = e^r\sec\theta = x$, $\quad x_\theta = e^r\sec\theta\tan\theta$

$y_r = e^r\tan\theta = y$, $\quad y_\theta = e^r\sec^2\theta$

$x^2 - y^2 = e^{2r}(\sec^2\theta - \tan^2\theta)$

⇨ $2r = \log(x^2 - y^2)$

⇨ $r = ½\log(x^2 - y^2)$

$$r_x = \frac{1}{2}\frac{1}{x^2 - y^2}(2x) = \frac{x}{(x^2 - y^2)}$$

$$r_y = \frac{1}{2}\frac{1}{x^2 - y^2}(-2y) = \frac{-y}{(x^2 - y^2)}$$

$$\frac{x}{y} = \frac{\sec\theta}{\tan\theta} = \frac{1/\cos\theta}{\sin\theta/\cos\theta} = \frac{1}{\sin\theta}$$

⇨ $\sin\theta = \frac{y}{x}$, $\theta = \sin^{-1}\left(\frac{y}{x}\right)$

$$\theta_x = \frac{1}{\sqrt{1 - \frac{y^2}{x^2}}}\, y\left(-\frac{1}{x^2}\right) = \frac{-y}{x\sqrt{x^2 - y^2}}$$

$$\theta_y = \frac{1}{\sqrt{1 - \frac{y^2}{x^2}}}(1/x) = \frac{1}{\sqrt{x^2 - y^2}}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} e^r\sec\theta\tan\theta \\ e^r\sec2\theta \end{vmatrix} = e^{2r}\sec^2\theta - y\,e^r\sec\theta\tan\theta$$

$$= e^{2r}\sec\theta[\sec^2\theta - \tan^2\theta] = e^{2r}\sec\theta$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{(x^2-y^2)} & \frac{-y}{(x^2-y^2)} \\ \frac{-y}{x\sqrt{x^2-y^2}} & \frac{1}{\sqrt{x^2-y^2}} \end{vmatrix}$$

$$=\left[\frac{x}{(x^2-y^2)\sqrt{x^2-y^2}} - \frac{y^2}{x(x^2-y^2)\sqrt{x^2-y^2}}\right]$$

$$= \frac{x^2-y^2}{x(x^2-y^2)\sqrt{x^2-y^2}} = \frac{1}{x\sqrt{x^2-y^2}} = \frac{1}{e^{2r}\sec\theta}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$$

Functional Dependence

Two functions u and v are functionally dependent if their Jacobian

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 0$$

If the Jacobian of u, v is not equal to zero then those functions u, v are functionally independent.

** **Maximum & Minimum for function of a single Variable:**

To find the Maxima & Minima of f(x) we use the following procedure.

(i) Find $f^1(x)$ and equate it to zero

(ii) Solve the above equation we get x_0, x_1 as roots.

(iii) Then find $f^{11}(x)$.

If $f^{11}(x)_{(x = x0)} > 0$, then f(x) is minimum at x_0

If $f^{11}(x)_{(x = x0)}$, < 0, f(x) is maximum at x_0. Similarly we do this for other stationary points.

PROBLEMS:

1. **Find the max & min of the function $f(x) = x^5 - 3x^4 + 5$ ('08 S-1)**

Sol: Given $f(x) = x^5 - 3x^4 + 5$

$f^1(x) = 5x^4 - 12x^3$

for maxima or minima $f^1(x) = 0$

$5x^4 - 12x^3 = 0$

$x = 0, x = 12/5$

$f^{11}(x) = 20x^3 - 36x^2$

At $x = 0 \Rightarrow f^{11}(x) = 0$. So f is neither maximum nor minimum at x = 0

At $x = (12/5) \Rightarrow f^{11}(x) = 20\,(12/5)^3 - 36(12/5)$

$= 144(48-36)/25 \quad = 1728/25 > 0$

So f(x) is minimum at x = 12/5

The minimum value is $f(12/5) = (12/5)^5 - 3(12/5)^4 + 5$

** **Maxima & Minima for functions of two Variables**:

Working procedure:

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ Equate each to zero. Solve these equations for x & y we get the pair of values $(a_1, b1)$ (a_2,b_2) (a_3, b_3) ………………

2. Find $l = \frac{\partial^2 f}{\partial x^2}, m = \frac{\partial^2 f}{\partial x\, \partial y}$, n = $\frac{\partial^2 f}{\partial y^2}$

3. i. If $ln - m^2 > 0$ and $l < 0$ at (a_1,b_1) then f(x ,y) is maximum at (a_1,b_1) and maximum value is $f(a_1,b_1)$

ii. If $ln - m^2 > 0$ and $l > 0$ at (a_1,b_1) then f(x ,y) is minimum at (a_1,b_1) and minimum value is $f(a_1,b_1)$.

iii. If $ln - m^2 < 0$ and at $(a_1, b1)$ then f(x, y) is neither maximum nor minimum at $(a_1, b1)$. In this case $(a_1, b1)$ is saddle point.

iv. If $ln - m^2 = 0$ and at $(a_1, b1)$, no conclusion can be drawn about maximum or minimum and needs further investigation. Similarly we do this for other stationary points.

PROBLEMS:

1. **Locate the stationary points & examine their nature of the following functions.**

$u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$, $(x > 0, y > 0)$

Sol: Given $u(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

For maxima & minima $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$

$\frac{\partial u}{\partial x} = 4x^3 - 4x + 4y = 0 \Rightarrow x^3 - x + y = 0$ -------------------> (1)

$\frac{\partial u}{\partial y} = 4y^3 + 4x - 4y = 0 \Rightarrow y^3 + x - y = 0$ -------------------> (2)

Adding (1) & (2),

$$x^3 + y^3 = 0$$

$\Rightarrow$ x = – y -------------------> (3)

(1) $\Rightarrow$ $x^3 - 2x \Rightarrow x = 0, \sqrt{2}, -\sqrt{2}$

Hence (3) $\Rightarrow$ y = 0, $-\sqrt{2}$, $\sqrt{2}$

$l = \dfrac{\partial^2 u}{\partial x^2} = 12x^2 - 4$, $m = \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = 4$ & $n = \dfrac{\partial^2 u}{\partial y^2} = 12y^2 - 4$

$ln - m^2 = (12x^2 - 4)(12y^2 - 4) - 16$

At $(-\sqrt{2}, \sqrt{2})$, $ln - m^2 = (24 - 4)(24 - 4) - 16 = (20)(20) - 16 > 0$ and $l = 20 > 0$

The function has minimum value at $(-\sqrt{2}, \sqrt{2})$

At (0,0) , $ln - m^2 = (0 - 4)(0 - 4) - 16 = 0$

(0,0) is not a extreme value.

2. Investigate the maxima & minima, if any, of the function $f(x) = x^3y^2 (1-x-y)$.

Sol: Given $f(x) = x^3y^2 (1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$

$\dfrac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$ $\quad \dfrac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$

For maxima & minima $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$

⇨ $3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$ => $x^2y^2(3 - 4x - 3y) = 0$ ---------------> (1)

⇨ $2x^3y - 2x^4y - 3x^3y^2 = 0$ => $x^3y(2 - 2x - 3y) = 0$ ----------------> (2)

From (1) & (2) $\quad 4x + 3y - 3 = 0$

$2x + 3y - 2 = 0$

$2x = 1$ => $x = ½$

$4(½) + 3y - 3 = 0$ => $3y = 3 - 2$, $y = (1/3)$

$l = \dfrac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$

$\left(\dfrac{\partial^2 f}{\partial x^2}\right)_{(1/2,1/3)} = 6(1/2)(1/3)^2 - 12(1/2)^2(1/3)^2 - 6(1/2)(1/3)^3 = 1/3 - 1/3 - 1/9 = -1/9$

$m = \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = 6x^2y - 8x^3y - 9x^2y^2$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(1/2\ ,1/3)} = 6(1/2)^2(1/3) -8\ (1/2)^3(1/3) -9(1/2)^2(1/3)^3 \ = \frac{6-4-3}{12} = \frac{-1}{12}$$

$$n = \frac{\partial^2 f}{\partial y^2} = 2x^3 -2x^4 -6x^3y$$

$$\left(\frac{\partial^2 f}{\partial y^2}\right)_{(1/2,1/3)} = 2(1/2)^3 -2(1/2)^4 -6(1/2)^3(1/3) = \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$$

$$ln- m^2 =(-1/9)(-1/8) –(-1/12)^2 = \frac{1}{72} - \frac{1}{144} = \frac{2-1}{144} = \frac{1}{144} > 0 \text{ and } l = \frac{-1}{9} < 0$$

The function has a maximum value at (1/2 , 1/3)

$$\therefore \text{Maximum value is } f\left(\frac{1}{2},\frac{1}{3}\right) = \left(\frac{1}{8}\times\frac{1}{9}\right)\left(1-\frac{1}{2}-\frac{1}{3}\right) = \frac{1}{72}\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{1}{432}$$

3. **Find three positive numbers whose sum is 100 and whose product is maximum.**

Sol: Let x ,y ,z be three +ve numbers.

Then $x + y + z = 100$

⇨ $z = 100 – x – y$

Let $f(x,y) = xyz = xy(100 – x – y) = 100xy – x^2y - xy^2$

For maxima or minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 100y – 2xy - y^2 = 0$ => $y(100 - 2x – y) = 0$ ----------------> (1)

$\frac{\partial f}{\partial y} = 100x – x^2 - 2xy = 0$ => $x(100 – x - 2y) = 0$ ------------------> (2)

From (1) & (2)

100 -2x –y = 0

200 -2x -4y =0

-100 + 3y = 0 => 3y =100 => y =100/3

100 – x –(200/3) = 0 => x = 100/3

$$l = \frac{\partial^2 f}{\partial x^2} = -2y$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)(100/3\ ,\ 100/3\ ) = -\ 200/3$$

$$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 100 -2x -2y$$

$\left(\frac{\partial^2 f}{\partial x \partial y}\right)$ (100/3 , 100/3) = 100 –(200/3) –(200/3) = -(100/3)

n = $\frac{\partial^2 f}{\partial y^2}$ = -2x

$\left(\frac{\partial^2 f}{\partial y^2}\right)$ (100/3 , 100/3) = - 200/3

ln -m^2 = (-200/3) (-200/3) - (-100/3)^2 = (100)^2 /3

The function has a maximum value at (100/3 , 100/3)

i.e. at x = 100/3, y = 100/3 $\quad \therefore$ z $= 100 - \frac{100}{3} - \frac{100}{3} = \frac{100}{3}$

The required numbers are x = 100/3, y = 100/3, z = 100/3

4. Find the maxima & minima of the function f(x) = 2(x^2 –y^2) –x^4 +y^4

Sol: Given f(x) = 2(x^2 –y^2) –x^4 +y^4 = 2x^2 –2y^2 –x^4 +y^4

For maxima & minima $\frac{\partial f}{\partial x}$ = 0 and $\frac{\partial f}{\partial y}$ = 0

$\frac{\partial f}{\partial x}$ = 4x - 4x^3 = 0 => 4x(1-x^2) = 0 => x = 0 , x = ± 1

$\frac{\partial f}{\partial y}$ = -4y + 4y^3 = 0 => -4y (1-y^2) = 0 =>y = 0, y = ± 1

l = $\left(\frac{\partial^2 f}{\partial x^2}\right)$ = 4-12x^2

m = $\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ = 0

n = $\left(\frac{\partial^2 f}{\partial y^2}\right)$ = -4 +12y^2

we have ln – m^2 = (4-12x^2)(-4 +12y^2) – 0

= -16 +48x^2 +48y^2 -144x^{2}y^2

= 48x^2 +48y^2 -144x^{2}y^2 -16

i) At (0 , ± 1)

ln – m^2 = 0 + 48 - 0 -16 =32 > 0

l = 4-0 = 4 > 0

f has minimum value at (0 , ± 1)

f (x ,y) = 2(x^2 –y^2) –x^4 +y^4

$f(0, \pm 1) = 0 - 2 - 0 + 1 = -1$

The minimum value is '-1 '.

ii) At $(\pm 1, 0)$

$ln - m^2 = 48 + 0 - 0 - 16 = 32 > 0$

$l = 4-12 = -8 < 0$

f has maximum value at $(\pm 1, 0)$

$f(x, y) = 2(x^2 - y^2) - x^4 + y^4$

$f(\pm 1, 0) = 2 - 0 - 1 + 0 = 1$

The maximum value is '1 '.

iii) At $(0,0)$, $(\pm 1, \pm 1)$

$ln - m^2 < 0$

$l = 4 - 12x^2$

$(0, 0)$ & $(\pm 1, \pm 1)$ are saddle points.

f has no max & min values at $(0, 0)$, $(\pm 1, \pm 1)$.

***Extremum** : A function which have a maximum or minimum or both is called 'extremum'

***Extreme value** :- The maximum value or minimum value or both of a function is Extreme value.

***Stationary points:** - To get stationary points we solve the equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ i.e the pairs (a_1, b_1), (a_2, b_2) ………….. are called Stationary.

*Maxima & Minima for a function with constant condition :Lagranges Method

Suppose $f(x, y, z) = 0$ ------------(1)

$\emptyset(x, y, z) = 0$ ------------- (2)

$F(x, y, z) = f(x, y, z) + \gamma\, \emptyset(x, y, z)$ where γ is called Lagrange's constant.

1. $\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \gamma \frac{\partial \emptyset}{\partial x} = 0$ --------------- (3)

$$\frac{\partial F}{\partial y} = 0 \quad => \frac{\partial f}{\partial y} + \gamma \frac{\partial \emptyset}{\partial y} = 0 \text{ --------------- } (4)$$

$$\frac{\partial F}{\partial z} = 0 \quad => \frac{\partial f}{\partial z} + \gamma \frac{\partial \emptyset}{\partial z} = 0 \text{ --------------- } (5)$$

2. Solving the equations (2) (3) (4) & (5) we get the stationary point (x, y, z).
3. Substitute the value of x , y , z in equation (1) we get the extremum.

Problem:

1. **Find the minimum value of $x^2 + y^2 + z^2$, given $x + y + z = 3a$ ('08 S-2)**

Sol: $u = x^2 + y^2 + z^2$

$\emptyset = x + y + z - 3a = 0$

Using Lagrange's function

$F(x , y , z) = u(x , y , z) + \gamma\, \emptyset(x , y , z)$

For maxima or minima

$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x} + \gamma \frac{\partial \emptyset}{\partial x} = 2x + \gamma = 0 \text{ ------------ } (1)$$

$$\frac{\partial F}{\partial y} = \frac{\partial u}{\partial y} + \gamma \frac{\partial \emptyset}{\partial y} = 2y + \gamma = 0 \text{ ------------ } (2)$$

$$\frac{\partial F}{\partial z} = \frac{\partial u}{\partial z} + \gamma \frac{\partial \emptyset}{\partial z} = 2z + \gamma = 0 \text{ ------------ } (3)$$

From (1) , (2) & (3)

$\gamma = -2x = -2y = -2z$

$x = y = z$

$\emptyset = x + x + x - 3a = 0$

$x = a$

$x = y = z = a$

Minimum value of $u = a^2 + a^2 + a^2 = 3a^2$