LECTURENOTES

MATHEMATICAL TRANSFORM TECHNIQUES
(MTT)

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ECE-R16

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UNIT-I
FOURIER SERIES

- Definition of periodic function
- Determination of Fourier coefficients
- Fourier expansion of periodic function in a given interval of length 2π
- Fourier series of even and odd functions
- Fourier series in an arbitrary interval
- Half- range Fourier sine and cosine expansions

INTRODUCTION:

Fourier series which was named after the French mathematician "Jean-Baptise Joseph Fourier" (1768-1830).Fourier series is an infinite series representation of periodic function in terms of trigonometric sine and cosine functions. It is very powerful method to solve ordinary and partial differential equations particularly with periodic functions appearing as non-homogeneous terms. We know that Taylor's series expansion is valid only for functions which are continuous and differentiable. Fourier series is possible not only for continuous functions but also for periodic functions, functions which are discontinuous in their values and derivatives because of the periodic nature Fourier series constructed for one period is valid for all. Fourier series has been an important tool in solving problems in many fields like current and voltage in alternating circuit, conduction of heat in solids, electrodynamics etc.

Periodic function

A function f:R→R is said to be periodic if there exists a positive number t such that f(x+T)=f(x) for all x belongs to R .

- T is called the **period** of f(x).
- If a function f(x) has a smallest period T(>0) then this is called **fundamental period of f(x) or primitive period of f(x)**

EXAMPLE

- Sin x, cos x are periodic functions with primitive period 2π
- Sinnx,cosnx are periodic functions with primitive period $\frac{2\pi}{n}$
- Tanx are periodic functions with primitive period π
- Tannx are periodic functions with primitive period $\frac{\pi}{n}$
- If f(x)= constant is a periodic function but it has no primitive period

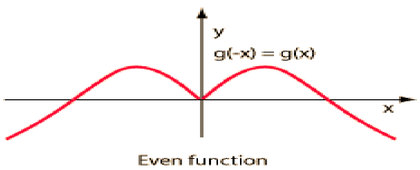
NOTE

- Any integral multiple of T is also a period i.e. if f(x) is a periodic then f(x+nT)=f(x).where T is a period and n ϵ Z
- If f_1 and f_2 are periodic functions having same period T then $f(x)=c_1f_1(x)+c_2f_2(x)$,[c_1,c_2 are constants] is also the periodic function of period T
- If T is the period of f then $f(c_1x+c_2)$also has the period T [c_1,c_2 are constants]
- If f(x) is a periodic function of x of period T
  - (1) f(ax),a≠0 is periodic function of x of period T/a
  - (2) f(x/b), b≠0 is periodic function of x of period Tb

EVEN FUNCTION:

A function f(x) is even function if f(-x)=f(x)

Ex: f(x)= cos x, x^2

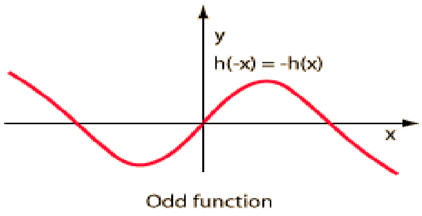


- The graph of even function y=f(x) is symmetric about Y-axis
- If f(x) is even function $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

ODD FUNCTION:

A function f(x) is odd function if f(-x)= -f(x)

Ex: f(x)=sin x, x^3



- The graph of odd function y=f(x) is symmetric about the origin
- If f(x) is odd function $\int_{-a}^{a} f(x)dx = 0$

NOTE

- There may be some functions which are neither even nor odd

  Ex: f(x) =4sinx +3tanx-e^x
- The product of two even functions is even
- The product of two odd functions is even
- The product of an even and odd function is odd

TRIGONOMETRIC SERIES: A series of the form

f(x)=a_0+$a_1$cosx+$a_2$cos2x+$a_3$cos3x+……………..$a_n$cosx+…….$b_1$sinx+$b_2$sin2x+$b_3$sin3x+……
……$b_n$sinx+………..

f(x)= $a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

Where $a_0,a_1,a_2,\ldots\ldots a_n$ and $b_1,b_2,\ldots\ldots b_n$ are coefficient of the series.Since each term of the trigonometric series is a function of period 2π it can be observed that if the series is convergent then its sum is also a function of period 2π

CONDITIONS FOR FOURIER EXPANSION (DIRICHLET CONDITIONS)

A function f(x) defined in [0,2π] has a valid Fourier series expansion of the form

$$\mathbf{f(x)= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)}$$

Where a_0, a_n, b_n are constants, provided

- f(x) is well defined and single-valued, except possibly at a finite number of point in the
  interval [0,2π] .
- f(x) has finite number of finite discontinuities in the interval in [0,2π] .
- f(x) has finite number of finite maxima and minima.

Note: The above conditions are valid for the function defined in the Intervals [-π,π],[0,2l], [-l,l]

- {1,cos 1x,cos 2x,….,cosnx,…..,sin 1x,sin 2x,….,sin nx,….}

  Consider any two, All these have a common period 2π . Here 1=cos 0x
- {1,cos $\frac{\pi x}{l}$,cos$\frac{2\pi x}{l}$,….,cos $\frac{n\pi x}{l}$,…..,sin $\frac{\pi x}{l}$,sin $\frac{2\pi x}{l}$,….,sin $\frac{n\pi x}{l}$,….}

  All these have a common period 2l .

These are called complete set of orthogonal functions.

Then the Fourier series converges to f(x) at all points where f(x) is continuous. Also the series converges to the average of the left limit and right limit of f(x) at each point of discontinuity of f(x).

Example

- $\sin^{-1} x$ cannot be expanded as fourier series since it is not single valued
- Tan x cannot be expanded as Fourier series in (0,2π) since tan x is infinite at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$

EULER'S FORMULAE

The Fourier series for the function f(x) in the interval c≤x≤c+2π is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

Where $a_0=\frac{1}{\pi}\int_c^{c+2\pi} f(x)dx$

$$a_n=\frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\, dx$$

$$b_n=\frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx dx$$

These values are known as Euler's Formulae.

Proof: consider $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$----------(1)

Integrating eq(1) with respective x from x=c, x=c+2π on both sides

$$\int_c^{c+2\pi} f(x)dx=\int_c^{c+2\pi}\frac{a_0}{2}dx+\int_c^{c+2\pi}\sum_{n=1}^{\infty}a_n\cos nxdx+\int_c^{c+2\pi}\sum_{n=1}^{\infty}b_n\sin nxdx$$

$$\int_c^{c+2\pi} f(x)dx=\int_c^{c+2\pi}\frac{a_0}{2}dx+\sum_{n=1}^{\infty}a_n\int_c^{c+2\pi}\cos nxdx+\sum_{n=1}^{\infty}b_n\int_c^{c+2\pi}\sin nxdx$$

$$\int_c^{c+2\pi} f(x)dx=\frac{a_0}{2}[c+2\pi-c]$$

$$a_0=\frac{1}{\pi}\int_c^{c+2\pi} f(x)dx$$

Multiplying cosnx and Integrating eq(1) with respective x from x=c, x=c+2π on both sides

$$\int_c^{c+2\pi} f(x)\cos nx\,dx=\int_c^{c+2\pi}\frac{a_0}{2}\cos nxdx+\int_c^{c+2\pi}\sum_{n=1}^{\infty}a_n\cos nx\cos nxdx+\int_c^{c+2\pi}\sum_{n=1}^{\infty}b_n\sin nx\cos nxdx$$

$$\int_c^{c+2\pi} f(x)\cos nx\,dx=\int_c^{c+2\pi}\frac{a_0}{2}\cos nxdx+\sum_{n=1}^{\infty}a_n\int_c^{c+2\pi}\cos^2 nxdx+\sum_{n=1}^{\infty}b_n\int_c^{c+2\pi}\sin nx\cos nxdx$$

$$\int_c^{c+2\pi} f(x)\cos nx\,dx=a_n\pi$$

$$a_n=\frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

Multiplying sinnx and Integrating eq(1) with respective x from x=c, x=c+2π on both sides

$$\int_c^{c+2\pi} f(x)\sin nx\,dx = \int_c^{c+2\pi} \frac{a_0}{2}\sin nx dx + \int_c^{c+2\pi} \sum_{n=1}^{\infty} a_n \cos nx \sin nx dx + \int_c^{c+2\pi} \sum_{n=1}^{\infty} b_n \sin nx \sin nx dx$$

$$\int_c^{c+2\pi} f(x)\sin nx\,dx = \int_c^{c+2\pi} \frac{a_0}{2}\sin nx dx + \sum_{n=1}^{\infty} a_n \int_c^{c+2\pi} \cos nx \sin nx dx + \sum_{n=1}^{\infty} b_n \int_c^{c+2\pi} \sin^2 nx dx$$

$$\int_c^{c+2\pi} f(x)\sin nx\,dx = b_n \pi$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

DEFINITION OF FOURIER SERIES

- Let f(x) be a function defined in [0,2π] . Let f(x+2π)=f(x) then the Fourier Series of is given by
  $f(x)= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

  Where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$

  $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx$

  $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$

  These values a_0, a_n, b_n are called as Fourier coefficients of f(x)in [0,2π].

- Let f(x) be a function defined in [-π,π] . Let f(x+2π)=f(x) then the Fourier Series of is given by
  $f(x)= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

  Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

  $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$

  $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

These values a_0, a_n, b_n are called as Fourier coefficients of f(x)in [-π,π]

- Let f(x) is a function defined in [0,2*l*]. Let f(x+2*l*)=f(x) then the Fourier Series of is given by
  $f(x)= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l})$

  Where $a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$

  $a_n = \frac{1}{l}\int_0^{2l} f(x) \cos\frac{n\pi x}{l}\, dx$

  $b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l} dx$

  These values a_0, a_n, b_n are called as Fourier coefficients of f(x) in [0,2*l*]

- Let f(x) be a function defined in [-*l*,*l*] . Let f(x+2*l*)=f(x) then the Fourier Series of is given by
  $f(x)= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l})$

Where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$

$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$

$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$

These values a_0, a_n, b_n are called as Fourier coefficients of f(x) in [-*l*,*l*]

FOURIER SERIES FOR EVEN AND ODD FUNCTIONS

We know that if *f(x)* be a function defined in [-π, π]. Let f(x+2π) =f(x), then the Fourier series *f(x)* of is given by

$f(x)= \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

These values a_0, a_n, b_n are called as Fourier coefficients of f(x) in [-π, π]

<u>Case (i):</u> When f(x) is an even function

- $f(x)= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx$

Where $a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$

$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$

- $f(x)= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}$

  Where $a_0 = \frac{2}{l}\int_{0}^{l} f(x)dx$

  $a_n = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}dx$

<u>Case (ii):</u> When f(x) is an Odd Function

- $f(x)= \sum_{n=1}^{\infty} b_n\sin nx$

  Where $b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx dx$

- $f(x)= \sum_{n=1}^{\infty} b_n\sin nx$

  Where $b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}dx$

FOURIER SERIES FOR DISCONTINUOUS FUNCTIONS

Let f(x) be defined by $f(x) = \begin{cases} f_1(x), c < x < x_0 \\ f_2(x), x_0 < x < c + 2\pi \end{cases}$

Where x_0 is the point of discontinuity in (c, c+2π)
Then the Fourier coefficient is given by

$$\boldsymbol{a_0 = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)dx + \int_{x_0}^{c+2\pi} f_2(x)\, dx\right]}$$

$$\boldsymbol{a_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\cos nx\, dx + \int_{x_0}^{c+2\pi} f_2(x)\cos nx\;\, dx\right]}$$

$$\boldsymbol{b_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\sin nx\, dx + \int_{x_0}^{c+2\pi} f_2(x)\sin nx\, dx\right]}$$

The Fourier series converges to $\frac{f(x+0)+f(x-0)}{2}$ if *x* is a point of discontinuity of *f(x)*

PROBLEMS

1 **Find the Fourier series expansion of f(x) = x², 0 < x < 2π. Hence deduce that**

$(i)\quad \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots\ldots\ldots = \frac{\pi^2}{6}$

$(ii)\quad \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\ldots\ldots\ldots\ldots = \frac{\pi^2}{12}$

$(iii)\quad \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots\ldots\ldots = \frac{\pi^2}{8}$

Sol Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{8\pi^3}{3} - 0\right] = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left\{0 + \frac{(4\pi)(1)}{n^2} - 0\right\} - \{0 + 0 - 0\}\right]$$

$$= \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x^2)\left(\frac{-\cos nx}{n}\right) - (2x)\left(\frac{-\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left\{-\frac{4\pi^2}{n} + 0 + \frac{2}{n^3}\right\} - \left\{0 + 0 + \frac{2}{n^3}\right\}\right]$$

$$= -\frac{4\pi}{n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{1}{2}\left(\frac{8\pi^2}{3}\right) + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right]$$

$$f(x) = \frac{4\pi^2}{3} + 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots\ldots\ldots\ldots\right] - 4\pi\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\right.$$

Put x = 0 in the above series we get

$$f(0) = \frac{4\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\ldots\ldots\right] - 4\pi(0) \quad \text{---------------} \; (1)$$

But x = 0 is the point of discontinuity. So we have

$$f(0) = \frac{f(0) + f(2\pi)}{2} = \frac{(0) + (4\pi^2)}{2} = 2\pi^2$$

Hence equation (1) becomes

$$2\pi^2 = \frac{4\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\ldots\ldots\right]$$

$$2\pi^2 - \frac{4\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\ldots\ldots\right]$$

$$\frac{2\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\ldots\ldots\right]$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\ldots\ldots\ldots \qquad -------(2)$$

Now, put x = π (which is point of continuity) in the above series we get

$$\pi^2 = \frac{4\pi^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \ldots\ldots\ldots\ldots\right] - 4\pi(0)$$

$$\pi^2 - \frac{4\pi^2}{3} = -4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots\dots\dots\right]$$

$$-\frac{\pi^2}{3} = -4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots\dots\dots\right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots\dots\dots\dots \quad -------(3)$$

Adding (2) and (3), we get

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots\right]$$

$$\frac{3\pi^2}{12} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots\dots$$

2 **Expand in Fourier series of f$(x) = x \sin x$ for $0 < x < 2\pi$ and deduce the result**

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots\dots\dots = \frac{\pi - 2}{4}$$

Sol Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x\,dx$$

$$= \frac{1}{\pi}\left[x(-\cos x) - (1)(-\sin x)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[(-2\pi + 0) - (0 + 0)\right]$$

$$= -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\cos nx \sin x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx \ , \quad n \neq 1$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \sin(n+1)x\ dx - \frac{1}{2\pi}\int_0^{2\pi} x \sin(n-1)x\ dx$$

$$= \frac{1}{2\pi}\left[(x)\left(\frac{-\cos(n+1)x}{n+1}\right) - (1)\left(\frac{-\sin(n+1)x}{(n+1)^2}\right)\right]_0^{2\pi}$$

$$-\frac{1}{2\pi}\left[(x)\left(\frac{-\cos(n-1)x}{n-1}\right) - (1)\left(\frac{-\sin(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{\frac{-2\pi(-1)^{2n+2}}{n+1}+0\right\}-\{0+0\}\right]-\frac{1}{2\pi}\left[\left\{\frac{-2\pi(-1)^{2n-2}}{n-1}+0\right\}-\{0+0\}\right]$$

$$= \frac{-1}{n+1}+\frac{1}{n-1}$$

$$a_n = \frac{-(n-1)+(n+1)}{(n+1)(n-1)}$$

$$a_n = \frac{2}{n^2-1}\ , \quad n \neq 1$$

When n = 1, we have

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\, dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\, dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right)-(1)\left(\frac{-\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{2\pi\left(\frac{-1}{2}\right)+0\right\}-(0+0)\right]$$

$$= -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin nx\sin x)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n-1)x-\cos(n+1)x\right] dx\ , \quad n \neq 1$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\cos(n-1)x\ dx - \frac{1}{2\pi}\int_0^{2\pi} x\cos(n+1)x\ dx$$

$$= \frac{1}{2\pi}\left[(x)\left(\frac{\sin(n-1)x}{n-1}\right)-(1)\left(\frac{-\cos(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}$$

$$-\frac{1}{2\pi}\left[(x)\left(\frac{\sin(n+1)x}{n+1}\right)-(1)\left(\frac{-\cos(n+1)x}{(n+1)^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{0+\frac{(-1)^{2n-2}}{(n-1)^2}\right\}-\left\{0+\frac{1}{(n-1)^2}\right\}\right]-\frac{1}{2\pi}\left[\left\{0+\frac{(-1)^{2n+2}}{(n+1)^2}\right\}-\left\{0+\frac{1}{(n+1)^2}\right\}\right]$$

$$= \frac{1}{2\pi}\left[\left\{0+\frac{1}{(n-1)^2}\right\}-\left\{0+\frac{1}{(n-1)^2}\right\}\right]-\frac{1}{2\pi}\left[\left\{0+\frac{1}{(n+1)^2}\right\}-\left\{0+\frac{1}{(n+1)^2}\right\}\right]$$

$$b_n = 0\ , \quad n \neq 1$$

When n = 1, we have

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x\,dx \;= \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2} - \left\{x\left(\frac{\sin 2x}{2}\right) - (1)\left(\frac{-\cos 2x}{4}\right)\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{2\pi^2 - 0 - \left(\frac{1}{2}\right)\right\} - \left\{0 - 0 - \frac{1}{2}\right\}\right]$$

$$= \pi$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n\cos nx + b_1\sin x + \sum_{n=2}^{\infty} b_n\sin nx$$

$$= \frac{-2}{2} - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2}{(n-1)(n+1)}\cos nx + \pi\sin x + 0$$

$$x\sin x = -1 - \frac{1}{2}\cos x + \pi\sin x + 2\left[\frac{\cos 2x}{1.3} + \frac{\cos 3x}{2.4} + \frac{\cos 4x}{3.5} + \frac{\cos 5x}{4.6} + \ldots\ldots\ldots\ldots\right]$$

Put $x = \frac{\pi}{2}$ in the above series we get

$$\frac{\pi}{2}(1) = -1 - 0 + \pi(1) + 2\left[\frac{-1}{1.3} + 0 + \frac{1}{3.5} + 0 + \frac{-1}{5.7} + 0 + \ldots\ldots\ldots\ldots\right]$$

$$\frac{\pi}{2} - \pi + 1 = -2\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots\ldots\ldots\right]$$

$$\frac{\pi - 2\pi + 2}{2} = -2\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots\ldots\ldots\right]$$

$$\frac{-\pi + 2}{2} = -2\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots\ldots\ldots\right]$$

$$\frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots\ldots\ldots$$

3 **Obtain the Fourier expansion of $f(x)=e^{-ax}$ in the interval $(-\pi, \pi)$.**

Sol Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$= \frac{e^{a\pi} - e^{-a\pi}}{a\pi} = \frac{2\sinh a\pi}{a\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx dx$$

$$a_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\{-a\cos nx + n\sin nx\}\right]_{-\pi}^{\pi}$$

$$= \frac{2a}{\pi}\left[\frac{(-1)^n \sinh a\pi}{a^2+n^2}\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}\{-a\sin nx - n\cos nx\}\right]_{-\pi}^{\pi}$$

$$= \frac{2n}{\pi}\left[\frac{(-1)^n \sinh a\pi}{a^2+n^2}\right]$$

$$f(x) = \frac{\sinh a\pi}{a\pi} + \frac{2a\sinh a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}\cos nx + \frac{2}{\pi}\sinh a\pi\sum_{n=1}^{\infty}\frac{n(-1)^n}{a^2+n^2}\sin nx$$

For x=0, a=1, the series reduces to

$$f(0)=1 = \frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$$

$$1 = \frac{\sinh \pi}{\pi} + \frac{2\sinh \pi}{\pi}\left[-\frac{1}{2} + \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}\right]$$

$$1 = \frac{2\sinh \pi}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$$

4 **Find the Fourier series for the function $f(x) = 1 + x + x^2$ in $(-\pi, \pi)$. Deduce**

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\ldots\ldots = \frac{\pi^2}{6}$$

Sol The given function is neither an even nor an odd function.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(1 + x + x^2)\,dx$$

$$= \frac{1}{\pi}\left[x + \frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\pi + \frac{\pi^2}{2} + \frac{\pi^3}{3}\right\} - \left\{-\pi + \frac{\pi^2}{2} - \frac{\pi^3}{3}\right\}\right]$$

$$= \frac{1}{\pi}\left[2\pi + \frac{2\pi^3}{3}\right] = 2 + \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x+x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(1+x+x^2)\left(\frac{\sin nx}{n}\right) - (1+2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{0 + \frac{(1+2\pi)(-1)^n}{n^2} - 0\right\} - \left\{0 + \frac{(1-2\pi)(-1)^n}{n^2} - 0\right\}\right]$$

$$= \frac{(-1)^n}{\pi n^2}\left[1+2\pi-1+2\pi\right]$$

$$= \frac{(-1)^n}{\pi n^2}(4\pi) = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(1+x+x^2)\left(\frac{-\cos nx}{n}\right) - (1+2x)\left(\frac{-\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{-(1+\pi+\pi^2)\frac{(-1)^n}{n} + 0 + \frac{2(-1)^n}{n^3}\right\} - \left\{-(1-\pi+\pi^2)\frac{(-1)^n}{n} + 0 + \frac{2(-1)^n}{n^3}\right\}\right]$$

$$= \frac{(-1)^n}{n\pi}\left[-1-\pi-\pi^2+1-\pi+\pi^2\right]$$

$$= \frac{(-1)^n}{n\pi}(-2\pi) = \frac{-2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= \frac{1}{2}\left(2 + \frac{2\pi^2}{3}\right) + \sum_{n=1}^{\infty}\left[\frac{4(-1)^n}{n^2}\cos nx + \frac{2(-1)^{n+1}}{n}\sin nx\right]$$

$$= 1 + \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \ldots\ldots\ldots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\ldots\ldots\right]$$

$$f(x) = 1 + \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\ldots\ldots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\ldots\ldots\right]$$

Put $x = \pi$ in the above series we get

$$f(\pi) = 1 + \frac{\pi^2}{3} - 4\left[-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \ldots\ldots\ldots\right] + 2(0) \quad \text{---------------} \ (1)$$

But $x = \pi$ is the point of discontinuity. So we have

$$f(\pi) = \frac{f(-\pi) + f(\pi)}{2} = \frac{(1-\pi+\pi^2) + (1+\pi+\pi^2)}{2} = \frac{2+2\pi^2}{2} = 1 + \pi^2$$

Hence equation (1) becomes

$$1+\pi^2 = 1+\frac{\pi^2}{3} - 4\left[-\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}-..........\right]$$

$$\pi^2 - \frac{\pi^2}{3} = 4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+..........\right]$$

$$\frac{2\pi^2}{3} = 4\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+..........\right]$$

$$\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+..............$$

5 **Find the Fourier series expansion of $(\pi - x)^2$ in $-\pi < x < \pi$.**

Sol Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - x)^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{(\pi - x)^3}{-3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{-3\pi}\left[0 - 8\pi^3\right]$$

$$= \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - x)^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) - [2(\pi - x)(-1)]\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\{0+0-0\} - \left\{0 - \frac{(4\pi)(-1)^n}{n^2} - 0\right\}\right]$$

$$= \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - x)^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2\left(\frac{-\cos nx}{n}\right) - [2(\pi - x)(-1)]\left(\frac{-\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{0+0+\frac{2(-1)^n}{n^3}\right\} - \left\{-(4\pi^2)\frac{(-1)^n}{n} + 0 + \frac{2(-1)^n}{n^3}\right\}\right]$$

$$= \frac{4\pi(-1)^n}{n}$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

$$=\frac{1}{2}\left(\frac{8\pi^2}{3}\right)+\sum_{n=1}^{\infty}\left[\frac{4(-1)^n}{n^2}\cos nx+\frac{4\pi(-1)^n}{n}\sin nx\right]$$

$$f(x)=\frac{4\pi^2}{3}+4\left[-\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}-\frac{\cos 3x}{3^2}+\ldots\ldots\ldots\right]+4\pi\left[-\frac{\sin x}{1}+\frac{\sin 2x}{2}-\frac{\sin 3x}{3}+\ldots\ldots\ldots\right]$$

$$(i.e.)\ f(x)=\frac{4\pi^2}{3}-4\left[\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\ldots\ldots\ldots\right]-4\pi\left[\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\ldots\ldots\ldots\right]$$

6 **Find the Fourier series of periodicity 3 for *f(x) = 2x – x²* in 0 <*x*< 3.**

Sol Fourier series is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{3}+b_n\sin\frac{2n\pi x}{3}\right)$$

$$a_0=\frac{1}{(3/2)}\int_0^3 f(x)\,dx=\frac{2}{3}\int_0^3(2x-x^2)\,dx$$

$$=\frac{2}{3}\left[\frac{2x^2}{2}-\frac{x^3}{3}\right]_0^3$$

$$=\frac{2}{3}\left[\left(9-\frac{27}{3}\right)-(0-0)\right]$$

$$=0$$

$$a_n=\frac{1}{(3/2)}\int_0^3 f(x)\cos nx\,dx=\frac{2}{3}\int_0^3(2x-x^2)\cos\frac{2n\pi x}{3}\,dx$$

$$=\frac{2}{3}\left[(2x-x^2)\left(\frac{\sin\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right)-(2-2x)\left(\frac{-\cos\frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}}\right)+(-2)\left(\frac{-\sin\frac{2n\pi x}{3}}{\frac{8n^3\pi^3}{27}}\right)\right]_0^3$$

$$=\frac{2}{3}\left[\left\{0-(-4)\left(\frac{-9}{4n^2\pi^2}\right)+0\right\}-\left\{0-(2)\left(\frac{-9}{4n^2\pi^2}\right)+0\right\}\right]$$

$$=\frac{2}{3}\left[\frac{-54}{4n^2\pi^2}\right]$$

$$=\frac{-9}{n^2\pi^2}$$

$$b_n=\frac{1}{(3/2)}\int_0^3 f(x)\sin nx\,dx=\frac{2}{3}\int_0^3(2x-x^2)\sin\frac{2n\pi x}{3}\,dx$$

$$=\frac{2}{3}\left[(2x-x^2)\left(\frac{-\cos\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right)-(2-2x)\left(\frac{-\sin\frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}}\right)+(-2)\left(\frac{\cos\frac{2n\pi x}{3}}{\frac{8n^3\pi^3}{27}}\right)\right]_0^3$$

$$=\frac{2}{3}\left[\left\{(-3)\left(\frac{-3}{2n\pi}\right)+0-2\left(\frac{27}{8n^3\pi^3}\right)\right\}-\left\{0+0-2\left(\frac{27}{8n^3\pi^3}\right)\right\}\right]$$

$$=\frac{3}{n\pi}$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{3}+b_n\sin\frac{2n\pi x}{3}\right)$$

$$=0+\sum_{n=1}^{\infty}\left(\frac{-9}{n^2\pi^2}\cos\frac{2n\pi x}{3}+\frac{3}{n\pi}\sin\frac{2n\pi x}{3}\right)$$

$$(i.e.)\ f(x)=-\frac{9}{\pi^2}\left[\frac{1}{1^2}\cos\frac{2\pi x}{3}+\frac{1}{2^2}\cos\frac{4\pi x}{3}+\frac{1}{3^2}\cos\frac{6\pi x}{3}+\ldots\ldots\ldots\ldots\right]$$

$$+\frac{3}{\pi}\left[\frac{1}{1}\sin\frac{2\pi x}{3}+\frac{1}{2}\sin\frac{4\pi x}{3}+\frac{1}{3}\sin\frac{6\pi x}{3}+\ldots\ldots\ldots\ldots\right]$$

7 **Expand $f(x) = x - x^2$ as a Fourier series in $-l < x < l$ and using this series find the root square mean value of $f(x)$ in the interval.**

Sol Fourier series is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$$

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)\,dx=\frac{1}{l}\int_{-l}^{l}(x-x^2)\,dx$$

$$=\frac{1}{l}\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{-l}^{l}$$

$$=\frac{1}{l}\left[\left\{\frac{l^2}{2}-\frac{l^3}{3}\right\}-\left\{\frac{l^2}{2}+\frac{l^3}{3}\right\}\right]$$

$$=\frac{1}{l}\left(\frac{-2l^3}{3}\right)=\frac{-2l^2}{3}$$

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}\,dx=\frac{1}{l}\int_{-l}^{l}(x-x^2)\cos\frac{n\pi x}{l}\,dx$$

$$=\frac{1}{l}\left[(x-x^2)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(1-2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)+(-2)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_{-l}^{l}$$

$$=\frac{1}{l}\left[\left\{0+(1-2l)\left(\frac{(-1)^n l^2}{n^2\pi^2}\right)+0\right\}-\left\{0+(1+2l)\left(\frac{(-1)^n l^2}{n^2\pi^2}\right)+0\right\}\right]$$

$$=\frac{(-1)^n l^2}{l\,n^2\pi^2}\left[1-2l-1-2l\right]$$

$$=\frac{(-1)^n l}{n^2\pi^2}\left[-4l\right]=\frac{4\,l^2\,(-1)^{n+1}}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx = \frac{1}{l}\int_{-l}^{l}(x-x^2)\sin\frac{n\pi x}{l}dx$$

$$= \frac{1}{l}\left[(x-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\left\{-(l-l^2)\left(\frac{(-1)^n l}{n\pi}\right) + 0 - \frac{2(-1)^n l^3}{n^3\pi^3}\right\} - \left\{-(-l-l^2)\left(\frac{(-1)^n l}{n\pi}\right) + 0 - \frac{2(-1)^n l^3}{n^3\pi^3}\right\}\right]$$

$$= \frac{-(-1)^n l}{l\ n\pi}\left[l - l^2 + l + l^2\right]$$

$$= \frac{(-1)^{n+1}}{n\pi}[2l] = \frac{2l\ (-1)^{n+1}}{n\pi}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)$$

$$= \frac{1}{2}\left(\frac{-2l^2}{3}\right) + \sum_{n=1}^{\infty}\left(\frac{4l^2(-1)^{n+1}}{n^2\pi^2}\cos\frac{n\pi x}{l} + \frac{2l(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{l}\right)$$

$$(i.e.)\ \ f(x) = \frac{-l^2}{3} + \frac{4l^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} - \frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} - \frac{1}{4^2}\cos\frac{4\pi x}{l} + \ldots\ldots\ldots\right]$$

$$+ \frac{2l}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - \frac{1}{4}\sin\frac{4\pi x}{l} + \ldots\ldots\ldots\right]$$

8 **Obtain the Fourier series of f(x) = 1-x^2 over the interval (-1,1).**

Sol The given function is even, as f(-x) = f(x). Also period of f(x) is 1-(-1)=2

Here

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)dx = 2\int_{0}^{1} f(x)dx$$

$$= 2\int_{0}^{1}(1-x^2)dx = 2\left[x - \frac{x^3}{3}\right]_0^1$$

$$= \frac{4}{3}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)dx$$

$$= 2\int_{0}^{1} f(x)\cos(n\pi x)dx$$

$$= 2\int_{0}^{1}(1-x^2)\ \cos(n\pi x)dx$$

Integrating by parts, we get

$$a_n = 2\left[(1-x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(\frac{-\cos n\pi x}{(n\pi)^2}\right) + (-2)\left(\frac{-\sin n\pi x}{(n\pi)^3}\right)\right]_0^1$$

$$= \frac{4(-1)^{n+1}}{n^2\pi^2}$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)dx$$ =0, since f(x)sin(nπx) is odd.

The Fourier series of f(x) is

$$f(x) = \frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos(n\pi x)$$

9 **Find the Fourier series for the function** $f(x) = \begin{cases} 1+x, & -2 \le x \le 0 \\ 1-x, & 0 \le x \le 2 \end{cases}$

Deduce that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

Sol . *f(– x) = 1 – x* in (–2, 0)
= *f(x)* in (0, 2)
and *f(– x) = 1 + x* in (0, 2)
= *f(x)* in (–2, 0)
Hence *f(x)* is an even function.

$$\therefore\ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 (1-x)\,dx$$

$$= \left[x - \frac{x^2}{2}\right]_0^2$$

$$= \left[(2-2)-(0)\right]$$

$$= 0$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}dx = \int_0^2 (1-x)\cos\frac{n\pi x}{2}dx$$

$$= \left[(1-x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^2$$

$$= \left[\left\{0 - \frac{4(-1)^n}{n^2\pi^2}\right\} - \left\{0 - \frac{4}{n^2\pi^2}\right\}\right]$$

$$= \frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]$$

$$f(x) = \frac{0}{2} + \sum_{n=1}^{\infty} \frac{4[1-(-1)^n]}{n^2\pi^2} \cos \frac{n\pi x}{2}$$

$$= \frac{4}{\pi^2}\left[\frac{2}{1^2}\cos\frac{\pi x}{2} + 0 + \frac{2}{3^2}\cos\frac{3\pi x}{2} + 0 + \frac{2}{5^2}\cos\frac{5\pi x}{2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

$$f(x) = \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

Put $x = 0$ in the above series we get

$$f(0) = \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\ldots\ldots\ldots\right] \text{ --------------- (1)}$$

But $x = 0$ is the point of discontinuity. So we have

$$f(0) = \frac{f(0-) + f(0+)}{2} = \frac{(1)+(1)}{2} = \frac{2}{2} = 1$$

Hence equation (1) becomes

$$1 = \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\ldots\ldots\ldots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\ldots\ldots\ldots\ldots\ldots$$

$$(i.e.)\ \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

10 **Obtain the sine series for** $f(x) = \begin{cases} x & in \quad 0 \le x \le \frac{l}{2} \\ l - x & in \quad \frac{l}{2} \le x \le l \end{cases}$

Sol Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\int_0^{l/2} x \sin \frac{n\pi x}{l}\, dx + \frac{2}{l}\int_{l/2}^{l} (l-x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[(x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{l/2} + \frac{2}{l}\left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{l/2}^{l}$$

$$= \frac{2}{l}\left[\left\{-\left(\frac{l}{2}\right)\left(\frac{l.\cos\frac{n\pi}{2}}{n\pi}\right)+\left(\frac{l^2.\sin\frac{n\pi}{2}}{n^2\pi^2}\right)\right\}-\{0+0\}\right]+\frac{2}{l}\left[\{0-0\}-\left\{-\left(\frac{l}{2}\right)\left(\frac{l.\cos\frac{n\pi}{2}}{n\pi}\right)-\left(\frac{l^2.\sin\frac{n\pi}{2}}{n^2\pi^2}\right)\right\}\right]$$

$$= \frac{2}{l}\left[\frac{2l^2.\sin\frac{n\pi}{2}}{n^2\pi^2}\right]$$

$$b_n = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty}\frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}$$

$$= \frac{4l}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi}{2}\sin\frac{\pi x}{l}+0+\frac{1}{3^2}\sin\frac{3\pi}{2}\sin\frac{3\pi x}{l}+0+\frac{1}{5^2}\sin\frac{5\pi}{2}\sin\frac{5\pi x}{l}+0+\ldots\ldots\ldots\ldots\right]$$

$$= \frac{4l}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{l}-\frac{1}{3^2}\sin\frac{3\pi x}{l}+\frac{1}{5^2}\sin\frac{5\pi x}{l}+\ldots\ldots\ldots\ldots\right]$$

11 **Find the Fourier series of** $f(x) = \begin{cases} 1, & 0 < x < \pi \\ 2, & \pi < x < 2\pi \end{cases}$

Sol Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}(1)\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2)\,dx$$

$$= \frac{1}{\pi}[x]_0^{\pi} + \frac{2}{\pi}[x]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}[(\pi-0)] + \frac{2}{\pi}[(2\pi-\pi)]$$

$$= 1 + 2 = 3$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}(1)\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} + \frac{2}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}(0-0) + \frac{2}{\pi}(0-0)$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi}(1)\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} + \frac{2}{\pi}\left[\frac{-\cos nx}{n}\right]_{\pi}^{2\pi}$$

$$= \frac{-1}{n\pi}[(-1)^n - 1] - \frac{2}{n\pi}[1-(-1)^n]$$

$$= \frac{1}{n\pi}\left[-(-1)^n + 1 - 2 + 2(-1)^n\right]$$

$$= \frac{(-1)^n - 1}{n\pi}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$$

$$= \frac{3}{2} + \sum_{n=1}^{\infty}\left[0.\cos nx + \frac{(-1)^n - 1}{n\pi}\sin nx\right]$$

$$= \frac{3}{2} - \frac{2}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots\dots\dots\dots\right]$$

12 **Find the Fourier series expansion of** $f(x) = \begin{cases} x, & 0 < x < \frac{l}{2} \\ l - x, & \frac{l}{2} < x < l \end{cases}$

Sol . Let $2L = l \Rightarrow L = \frac{l}{2}$, then the given function becomes

$$f(x) = \begin{cases} x, & 0 < x < L \\ 2L - x, & L < x < 2L \end{cases}$$

Fourier series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx = \frac{1}{L}\int_0^{L}(x)\,dx + \frac{1}{L}\int_L^{2L}(2L - x)\,dx$$

$$= \frac{1}{L}\left[\frac{x^2}{2}\right]_0^{L} + \frac{1}{L}\left[\frac{(2L-x)^2}{-2}\right]_L^{2L}$$

$$= \frac{1}{L}\left[\frac{L^2}{2} - 0\right] + \frac{1}{L}\left[0 - \frac{L^2}{-2}\right]$$

$$= \frac{L}{2} + \frac{L}{2} = L$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\int_0^{L} x\cos\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_L^{2L}(2L-x)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\left[(x)\left(\frac{\sin\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - (1)\left(\frac{-\cos\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right]_0^L + \frac{1}{L}\left[(2L-x)\left(\frac{\sin\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right]_L^{2L}$$

$$= \frac{1}{L}\left[\left\{0 + \frac{(-1)^n L^2}{n^2\pi^2}\right\} - \left\{0 + \frac{L^2}{n^2\pi^2}\right\}\right] + \frac{1}{L}\left[\left\{0 - \frac{L^2}{n^2\pi^2}\right\} - \left\{0 - \frac{(-1)^n L^2}{n^2\pi^2}\right\}\right]$$

$$= \frac{1}{L}\frac{L^2}{n^2\pi^2}\left[(-1)^n - 1 - 1 + (-1)^n\right] = \frac{2L}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\int_0^{L} x\sin\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_L^{2L}(2L-x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\left[(x)\left(\frac{-\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right]_0^L + \frac{1}{L}\left[(2L-x)\left(\frac{-\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right]_L^{2L}$$

$$= \frac{1}{L}\left[\left\{-\frac{(-1)^n L^2}{n\pi} + 0\right\} - \{0+0\}\right] + \frac{1}{L}\left[\{0-0\} - \left\{-\frac{(-1)^n L^2}{n\pi} - 0\right\}\right]$$

$$= 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$

$$= \frac{L}{2} + \sum_{n=1}^{\infty}\left(\frac{2L[(-1)^n - 1]}{n^2\pi^2}\cos\frac{n\pi x}{L} + 0\right)$$

$$= \frac{L}{2} + \frac{2L}{\pi^2}\left[-\frac{2}{1^2}\cos\frac{\pi x}{L} + 0 - \frac{2}{3^2}\cos\frac{3\pi x}{L} + 0 - \frac{2}{5^2}\cos\frac{5\pi x}{L} + 0 - \ldots\ldots\ldots\ldots\right]$$

$$= \frac{L}{2} - \frac{4L}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{L} + \frac{1}{3^2}\cos\frac{3\pi x}{L} + \frac{1}{5^2}\cos\frac{5\pi x}{L} + \ldots\ldots\ldots\ldots\right]$$

$$(i.e.)\;\; f(x) = \frac{l}{4} - \frac{2l}{\pi^2}\left[\frac{1}{1^2}\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{6\pi x}{l} + \frac{1}{5^2}\cos\frac{10\pi x}{l} + \ldots\ldots\ldots\ldots\right]$$

13 **Find the Fourier series expansion of** $f(x) = \begin{cases} l - x, & 0 < x < l \\ 0, & l < x < 2l \end{cases}$

Hence deduce the value of the series (i) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots$

(ii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots\ldots$

Sol

Fourier series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{1}{l}\int_0^{l}(l-x)\,dx + \frac{1}{l}\int_l^{2l}(0)\,dx$$

$$= \frac{1}{l}\left[\frac{(l-x)^2}{-2}\right]_0^l$$

$$= \frac{1}{-2l}\left[0 - l^2\right]$$

$$= \frac{l}{2}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_0^{l}(l-x)\cos\frac{n\pi x}{l}\,dx + 0$$

$$= \frac{1}{l}\left[(l-x)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{1}{l}\left[\left\{0 - \frac{(-1)^n l^2}{n^2\pi^2}\right\} - \left\{0 - \frac{l^2}{n^2\pi^2}\right\}\right]$$

$$= \frac{1}{l}\,\frac{l^2}{n^2\pi^2}\left[(-1)^{n+1} + 1\right]$$

$$= \frac{l}{n^2\pi^2}\left[(-1)^{n+1} + 1\right]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_0^{l}(l-x)\sin\frac{n\pi x}{l}\,dx + 0$$

$$= \frac{1}{l}\left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{1}{l}\left[\{0-0\} - \left\{-\frac{l^2}{n\pi} - 0\right\}\right]$$

$$= \frac{l}{n\pi}$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$$

$$=\frac{l}{4}+\sum_{n=1}^{\infty}\left(\frac{l[(-1)^{n+1}+1]}{n^2\pi^2}\cos\frac{n\pi x}{l}+\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)$$

$$=\frac{l}{4}+\frac{l}{\pi^2}\left[\frac{2}{1^2}\cos\frac{\pi x}{l}+0+\frac{2}{3^2}\cos\frac{3\pi x}{l}+0+\frac{2}{5^2}\cos\frac{5\pi x}{l}+0+\dots\dots\dots\right]$$

$$+\frac{l}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{l}+\frac{1}{2}\sin\frac{2\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}+\dots\dots\dots\right]$$

$$(i.e.)\ f(x)=\frac{l}{4}+\frac{2l}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l}+\frac{1}{3^2}\cos\frac{3\pi x}{l}+\frac{1}{5^2}\cos\frac{5\pi x}{l}+\dots\dots\dots\right]$$

$$+\frac{l}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{l}+\frac{1}{2}\sin\frac{2\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}+\dots\dots\dots\right]---------(1)$$

Put $x=\frac{l}{2}$ (which is point of continuity) in equation (1), we get

$$l-\frac{l}{2}=\frac{l}{4}+\frac{2l}{\pi^2}(0)+\frac{l}{\pi}\left[\frac{1}{1}\sin\frac{\pi}{2}+\frac{1}{2}\sin\pi+\frac{1}{3}\sin\frac{3\pi}{2}+\frac{1}{4}\sin 4\pi+\frac{1}{5}\sin\frac{5\pi}{2}+\dots\dots\dots\right]$$

$$\frac{l}{2}=\frac{l}{4}+\frac{l}{\pi}\left[1+0-\frac{1}{3}+0+\frac{1}{5}+0-\frac{1}{7}+\dots\dots\dots\right]$$

$$\frac{l}{2}-\frac{l}{4}=\frac{l}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots\dots\dots\right]$$

$$\frac{l}{4}=\frac{l}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots\dots\dots\right]$$

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots\dots\dots$$

Put $x=l$ in equation (1) we get

$$f(l)=\frac{l}{4}+\frac{2l}{\pi^2}\left[-\frac{1}{1^2}-\frac{1}{3^2}-\frac{1}{5^2}-\dots\dots\dots\right]\ \text{---------------}\ (2)$$

But $x=l$ is the point of discontinuity. So we have

$$f(l)=\frac{f(l-)+f(l+)}{2}=\frac{(0)+(0)}{2}=0$$

Hence equation (2) becomes

$$0=\frac{l}{4}-\frac{2l}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\dots\dots\right]$$

$$-\frac{l}{4}=-\frac{2l}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\dots\dots\right]$$

$$\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\dots\dots\dots$$

<u>**HALF RANGE FOURIER SERIES**</u>

- **Half Range Fourier Sine Series defined in $[\mathbf{0}, \boldsymbol{\pi}]$:**

  The Fourier half range sine series in $[\mathbf{0}, \boldsymbol{\pi}]$is given by ***f(x)***=$\sum_{n=1}^{\infty} b_n sinnx$
  Where $b_n=\frac{2}{\pi}\int_0^{\pi} \mathbf{f(x)sin\ nxdx}$

This is Similar to the Fourier series defined for odd function in $[-\pi, \pi]$

- **Half Range Fourier Sine Series defined in $[\mathbf{0}, \boldsymbol{l}]$:**

  The Fourier half range sine series in $[\mathbf{0}, \boldsymbol{\pi}]$is given by ***f(x)***= $\sum_{n=1}^{\infty} b_n sinnx$
  Where $b_n=\frac{2}{l}\int_0^{l} f(x) sin\ \frac{n\pi x}{l} dx$

This is Similar to the Fourier series defined for odd function in $[-l, l]$

- **Half Range Fourier cosine Series defined in $[\mathbf{0}, \boldsymbol{\pi}]$:**

  The Fourier half range cosine series in $[\mathbf{0}, \boldsymbol{\pi}]$is given by
  f(x)= $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx$
  Where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

  $$a_n = \frac{2}{\pi}\int_0^{\pi} \mathbf{f(x)cosnxdx}$$

This is Similar to the Fourier series defined for even function in $[-\pi, \pi]$

- **Half Range Fourier cosine Series defined in $[\mathbf{0}, \boldsymbol{l}]$:**

  The Fourier half range cosine series in $[\mathbf{0}, \boldsymbol{l}]$is given by
  f(x)= $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\frac{n\pi x}{l}$

  Where $a_0 = \frac{2}{l}\int_0^{l} f(x)dx$

  $$a_n = \frac{2}{l}\int_0^{l} \mathbf{f(x)cos}\frac{n\pi x}{l}\mathbf{dx}$$

This is Similar to the Fourier series defined for even function in $[-l, l]$

Problems

1 **Find the half range sine series for *f(x)* = 2 in 0 <*x* <π.**

Sol

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} 2\sin nx\,dx$$

$$= \frac{4}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} = \frac{-4}{n\pi}\left[(-1)^n - 1\right] = \frac{4}{n\pi}\left[1-(-1)^n\right]$$

Half range sine series is

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} \frac{4[1-(-1)^n]}{n\pi}\sin nx$$

$$=\frac{4}{\pi}\left[\frac{2\sin x}{1}+\frac{2\sin 3x}{3}+\frac{2\sin 5x}{5}+\ldots\ldots\ldots\ldots\right]$$

$$=\frac{8}{\pi}\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\ldots\ldots\ldots\ldots\right]$$

2 **Expand *f(x) = cos x, 0 < x < π* in a Fourier sine series.**

Sol Fourier sine series is

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\cos x \sin nx\,dx$$

$$=\frac{1}{\pi}\int_0^{\pi} 2\sin nx\cos x\,dx$$

$$=\frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x+\sin(n-1)x]\,dx\ , \quad n\neq 1$$

$$=\frac{1}{\pi}\left[\left(\frac{-\cos(n+1)x}{n+1}\right)+\left(\frac{-\cos(n-1)x}{n-1}\right)\right]_0^{\pi}$$

$$=-\frac{1}{\pi}\left[\left\{\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}\right\}-\left\{\frac{1}{n+1}+\frac{1}{n-1}\right\}\right]$$

$$=-\frac{1}{\pi}\left[(-1)^n\left\{\frac{-1}{n+1}+\frac{-1}{n-1}\right\}-\left\{\frac{1}{n+1}+\frac{1}{n-1}\right\}\right]$$

$$=\frac{1}{\pi}\left[(-1)^n\left\{\frac{1}{n+1}+\frac{1}{n-1}\right\}+\left\{\frac{1}{n+1}+\frac{1}{n-1}\right\}\right]$$

$$=\frac{1}{\pi}\left[(-1)^n\left\{\frac{2n}{n^2-1}\right\}+\left\{\frac{2n}{n^2-1}\right\}\right]$$

$$b_n = \frac{2n}{\pi(n^2-1)}\left[(-1)^n+1\right],\quad n\neq 1$$

When n = 1, we have

$$b_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\sin x\,dx = \frac{2}{\pi}\int_0^{\pi}\cos x\sin x\,dx$$

$$=\frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx$$

$$=\frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} = -\frac{1}{2\pi}(1-1)=0$$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx = b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= 0 + \sum_{n=2}^{\infty} \frac{2n[(-1)^n+1]}{\pi(n^2-1)} \sin nx$$

$$= \frac{2}{\pi}\left[\frac{4\sin 2x}{3} + 0 + \frac{8\sin 4x}{15} + 0 + \frac{12\sin 6x}{35} + 0 + \dots\dots\dots\right]$$

$$= \frac{8}{\pi}\left[\frac{\sin 2x}{3} + \frac{2\sin 4x}{15} + \frac{3\sin 6x}{35} + \dots\dots\dots\right]$$

3 **Find the half range cosine series for the function f*(x)* =*x* (π – *x*) in 0<*x*< π.**

Sol Half range Fourier cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\,dx$$

$$= \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi^3}{2} - \frac{\pi^3}{3}\right) - (0-0)\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{6}\right]$$

$$= \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{0 + \frac{(-\pi)(-1)^n}{n^2} + 0\right\} - \left\{0 + \frac{(\pi)(1)}{n^2} + 0\right\}\right]$$

$$= \frac{2\pi}{\pi n^2}\left[-(-1)^n - 1\right]$$

$$= -\frac{2}{n^2}\left[(-1)^n + 1\right]$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

$$=\frac{1}{2}\left(\frac{\pi^2}{3}\right)+\sum_{n=1}^{\infty}-\frac{2}{n^2}\left[(-1)^n+1\right]\cos nx$$

$$=\frac{\pi^2}{6}-2\left[0+\frac{2\cos 2x}{2^2}+0+\frac{2\cos 4x}{4^2}+0+\frac{2\cos 6x}{6^2}+0+\ldots\ldots\ldots\ldots\right]$$

$$=\frac{\pi^2}{6}-4\left[\frac{\cos 2x}{2^2}+\frac{\cos 4x}{4^2}+\frac{\cos 6x}{6^2}+\ldots\ldots\ldots\ldots\right]$$

4 **Find the half range cosine series for the function*f(x) = x* in0 < x <*l*.**

Sol

Half range Fourier cosine series is $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$

$$a_0=\frac{2}{l}\int_0^l f(x)\,dx=\frac{2}{l}\int_0^l x\,dx=\frac{2}{l}\left[\frac{x^2}{2}\right]_0^l=\frac{2}{l}\left[\frac{l^2}{2}-0\right]=l$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx=\frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{l}\left[(x)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(1)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$=\frac{2}{l}\left[\left\{0+\frac{(-1)^n l^2}{n^2\pi^2}\right\}-\left\{0+\frac{l^2}{n^2\pi^2}\right\}\right]$$

$$=\frac{2l}{n^2\pi^2}\left[(-1)^n-1\right]$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$$

$$=\frac{l}{2}+\sum_{n=1}^{\infty}\frac{2l[(-1)^n-1]}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

$$=\frac{l}{2}+\frac{2l}{\pi^2}\left[-\frac{2}{1^2}\cos\frac{\pi x}{l}+0-\frac{2}{3^2}\cos\frac{3\pi x}{l}+0-\frac{2}{5^2}\cos\frac{5\pi x}{l}+0-\ldots\ldots\ldots\ldots\ldots\right]$$

$$(i.e.)\ f(x)=\frac{l}{2}-\frac{4l}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l}+\frac{1}{3^2}\cos\frac{3\pi x}{l}+\frac{1}{5^2}\cos\frac{5\pi x}{l}+\ldots\ldots\ldots\ldots\ldots\right]$$

5 **Find the half range sine series of *f(x) = x cos x* in *(0, π)*.**

Sol

Fourier sine series is $f(x)=\sum_{n=1}^{\infty}b_n\sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos x\,\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,(2\sin nx\cos x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x + \sin(n-1)x]\,dx\ ,\quad n\neq 1$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin(n+1)x\,dx + \frac{1}{\pi}\int_0^{\pi} x\sin(n-1)x\,dx\ ,\ n\neq 1$$

$$b_n = \frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1}\right)-(1)\left(\frac{-\sin(n+1)x}{(n+1)^2}\right)\right]_0^{\pi} + \frac{1}{\pi}\left[x\left(\frac{-\cos(n-1)x}{n-1}\right)-(1)\left(\frac{-\sin(n-1)x}{(n-1)^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\frac{-\pi(-1)^{n+1}}{n+1}+0\right\}-\{0+0\}\right] + \frac{1}{\pi}\left[\left\{\frac{-\pi(-1)^{n-1}}{n-1}+0\right\}-\{0+0\}\right]$$

$$= \frac{(-1)^{n+2}}{n+1} + \frac{(-1)^n}{n-1}$$

$$= (-1)^n\left[\frac{1}{n+1}+\frac{1}{n-1}\right]$$

$$= (-1)^n\left[\frac{2n}{(n+1)(n-1)}\right]$$

$$(i.e.)\, b_n = \frac{2n(-1)^n}{n^2-1}\ ,\ n\neq 1$$

When n = 1, we have

$$b_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\sin x\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos x\sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right)-(1)\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\pi\left(\frac{-1}{2}\right)+0\right\}-\{0+0\}\right] = -\frac{1}{2}$$

$$f(x) = \sum_{n=1}^{\infty} b_n\sin nx = b_1\sin x + \sum_{n=2}^{\infty} b_n\sin nx$$

$$= -\frac{1}{2}\sin x + \sum_{n=2}^{\infty}\frac{2n(-1)^n}{n^2-1}\sin nx$$

$$= -\frac{1}{2}\sin x + 2\left[\frac{2\sin 2x}{3} - \frac{3\sin 3x}{8} + \frac{4\sin 4x}{15} + \ldots\ldots\ldots\ldots\right]$$

6 **Obtain the half range cosine series for *f(x)* = (*x* – 2)^2 in the interval 0 <*x* < 2. Deduce that** $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

Sol Half range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 (x-2)^2\,dx$$

$$= \left[\frac{(x-2)^3}{3}\right]_0^2$$

$$= \left[0 - \left\{\frac{-8}{3}\right\}\right] = \frac{8}{3}$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx = \int_0^2 (x-2)^2 \cos\frac{n\pi x}{2}\,dx$$

$$= \left[(x-2)^2\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - [2(x-2)]\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right) + (2)\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^3\pi^3}{8}}\right)\right]_0^2$$

$$= \left[\{0+0-0\} - \left\{0 - \frac{16}{n^2\pi^2} - 0\right\}\right]$$

$$= \frac{16}{\pi^2 n^2}$$

$$f(x) = \frac{8}{6} + \sum_{n=1}^{\infty} \frac{16}{n^2\pi^2}\cos\frac{n\pi x}{2}$$

$$(i.e.)\ f(x) = \frac{4}{3} + \frac{16}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{2^2}\cos\frac{2\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \ldots\ldots\ldots\ldots\ldots\ldots\right]$$

Put $x = 0$ in equation (1) we get

$$f(0) = \frac{4}{3} + \frac{16}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\ldots\ldots\ldots\ldots\right]$$

But $x = 0$ is the point of discontinuity. So we have

$$f(x) = \frac{(x+2)^2 + (x-2)^2}{2}$$

$$f(0) = \frac{(0+2)^2 + (0-2)^2}{2} = \frac{(4)+(4)}{2} = 4$$

Hence equation becomes

$$4=\frac{4}{3}+\frac{16}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\dots\dots\dots\right]$$

$$4-\frac{4}{3}=\frac{16}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\dots\dots\dots\right]$$

$$\frac{8}{3}=\frac{16}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\dots\dots\dots\right]$$

$$\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\dots\dots\dots$$

Put $x = 2$ in equation we get

$$f(2)=\frac{4}{3}+\frac{16}{\pi^2}\left[-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\dots\dots\dots\dots\right]$$

But $x = 2$ is the point of discontinuity. So we have

$$f(x)=\frac{(x-2)^2+(2-x)^2}{2}$$

$$f(2)=\frac{(2-2)^2+(2-2)^2}{2}=0$$

Hence equation becomes

$$0=\frac{4}{3}+\frac{16}{\pi^2}\left[-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\dots\dots\dots\dots\right]$$

$$-\frac{4}{3}=-\frac{16}{\pi^2}\left[\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\dots\dots\dots\dots\right]$$

$$\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\dots\dots\dots\dots$$

we get

$$\frac{\pi^2}{6}+\frac{\pi^2}{12}=2\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\dots\right]$$

$$\frac{3\pi^2}{12}=2\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\dots\right]$$

$$\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\dots\dots$$

$$(i.e.)\ \frac{\pi^2}{8}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

UNIT-II
FOURIER TRANSFORMS

- Fourier integral theorem,
- Fourier sine and cosine integrals
- Fourier transforms
- Fourier sine and cosine transform
- Inverse transforms
- Finite Fourier transforms

Introduction

The Fourier transform named after Joseph Fourier, is a mathematical transformation employed to transform signals between time (or spatial) domain and frequency domain, which has many applications in physics and engineering. It is reversible, being able to transform from either domain to the other. The term itself refers to both the transform operation and to the function it produces.

In the case of a periodic function over time (for example, a continuous but not necessarily sinusoidal musical sound), the Fourier transform can be simplified to the calculation of a discrete set of complex amplitudes, called Fourier series coefficients. They represent the frequency spectrum of the original time-domain signal. Also, when a time-domain function is sampled to facilitate storage or computer-processing, it is still possible to recreate a version of the original Fourier transform according to the Poisson summation formula, also known as discrete-time Fourier transform. See also Fourier analysis and List of Fourier-related transforms.

Integral Transform

The integral transform of a function f(x) is given by

$$I\,[f(x)] \text{ or } F(s) = \int_a^b f(x)k(s,x)dx$$

Where k(s, x) is a known function called **kernel of the transform**

s is called the **parameter of the transform**

f(x) is called the **inverse transform of F(s)**

Fourier transform

$k(s,x) = e^{isx}$

$$F[f(x)] = F(s) = \int_{-\infty}^{\infty} f(x)e^{isx}dx$$

Laplace transform

$k(s,x) = e^{-sx}$

$$L[f(x)] = F(s) = \int_0^{\infty} f(x)e^{-sx}dx$$

Henkel transform

$k(s,x) = xJ_n(sx)$

$$H[f(x)] = H(s) = \int_0^{\infty} f(x)xJ_n(sx)dx$$

Mellin transform

$k(s,x) = x^{s-1}$

$$M[f(x)] = M(s) = \int_0^{\infty} f(x)x^{s-1}dx$$

DIRICHLET'S CONDITION

A function f(x) is said to satisfy Dirichlet's conditions in the interval (a,b) if

1. f(x) defined and is single valued function except possibly at a finite number of points in the interval (a,b)
2. f(x) and $f^1(x)$ are piecewise continuous in (a,b)

Fourier integral theorem

If f(x) is a given function defined in (-*l*,*l*) and satisfies the Dirichlet conditions then

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dtd\lambda$$

Proof:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(\frac{n\pi x}{L})+\sum_{n=1}^{\infty}b_n\sin(\frac{n\pi x}{L})$$

where

$$a_0=\frac{1}{L}\int_{-L}^{L}f(t)\,dt$$

$$a_n=\frac{1}{L}\int_{-L}^{L}f(t)\cos(\frac{n\pi x}{L})\,dt$$

$$b_n=\frac{1}{L}\int_{-L}^{L}f(t)\sin(\frac{n\pi x}{L})\,dt$$

Substituting the values in f(x)

$$f(x)=\frac{1}{L}\int_{-L}^{L}f(t)[1+2\sum_{n=1}^{\infty}\cos(\frac{n\pi}{L})(t-x)]\,dt----------(1)$$

But cosine functions are even functions

$$\sum_{n=-\infty}^{\infty}\cos(\frac{n\pi}{L})(t-x)=1+2\sum_{n=1}^{\infty}\cos(\frac{n\pi}{L})(t-x)--------(2)$$

Substituting equation (2) in (1)

$$f(x)=\frac{1}{2\pi}\int_{-L}^{L}f(t)\frac{\pi}{L}\sum_{n=-\infty}^{\infty}\cos(\frac{n\pi}{L})(t-x)\,dt$$

$$\frac{n\pi}{L}=\lambda$$

$$\operatorname{Lt}_{n\to\infty}\frac{\pi}{L}\sum_{n=-\infty}^{\infty}\cos(\frac{n\pi}{L})(t-x)=\int_{-\infty}^{\infty}\cos\lambda(t-x)\,d\lambda=2\int_{0}^{\infty}\cos\lambda(t-x)\,d\lambda$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)[2\int_{0}^{\infty}\cos\lambda(t-x)\,d\lambda]\,dt$$

$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty}f(t)\cos\lambda(t-x)\,d\lambda dt$$

Fourier Sine Integral

If f (t) is an odd function

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}\sin\lambda x\int_{0}^{\infty}f(t)\sin\lambda t dtd\lambda$$

Fourier Cosine Integral

If f (t) is an even function

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}\cos\lambda x\int_{0}^{\infty}f(t)\cos\lambda t dtd\lambda$$

Problems

1 **Express** $f(x)=\{\begin{matrix}1,|x|<1\\0,|x|>1\end{matrix}$ **as a Fourier integral. Hence evaluate** $\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}$ **and also find the value of** $\int_0^\infty \frac{\sin\lambda}{\lambda}$

Sol

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,d\lambda dt$$

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-1}^{1}\cos\lambda(t-x)\,d\lambda dt$$

$$f(x)=\frac{1}{\pi}\int_0^\infty \frac{2}{\lambda}\sin\lambda\cos\lambda\, xd\lambda$$

$$\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda=\frac{\pi}{2}f(x)=\{\begin{matrix}\frac{\pi}{2},|x|<1\\0,|x|>1\end{matrix}$$

$$|x|=1$$

$$\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda=\frac{\pi}{2}\left[\frac{1+0}{2}\right]=\frac{\pi}{4}$$

$$x=0$$

$$\int_0^\infty \frac{\sin\lambda}{\lambda}d\lambda=\frac{\pi}{2}$$

2 **Using Fourier Integral show that** $e^{-x}\cos x=\frac{2}{\pi}\int_0^\infty \frac{\lambda^2+2}{\lambda^4+2}\cos\lambda xd\lambda$

Sol

$$f(x)=e^{-x}\cos x$$

$$f(x)=\frac{2}{\pi}\int_0^\infty \cos\lambda x[\int_0^\infty f(t)\cos\lambda\, t\,dt]\,d\lambda$$

$$f(x)=\frac{2}{\pi}\int_0^\infty \cos\lambda x[\int_0^\infty e^{-t}\cos t\cos\lambda\, t\,dt]\,d\lambda$$

$$f(x)=\frac{1}{\pi}\int_0^\infty \cos\lambda x[\int_0^\infty e^{-t}(\cos(\lambda+1)\,t+\cos(\lambda-1)\,t\,dt]\,d\lambda$$

$$f(x)=\frac{1}{\pi}\int_0^\infty \cos\lambda x[\frac{1}{(\lambda+1)^2+1}+\frac{1}{(\lambda-1)^2+1}]\,d\lambda$$

$$f(x)=\frac{2}{\pi}\int_0^\infty \frac{\lambda^2+2}{\lambda^4+2}\cos\lambda xd\lambda$$

FOURIER TRANSFORMS

The complex form of Fourier integral of any function f(x) is in the form

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\int_{-\infty}^{\infty} f(t)e^{i\lambda t}dtd\lambda$$

Replacing λ by s

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}ds\int_{-\infty}^{\infty} f(t)e^{ist}dt$$

Let

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{ist}dt$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$$

Here F(s) is called Fourier transform of f(x) and f(x) is called inverse Fourier transform of F(s)

Alternative Definitions

$$F[f(t)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ist}dt,\ f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$$

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{-isx}dx,\ f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{isx}ds$$

Fourier Cosine Transform
Infinite

$$F_C[f(t)] = F_C(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\cos stdt$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_C[f(t)]\cos sxds$$

Finite

$$F_C[f(t)] = F_C(s) = \sqrt{\frac{2}{l}}\int_0^{l} f(t)\cos(\frac{n\pi t}{l})dt$$

$$f(x) = \frac{1}{l}F_C(0) + \sqrt{\frac{2}{l}}\sum_{n=1}^{\infty} F_C(s)\cos(\frac{n\pi x}{l})$$

Fourier Sine Transform
Infinite

$$F_S[f(t)] = F_S(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin stdt$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_S[f(t)]\sin sxds$$

Finite

$$F_S[f(t)] = F_S(s) = \sqrt{\frac{2}{l}}\int_0^l f(t)\sin(\frac{n\pi t}{l})dt$$

$$f(x) = \sqrt{\frac{2}{l}}\sum_{n=1}^{\infty} F_S(s)\sin(\frac{n\pi x}{l})$$

Alternative Definitions:

$$1. F_C(s) = \int_0^{\infty} f(x)\cos sx dx, f(x) = \frac{2}{\pi}\int_0^{\infty} F_C(s)\cos sx ds$$

$$2. F_S(s) = \int_0^{\infty} f(x)\sin sx dx, f(x) = \frac{2}{\pi}\int_0^{\infty} F_S(s)\sin sx ds$$

Properties of Fourier Transforms

1 **Linear Property:** $F[af_1(x) + bf_2(x)] = aF_1(s) + bF_2(s)$

Proof

$$F[af_1(x) + bf_2(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[af_1(x) + bf_2(x)]e^{ist}dt$$

$$F[af_1(x) + bf_2(x)] = \frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(x)e^{ist}dt + \frac{b}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_2(x)e^{ist}dt$$

$$F[af_1(x) + bf_2(x)] = aF_1(s) + bF_2(s)$$

2 **Shifting Theorem:** (a) $F[f(x-a)] = e^{ias}F(s)$

(b) $F[e^{iax}f(x)] = F(s+a)$

Proof (a)

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t-a)e^{ist}dt$$

$t - a = z$

$dt = dz$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)e^{isz}e^{ias}dz$$

$$F[f(x-a)] = e^{ias}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)e^{isz}dz$$

$$F[f(x-a)] = e^{ias}F(s)$$

(b)

$$F[e^{iax}f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}e^{iat}dt$$

$$F[e^{iax}f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i(a+s)t}dt$$

$$F[e^{iax}f(x)] = F(s+a)$$

3 **Change of scale property:** $F[f(ax)] = \frac{1}{a}F(\frac{s}{a})(a > 0)$

Proof

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(at)e^{ist}dt$$

$$at = z$$

$$dt = \frac{1}{a}dz$$

$$F[f(ax)] = \frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)e^{i\left(\frac{s}{a}\right)z}dz$$

$$F[f(ax)] = \frac{1}{a}F(\frac{s}{a})$$

4 **Multiplication Property:** $F[x^n f(x)] = (-i)^n \frac{d^n F}{ds^n}$

Proof

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}dt$$

$$\frac{dF}{ds} = \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t.f(t)e^{ist}dt$$

$$\frac{d^2F}{ds^2} = \frac{i^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^2.f(t)e^{ist}dt$$

continuing

$$\frac{d^nF}{ds^n} = \frac{i^n}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^n.f(t)e^{ist}dt$$

$$F[x^n f(x)] = (-i)^n \frac{d^nF}{ds^n}$$

5 **Modulation Theorem:** $F[f(x)\cos ax] = \frac{1}{2}\left[F(s+a)+F(s-a)\right], F[s] = F[f(x)]$

Proof

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\cos at\, e^{ist}dt$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\left[\frac{e^{iat}+e^{-iat}}{2}\right]e^{ist}dt$$

$$F[f(x)] = \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i(s+a)t}dt + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i(s-a)t}dt\right]$$

$$F[f(x)\cos ax] = \frac{1}{2}\left[F(s+a)+F(s-a)\right]$$

Problems

1 **Find the Fourier transform of** $f(x)=\{\begin{matrix}1,|x|<1\\0,|x|>1\end{matrix}$

Hence evaluate $\int_0^{\infty}\frac{\sin x}{x}dx$

Sol:

$$F[f(x)]=\int_{-\infty}^{\infty}f(x)e^{isx}dx$$

$$F[f(x)]=\int_{-1}^{1}1.e^{isx}dx$$

$$F[f(x)]=\left|\frac{e^{isx}}{is}\right|_{-1}^{1}$$

$$F[f(x)]=\frac{e^{is}-e^{-is}}{is}=2\frac{\sin s}{s}$$

$$f(x)=\frac{1}{2\Pi}\int_{-\infty}^{\infty}F[s]e^{-isx}ds$$

$$f(x)=\frac{1}{2\Pi}\int_{-\infty}^{\infty}2\frac{\sin s}{s}e^{-isx}ds$$

$$f(x)=\frac{1}{\Pi}\int_{-\infty}^{\infty}\frac{\sin s}{s}e^{-isx}ds$$

$$\frac{1}{\Pi}\int_{-\infty}^{\infty}\frac{\sin s}{s}e^{-isx}ds=\{\begin{matrix}1,|x|<1\\0,|x|>1\end{matrix}$$

$$x=0$$

$$\int_{-\infty}^{\infty}\frac{\sin s}{s}ds=\Pi$$

$$\int_0^{\infty}\frac{\sin s}{s}ds=\frac{\Pi}{2}$$

2 **Find the Fourier transform of** $f(x)=\{\begin{matrix}1-x^2,|x|\le 1\\0,|x|>1\end{matrix}$

Hence evaluate $\int_0^{\infty}\frac{x\cos x-\sin x}{x^3}\cos\frac{x}{2}dx$

Sol:

$$F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F[f(x)] = \int_{-1}^{1} (1-x^2) e^{isx} dx$$

$$F[f(x)] = \left| (1-x^2)\frac{e^{isx}}{is} - 2x\frac{e^{isx}}{(is)^2} + 2\frac{e^{isx}}{(is)^3} \right|_{-1}^{1}$$

$$F[f(x)] = 2\left(\frac{e^{is}+e^{-is}}{-s^2}\right) - 2\left(\frac{e^{is}-e^{-is}}{-is^3}\right)$$

$$F[f(x)] = \frac{-4}{s^3}(s\cos s - \sin s)$$

$$f(x) = \frac{1}{2\Pi}\int_{-\infty}^{\infty} F[s] e^{-isx} ds$$

$$f(x) = \frac{1}{2\Pi}\int_{-\infty}^{\infty} \frac{-4}{s^3}(s\cos s - \sin s) e^{-isx} ds$$

$$\frac{1}{2\Pi}\int_{-\infty}^{\infty} \frac{-4}{s^3}(s\cos s - \sin s) e^{-isx} ds = \begin{cases} 1-x^2, |x| \le 1 \\ 0, |x| > 1 \end{cases}$$

$$x = 1/2$$

$$\frac{1}{2\Pi}\int_{-\infty}^{\infty} \frac{-4}{s^3}(s\cos s - \sin s) e^{-isx} ds = \frac{3}{4}$$

$$\int_{-\infty}^{\infty} \frac{(s\cos s - \sin s)}{s^3}\left[\cos\frac{s}{2} - i\sin\frac{s}{2}\right] ds = -\frac{3\Pi}{8}$$

$$\int_{0}^{\infty} \frac{(s\cos s - \sin s)}{s^3}\cos\frac{s}{2} ds = -\frac{3\Pi}{16}$$

3 **Find the Fourier transform of $e^{-a^2x^2}$ .Hence deduce that $e^{x^2/2}$ is self-reciprocal in respect of Fourier transform**

Sol:

$$F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F[f(x)] = \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx$$

$$F[f(x)] = \int_{-\infty}^{\infty} e^{-a^2(x^2 - isx/a^2)} dx$$

$$F[f(x)] = \int_{-\infty}^{\infty} e^{-a^2(x - isx/2a^2)^2} e^{-s^2/4a^2} dx$$

$$t = a(x - isx/2a^2)$$

$$dx = dt/a$$

$$F[f(x)] = \int_{-\infty}^{\infty} e^{-t^2} e^{-s^2/4a^2} \frac{dt}{a}$$

$$F[f(x)] = \frac{e^{-s^2/4a^2}}{a} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$F[f(x)] = \frac{e^{-s^2/4a^2}}{a} \sqrt{\Pi}$$

$$F[f(x)] = \frac{\sqrt{\Pi}}{a} e^{-s^2/4a^2}$$

$$a^2 = 1/2$$

$$F[e^{-x^2/2}] = \sqrt{2\Pi} e^{-s^2/2}$$

Hence $e^{x^2/2}$ is self-reciprocal in respect of Fourier transform

4 **Find the Fourier cosine transform** e^{-x^2} **.**

Sol:

$$F_c(e^{-x^2}) = \int_0^\infty e^{-x^2}\cos sxdx = I$$

$$\frac{dI}{ds} = -\int_0^\infty xe^{-x^2}\,\text{sins}\,xdx = \frac{1}{2}\int_0^\infty (-2xe^{-x^2})\,\text{sins}\,xdx$$

$$\frac{dI}{ds} = \frac{-s}{2}\int_0^\infty e^{-x^2}\cos sxdx = \frac{-s}{2}I$$

$$\frac{dI}{I} = \frac{-s}{2}ds$$

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$$\log I = \int \frac{-s}{2}ds + \log c = \frac{-s^2}{4} + \log c = \log(ce^{-s^2/4})$$

$$I = ce^{-s^2/4}$$

$$\int_0^\infty e^{-x^2}\cos sxdx = ce^{-s^2/4}$$

$$s = 0$$

$$c = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\Pi}}{2}$$

$$\int_0^\infty e^{-x^2}\cos sxdx = \frac{\sqrt{\Pi}}{2}e^{-s^2/4}$$

5 **Find the Fourier sine transform** $e^{-|x|}$ **.Hence show that**

$$\int_0^\infty \frac{x\sin mx}{1+x^2}dx = \frac{\Pi e^{-m}}{2}\text{ ,m>0}$$

Sol: x being positive in the interval $(0,\infty)$

$$e^{-|x|} = e^{-x}$$

$$F_s(e^{-x}) = \int_0^\infty e^{-x}\sin sxdx = \frac{s}{1+s^2}$$

$$f(x) = \frac{2}{\Pi}\int_0^\infty F_s(e^{-x})\sin sxds$$

$$f(x) = \int_0^\infty \frac{s}{1+s^2}\sin sxds$$

$$e^{-x} = \int_0^\infty \frac{s}{1+s^2}\sin sxds$$

Replace x by m

$$e^{-m} = \frac{2}{\Pi}\int_0^\infty \frac{s}{1+s^2}\sin smds$$

$$\int_0^\infty \frac{s}{1+s^2}\sin smds = \frac{\Pi}{2}e^{-m}$$

$$\int_0^\infty \frac{x}{1+x^2}\sin mxds = \frac{\Pi}{2}e^{-m}$$

6 **Find the Fourier cosine transform** $f(x)=\begin{cases} x, 0<x<1 \\ 2-x, 1<x<2 \\ 0, x>2 \end{cases}$.

Sol:
$$F_c(f(x))=\int_0^\infty f(x)\cos sx\,dx$$
$$F_c(f(x))=\int_0^1 x\cos sx\,dx+\int_1^2 (2-x)\cos sx\,dx+\int_2^\infty 0.\cos sx\,dx$$
$$F_c(f(x))=\left(\frac{\sin s}{s}+\frac{\cos s}{s^2}-\frac{1}{s^2}\right)+\left(-\frac{\sin s}{s}-\frac{\cos 2s}{s^2}+\frac{\cos s}{s^2}\right)$$
$$F_c(f(x))=\frac{2\cos s}{s^2}-\frac{1}{s^2}-\frac{\cos 2s}{s^2}$$

7 **If the Fourier sine transform of** $f(x)=\frac{1-\cos n\Pi}{(n\Pi)^2}$ **then find f(x).**

Sol:
$$f(x)=\frac{2}{\Pi}\sum_{n=1}^{\infty}F_s(n)\sin nx$$
$$F_s(n)=\frac{1-\cos n\Pi}{(n\Pi)^2}$$
$$f(x)=\frac{2}{\Pi}\sum_{n=1}^{\infty}\frac{1-\cos n\Pi}{(n\Pi)^2}\sin nx$$
$$f(x)=\frac{2}{\Pi^3}\sum_{n=1}^{\infty}\frac{1-\cos n\Pi}{n^2}\sin nx$$

UNIT-III
LAPLACE TRANSFORM

- Definition of Laplace transform
- Properties of Laplace transform
- Laplace transforms of derivatives and integrals
- Inverse Laplace transform
- Properties of Inverse Laplace transform
- Convolution theorem and applications

Introduction

In mathematics the Laplace transform is an integral transform named after its discoverer Pierre-Simon Laplace . It takes a function of a positive real variable t (often time) to a function of a complex variable s (frequency).The Laplace transform is very similar to the Fourier transform. While the Fourier transform of a function is a complex function of a real variable (frequency), the Laplace transform of a function is a complex function of a complex variable. Laplace transforms are usually restricted to functions of t with t > 0. A consequence of this restriction is that the Laplace transform of a function is a holomorphic function of the variable s. Unlike the Fourier transform, the Laplace transform of a distribution is generally a well-behaved function. Also techniques of complex variables can be used directly to study Laplace transforms. As a holomorphic function, the Laplace transform has a power series representation. This power series expresses a function as a linear superposition of moments of the function. This perspective has applications in probability theory.

Introduction

Let f(t) be a given function which is defined for all positive values of t, if

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

exists, then F(s) is called Laplace transform of f(t) and is denoted by

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

The inverse transform, or inverse of L{f(t)} or F(s), is

$$f(t) = L^{-1}\{F(s)\}$$

where s is real or complex value.

Laplace Transform of Basic Functions

1. $L\ [1] = \int_0^\infty e^{-st} dt = -\frac{1}{s} e^{-st}\Big|_0^\infty = \frac{1}{s}$

2. $L\ [t^a] = \int_0^\infty t^a e^{-st} dt = \int_0^\infty (\frac{u}{s})^a e^{-u} \frac{du}{s} = \frac{1}{s^{a+1}} \int_0^\infty u^a e^{-u} du = \frac{\Gamma(a+1)}{s^{a+1}}$

3. $L\ [e^{at}] = \int_0^\infty e^{at} e^{-st} dt = \frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^\infty = \frac{1}{s-a}$

4. $L\ [e^{iat}] = \frac{1}{s-ia} \Rightarrow L\ [\cos at + i \sin at] = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$

$\therefore L\ [\cos at] = \frac{s}{s^2+a^2}$, and $L\ [\sin at] = \frac{a}{s^2+a^2}$

5. $L\ [\sinh at] = L\ [\frac{e^{at} - e^{-at}}{2}] = \frac{1}{2}(\frac{1}{s-a} - \frac{1}{s+a}) = \frac{a}{s^2-a^2}$

$L\ [\cosh at] = L\ [\frac{e^{at} + e^{-at}}{2}] = \frac{1}{2}(\frac{1}{s-a} + \frac{1}{s+a}) = \frac{s}{s^2-a^2}$

1. Linearity

L $[af(t)+bg(t)]=\int_0^\infty [af(t)+bg(t)]e^{-st}dt = a\int_0^\infty f(t)e^{-st}dt + b\int_0^\infty g(t)e^{-st}dt = aF(s)+bG(s)$

EX: Find the Laplace transform of $\cos^2 t$.

$$\text{Solution}: \mathrm{L}[\cos^2 t] = \mathrm{L}[\frac{1+\cos 2t}{2}] = \frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+2^2}\right) = \frac{s^2+2}{s(s^2+4)}$$

2. Shifting

$$(a)\ \mathrm{L}\ [f(t-a)u(t-a)] = \int_0^\infty f(t-a)u(t-a)e^{-st}dt = \int_a^\infty f(t-a)e^{-st}dt$$

Let $\tau = t-a$, then

$$\mathrm{L}\ [f(t-a)u(t-a)] = \int_0^\infty f(\tau)e^{-s(\tau+a)}d\tau = e^{-sa}\int_0^\infty f(\tau)e^{-s\tau}d\tau = e^{-sa}F(s)$$

$$(b)\ F(s-a) = \int_0^\infty f(t)e^{-(s-a)t}dt = \int_0^\infty [e^{at}f(t)]e^{-st}dt = \mathrm{L}\ [e^{at}f(t)]$$

EX: What is the Laplace transform of the function $f(t) = \begin{cases} 0, & t<4 \\ 2t^3, & t\ge 4 \end{cases}$

Solution: $f(t)=2t^3u(t-4)$

L $[f(t)]=$ L $\{2[(t-4)^3+12(t-4)^2+48(t-4)+64]u(t-4)\}$

$$= 2e^{-4s}\left(\frac{3!}{s^4}+12\times\frac{2!}{s^3}+48\times\frac{1}{s^2}+\frac{64}{s}\right) = 4e^{-4s}\left(\frac{3}{s^4}+\frac{12}{s^3}+\frac{24}{s^2}+\frac{32}{s}\right)$$

3. Scaling

$$\mathrm{L}\ [f(at)] = \int_0^\infty f(at)e^{-st}dt$$

Let $\tau = at$, then

$$\mathrm{L}\ [f(at)] = \int_0^\infty f(\tau)e^{-s\frac{\tau}{a}}d\frac{\tau}{a} = \frac{1}{a}\int_0^\infty f(\tau)e^{-\frac{s}{a}\tau}d\tau = \frac{1}{a}F(\frac{s}{a})$$

EX: Find the Laplace transform of $\cos 2t$.

$$\text{Solution}: \because \mathrm{L}\ [\cos t] = \frac{s}{s^2+1}$$

$$\therefore \mathrm{L}\ [\cos 2t] = \frac{1}{2}\frac{\frac{s}{2}}{(\frac{s}{2})^2+1} = \frac{s}{s^2+4}$$

4. Derivative

(a) Derivative of original function

L $[f'(t)]=\int_0^\infty f'(t)e^{-st}dt = f(t)e^{-st}\Big|_0^\infty - (-s)\int_0^\infty f(t)e^{-st}dt$

(1) If $f(t)$ is continuous, equation (2.1) reduces to

L $[f'(t)]=-f(0)+sF(s)=sF(s)-f(0)$

(2) If $f(t)$ is not continuous at $t=a$, equation reduces to

L $[f'(t)]= f(t)e^{-st}\Big|_0^{a^-} + f(t)e^{-st}\Big|_{a^+}^\infty + sF(s) = [f(a^-)e^{-sa}-f(0)]+[0-f(a^+)e^{-sa}]+sF(s)$

$=sF(s)-f(0)-e^{-sa}[f(a^+)-f(a^-)]$

(3) Similarly, if $f(t)$ is not continuous at $t=a_1, a_2, \cdots,\cdots,a_n$, equation reduces to

L $[f'(t)]= sF(s) - f(0) - \sum_{i=1}^{n} e^{-sa_i}[f(a_i^+) - f(a_i^-)]$

If $f(t), f'(t), f''(t), \cdots, f^{(n-1)}(t)$ are continuous, and $f^{(n)}(t)$ is piecewise continuous, and all of them are exponential order functions, then

$$\mathcal{L}\,[f^{(n)}(t)] = s^n F(s) - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0)$$

(*b*) Derivative of transformed function

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt = \int_0^\infty \frac{\partial}{\partial s}[f(t)e^{-st}]dt = \int_0^\infty (-t)f(t)e^{-st}dt = \mathcal{L}\,[(-t)f(t)]$$

$$[\text{Deduction}]\quad \frac{d^n F(s)}{ds^n} = \mathcal{L}\,[(-t)^n f(t)]$$

EX: Find the Laplace transform of te^t.

$$\text{Solution}:\ \mathcal{L}\,(e^t) = \frac{1}{s-1} \Rightarrow \mathcal{L}\,(te^t) = -\frac{d}{ds}\left(\frac{1}{s-1}\right) = \frac{1}{(s-1)^2}$$

EX: $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$, find $\mathcal{L}\,[f'(t)]$.

$$\text{Solution}:\ f(t) = t^2[u(t) - u(t-1)]$$

$$\mathcal{L}\,[f(t)] = \mathcal{L}\,[t^2u(t)] - \mathcal{L}\,[t^2u(t-1)] = \frac{2!}{s^3} - \mathcal{L}\,\{[(t-1)+1]^2 u(t-1)\}$$

$$= \frac{2}{s^3} - \mathcal{L}\,\{[(t-1)^2 + 2(t-1) + 1]u(t-1)\}$$

$$= \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + 2\frac{1}{s^2} + \frac{1}{s}\right)$$

$$\mathcal{L}\,[f'(t)] = sF(s) - f(0) - e^{-s}[f(1^+) - f(1^-)]$$

$$= [\frac{2}{s^2} - e^{-s}(\frac{2}{s^2} + \frac{2}{s} + 1)] - 0 - e^{-s}(0-1) = \frac{2}{s^2} - e^{-s}(\frac{2}{s^2} + \frac{2}{s})$$

5. Integration

(*a*) Integral of original function

$$\mathcal{L}\,[\int_0^t f(\tau)d\tau] = \int_0^\infty \int_0^t f(\tau)d\tau e^{-st}dt$$

$$= \frac{1}{-s}\left[e^{-st}\int_0^t f(\tau)d\tau\Big|_0^\infty - \int_0^\infty f(t)e^{-st}dt\right] = \frac{1}{s}F(s)$$

$$\Rightarrow \mathcal{L}\,[\int_0^t\int_0^t \cdots \int_0^t f(t)dtdt\cdots dt] = \frac{1}{s^n}F(s)$$

(*b*) Integration of Laplace transform

$$\int_s^\infty F(s)ds = \int_s^\infty \int_0^\infty f(t)e^{-st}dtds = \int_0^\infty f(t)\int_s^\infty e^{-st}dsdt$$

$$= \int_0^\infty f(t)\frac{e^{-st}}{-t}\Big|_s^\infty dt = \int_0^\infty \frac{f(t)}{t}e^{-st}dt = \mathcal{L}\,[\frac{f(t)}{t}]$$

$$\Rightarrow \int_s^\infty \int_s^\infty \cdots \int_s^\infty F(s)dsds\cdots ds = \mathcal{L}\,[\frac{1}{t^n}f(t)]$$

EX: Find $(a)\ \mathcal{L}\left[\frac{1-e^{-t}}{t}\right]\quad (b)\ \mathcal{L}\left[\frac{1-e^{-t}}{t^2}\right].$

$$\text{Solution}: (a)\mathcal{L}[1-e^{-t}] = \frac{1}{s}-\frac{1}{s+1}$$

$$\mathcal{L}\left[\frac{1-e^{-t}}{t}\right] = \int_s^\infty \left(\frac{1}{s}-\frac{1}{s+1}\right)ds = \ln s - \ln(s+1)\Big|_s^\infty = \ln\frac{s}{s+1}\Big|_s^\infty$$

$$= 0 - \ln\frac{s}{s+1} = \ln\frac{s+1}{s}$$

$$(b)\mathcal{L}\left[\frac{1-e^{-t}}{t^2}\right] = \int_s^\infty \ln\frac{s+1}{s}\,ds = s\ln\frac{s+1}{s}\Big|_s^\infty - \int_s^\infty s\left(\frac{1}{s+1}-\frac{1}{s}\right)ds$$

$$= s\ln\frac{s+1}{s}\Big|_s^\infty + \int_s^\infty \frac{1}{s+1}\,ds = \left[s\ln\frac{s+1}{s} + \ln(s+1)\right]_s^\infty$$

$$= \left[(s+1)\ln(s+1) - s\ln s\right]_s^\infty = s\ln s - (s+1)\ln(s+1)$$

EX: Find $(a)\ \int_0^\infty \frac{\sin kt\, e^{-st}}{t}\,dt \qquad (b)\ \int_{-\infty}^\infty \frac{\sin x}{x}\,dx\ .$

$$\text{Solution}: (a)\int_0^\infty \frac{\sin kte^{-st}}{t}\,dt = \mathcal{L}\left[\frac{\sin kt}{t}\right]$$

$$\because \mathcal{L}[\sin kt] = \frac{k}{s^2+k^2}$$

$$\mathcal{L}\left[\frac{\sin kt}{t}\right] = \int_s^\infty \frac{k}{s^2+k^2}\,ds = \frac{1}{k}\int_s^\infty \frac{1}{(\frac{s}{k})^2+1}\,ds$$

$$= \tan^{-1}\frac{s}{k}\Big|_s^\infty = \frac{\pi}{2} - \tan^{-1}\frac{s}{k}$$

$$(b)\int_{-\infty}^\infty \frac{\sin x}{x}\,dx = 2\int_0^\infty \frac{\sin x}{x}\,dx$$

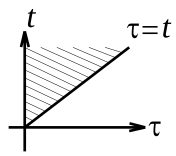
$$= 2\lim_{\substack{k\to 1\\ s\to 0}} \int_0^\infty \frac{\sin kte^{-st}}{t}\,dt$$

$$= 2\lim_{\substack{k\to 1\\ s\to 0}} \left(\frac{\pi}{2} - \tan^{-1}\frac{s}{k}\right) = \pi$$

6. Convolution theorem

$$\mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right] = \int_0^\infty \int_0^t f(\tau)g(t-\tau)d\tau e^{-st}dt$$

$$= \int_0^\infty \int_\tau^\infty f(\tau)g(t-\tau)e^{-st}dtd\tau = \int_0^\infty f(\tau)\int_\tau^\infty g(t-\tau)e^{-st}dtd\tau$$



Let $u = t-\tau$, $du = dt$, then

$$\mathcal{L}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right] = \int_0^\infty f(\tau)\int_0^\infty g(u)e^{-s(u+\tau)}dud\tau$$

$$= \int_0^\infty f(\tau)e^{-s\tau}d\tau\int_0^\infty g(u)e^{-su}du = F(s)G(s)$$

EX: Find the Laplace transform of $\int_0^t e^{t-\tau} \sin 2\tau \, d\tau$.

$$\text{Solution} : \because L\,[e^t] = \frac{1}{s-1},\; L\,[\sin 2t] = \frac{2}{s^2+4}$$

$$\therefore L\,[\int_0^t e^{t-\tau} \sin 2t \, d\tau] = L\,[e^t * \sin 2t] = L\,[e^t] \cdot L\,[\sin 2t]$$

$$= \frac{1}{s-1} \cdot \frac{2}{s^2+4} = \frac{2}{(s-1)(s^2+4)}$$

7. Periodic Function: *f* (*t* + *T*) =*f* (*t*)

$$L\,[f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^T f(t)e^{-st}dt + \int_T^{2T} f(t)e^{-st}dt + \cdots\cdots$$

$$\text{and} \int_T^{2T} f(t)e^{-st}dt = \int_0^T f(u+T)e^{-s(u+T)}du = e^{-sT}\int_0^T f(u)e^{-su}du$$

Similarly,

$$\int_{2T}^{3T} f(t)e^{-st}dt = e^{-2sT}\int_0^T f(u)e^{-su}du$$

$$\therefore L\,[f(t)] = (1 + e^{-sT} + e^{-2sT} + \cdots\cdots)\int_0^T f(t)e^{-st}dt$$

$$= \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt$$

EX: Find the Laplace transform of $f(t) = \frac{k}{p}t,\ 0 < t < p,\ f(t+p) = f(t)$.

$$\text{Solution} : L\,[f(t)] = \frac{1}{1-e^{-ps}}\int_0^p \frac{k}{p} te^{-st}dt$$

$$= \frac{1}{1-e^{-ps}} \frac{k}{p}[\frac{1}{-s}(te^{-st}\Big|_0^p - \int_0^p e^{-st}dt)]$$

$$= \frac{-k}{ps(1-e^{-ps})}(te^{-st} + \frac{1}{s}e^{-st})\Big|_0^p$$

$$= \frac{-k}{ps(1-e^{-ps})}(pe^{-sp} + \frac{e^{-sp}}{s} - \frac{1}{s})$$

8. Initial Value Theorem:

$$\because L\,[f'(t)] = sF(s) - f(0) \Rightarrow \lim_{s\to\infty}\int_0^\infty f'(t)e^{-st}dt = \lim_{s\to\infty} sF(s) - f(0) \Rightarrow 0 = \lim_{s\to\infty} sF(s) - f(0)$$

$$\text{we get initial value theorem} \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

$$\text{Deduce general initial value theorem} : \lim_{t\to 0}\frac{f(t)}{g(t)} = \lim_{s\to\infty}\frac{F(s)}{G(s)}$$

9. Final Value Theorem:

$$\mathcal{L}\ [f'(t)] = sF(s) - f(0) \Rightarrow \lim_{s\to 0}\int_0^\infty f'(t)e^{-st}dt = \lim_{s\to 0} sF(s) - f(0) \Rightarrow$$

$$\lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0} sF(s) - f(0) \Rightarrow \text{final value theorem}: \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$\text{General final value theorem}: \lim_{t\to\infty}\frac{f(t)}{g(t)} = \lim_{s\to 0}\frac{F(s)}{G(s)}$$

EX: Find $\mathcal{L}\ [\int_0^t \frac{\sin x}{x}dx]$.

$$\text{Solution : Let } f(t) = \int_0^t \frac{\sin x}{x}dx \Rightarrow f'(t) = \frac{\sin t}{t},\ f(0) = 0$$

$$\mathcal{L}\ [tf'(t)] = \mathcal{L}\ [\sin t] = \frac{1}{s^2+1}$$

$$-\frac{d}{ds}\mathcal{L}\ [f'(t)] = \frac{1}{s^2+1}$$

$$-\frac{d}{ds}[sF(s) - f(0)] = \frac{1}{s^2+1} \Rightarrow \frac{d}{ds}[sF(s)] = -\frac{1}{s^2+1}$$

$sF(s) = -\tan^{-1}s + C$

From the initial value theorem, we get

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

$$0 = -\frac{\pi}{2} + C \quad \therefore C = \frac{\pi}{2}$$

$$sF(s) = \frac{\pi}{2} - \tan^{-1}s = \tan^{-1}\frac{1}{s}$$

$$F(s) = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

EX: Find $\mathcal{L}\ \left[\int_t^\infty \frac{e^{-x}}{x}dx\right]$.

$$\text{Solution : Let } f(t) = \int_x^\infty \frac{e^{-x}}{x}dx \Rightarrow f'(t) = -\frac{e^{-t}}{t}, \lim_{t\to\infty} f(t) = 0$$

$$\mathcal{L}\ [tf'(t)] = \mathcal{L}\ [-e^{-t}] = -\frac{1}{s+1}$$

$$-\frac{d}{ds}[sF(s) - f(0)] = -\frac{1}{s+1}$$

$$\frac{d}{ds}[sF(s)] = \frac{1}{s+1}$$

$$sF(s) = \ln(s+1) + C$$

$$\text{From the final value theorem}: \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$0 = 0 + C \Rightarrow C = 0, \text{ and } F(s) = \frac{\ln(s+1)}{s}$$

Note: $\int_0^t \frac{\sin x}{x}dx$, and $\int_t^\infty \frac{e^{-x}}{x}dx$ are called sine, and exponential integral function, respectively.

I. Inversion from Basic Properties

1. Linearity

Ex. 1.

$(a) \mathcal{L}^{-1}[\frac{2s+1}{s^2+4}]$ $(b) \mathcal{L}^{-1}[\frac{4(s+1)}{s^2-16}]$.

Solution : $(a) \mathcal{L}^{-1}[\frac{2s+1}{s^2+4}] = \mathcal{L}^{-1}[2\frac{s}{s^2+2^2} + \frac{1}{2}\frac{2}{s^2+2^2}] = 2\cos 2t + \frac{1}{2}\sin 2t$

$(b) \mathcal{L}^{-1}[\frac{4(s+1)}{s^2-16}] = \mathcal{L}^{-1}[4\frac{s}{s^2-4^2} + \frac{4}{s^2-4^2}] = 4\cosh 4t + \sinh 4t$

2. Shifting

Ex. 2.

$(a) \mathcal{L}^{-1}[\frac{e^{-\pi s}}{s^2+2s+2}]$ $(b) \mathcal{L}^{-1}[\frac{2s+3}{s^2+3s+2}]$.

Solution : $(a) \mathcal{L}^{-1}[\frac{e^{-\pi s}}{s^2+2s+2}] = \mathcal{L}^{-1}[\frac{e^{-\pi s}}{(s+1)^2+1}]$

$\because \mathcal{L}^{-1}[\frac{1}{(s+1)^2+1}] = e^{-t}\sin t$

and $\mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s)$

$\therefore \mathcal{L}^{-1}[\frac{e^{-\pi s}}{(s+1)^2+1}] = e^{-(t-\pi)}\sin(t-\pi)u(t-\pi) = -e^{-(t-\pi)}\sin t u(t-\pi)$

$(b) \mathcal{L}^{-1}[\frac{2s+3}{s^2+3s+2}] = \mathcal{L}^{-1}[\frac{2(s+\frac{3}{2})}{(s+\frac{3}{2})^2-(\frac{1}{2})^2}] = 2e^{-\frac{3}{2}t}\cosh\frac{t}{2}$

3. Scaling

Ex. 3.

$\mathcal{L}^{-1}[\frac{4s}{16s^2-4}]$.

Solution: $\mathcal{L}^{-1}[\frac{4s}{16s^2-4}] = \mathcal{L}^{-1}[\frac{4s}{(4s)^2-2^2}] = \frac{1}{4}\cosh 2\cdot\frac{1}{4}t = \frac{1}{4}\cosh\frac{t}{2}$

4. Derivative

Ex. 4.

$(a) \mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]$ $(b) \mathcal{L}^{-1}[\ln\frac{s+a}{s+b}]$.

$$\text{solution}:(a)\mathcal{L}[\sin\omega t]=\frac{\omega}{s^2+\omega^2}\Rightarrow \mathcal{L}[t\sin\omega t]=-\frac{d}{ds}(\frac{\omega}{s^2+\omega^2})=\frac{2\omega s}{(s^2+\omega^2)^2}$$

$$\text{Let } F(t)=t\sin\omega t\Rightarrow \mathcal{L}[F'(t)]=s\cdot\frac{2\omega s}{(s^2+\omega^2)^2}-F(0)$$

$$\mathcal{L}[F'(t)]=2\omega\frac{s^2}{(s^2+\omega^2)^2}=2\omega[\frac{(s^2+\omega^2)-\omega^2}{(s^2+\omega^2)^2}]=2\omega[\frac{1}{s^2+\omega^2}-\frac{\omega^2}{(s^2+\omega^2)^2}]$$

$$=2\mathcal{L}[\sin\omega t]-\frac{2\omega^3}{(s^2+\omega^2)^2}$$

$$\frac{1}{(s^2+\omega^2)^2}=\frac{1}{2\omega^3}\cdot\mathcal{L}[2\sin\omega t-F'(t)]$$

$$\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]=\frac{1}{2\omega^3}\cdot[2\sin\omega t-F'(t)]=\frac{1}{2\omega^3}(\sin\omega t-\omega t\cos\omega t)$$

$$(b)\text{ Let } \mathcal{L}[f(t)]=\ln\frac{s+a}{s+b}=\ln(s+a)-\ln(s+b)$$

$$\mathcal{L}[tf(t)]=-\frac{d}{ds}[\ln(s+a)-\ln(s+b)]=\frac{1}{s+b}-\frac{1}{s+a}=\mathcal{L}[e^{-bt}-e^{-at}]$$

$$\therefore f(t)=\frac{e^{-bt}-e^{-at}}{t}$$

5. Integration

Ex. 5.

$$(a)\mathcal{L}^{-1}[\frac{1}{s^2}(\frac{s-1}{s+1})] \qquad (b)\mathcal{L}^{-1}[\ln\frac{s+a}{s+b}].$$

$$\text{Solution}:(a)\mathcal{L}^{-1}[\frac{1}{s^2}(\frac{s-1}{s+1})]=\mathcal{L}^{-1}[\frac{1}{s(s+1)}-\frac{1}{s^2(s+1)}]=\int_0^t e^{-t}dt-\int_0^t\int_0^t e^{-t}dtdt$$

$$=-(e^{-t}-1)+\int_0^t(e^{-t}-1)dt=-(e^{-t}-1)-(e^{-t}-1)-t=2-2e^{-t}-t$$

$$(b)\mathcal{L}[e^{-bt}-e^{-at}]=\frac{1}{s+b}-\frac{1}{s+a}$$

$$\mathcal{L}[\frac{e^{-bt}-e^{-at}}{t}]=\int_s^\infty(\frac{1}{s+b}-\frac{1}{s+a})ds=\ln\frac{s+b}{s+a}\bigg|_s^\infty=\ln\frac{s+a}{s+b}$$

$$\therefore \mathcal{L}^{-1}[\ln\frac{s+a}{s+b}]=\frac{e^{-bt}-e^{-at}}{t}$$

6. Convolution

Ex. 6.

$$(a)\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}] \qquad (b)\mathcal{L}^{-1}[\frac{s}{(s^2+\omega^2)^2}].$$

$$\text{Solution : }(a)\mathcal{L}[\sin\omega t]=\frac{\omega}{s^2+\omega^2}\Rightarrow \mathcal{L}[\frac{1}{\omega}\sin\omega t]=\frac{1}{s^2+\omega^2}$$

$$\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]=\frac{1}{\omega^2}\int_0^t \sin\omega\tau\sin\omega(t-\tau)d\tau$$

$$=\frac{1}{\omega^2}\int_0^t\frac{1}{2}[\cos(\omega\tau-\omega t+\omega\tau)-\cos(\omega\tau+\omega t-\omega\tau)]d\tau$$

$$=\frac{1}{2\omega^2}\int_0^t[\cos(2\omega\tau-\omega t)-\cos\omega t]d\tau=\frac{1}{2\omega^2}\left[\frac{1}{2\omega}\sin(2\omega\tau-\omega t)-\tau\cos\omega t\right]_0^t$$

$$=\frac{1}{2\omega^2}\{[\frac{1}{2\omega}(\sin\omega t-\sin(-\omega t)]-t\cos\omega t\}=\frac{1}{2\omega^3}(\sin\omega t-\omega t\cos\omega t)$$

$$(b)\mathcal{L}[\frac{1}{\omega}\sin\omega t]=\frac{1}{s^2+\omega^2}\qquad \mathcal{L}[\cos\omega t]=\frac{s}{s^2+\omega^2}$$

$$\mathcal{L}^{-1}[\frac{s}{(s^2+\omega^2)^2}]=\frac{1}{\omega}\int_0^t\sin\omega\tau\cos\omega(t-\tau)d\tau$$

$$=\frac{1}{\omega}\int_0^t\frac{1}{2}[\sin(\omega\tau+\omega t-\omega\tau)+\sin(\omega\tau-\omega t+\omega\tau)]d\tau$$

$$=\frac{1}{2\omega}\int_0^t[\sin\omega t+\sin(2\omega\tau-\omega t)]d\tau=\frac{1}{2\omega}\left[\tau\sin\omega t+\frac{-1}{2\omega}\cos(2\omega\tau-\omega t)\right]_0^t$$

$$=\frac{1}{2\omega}\{t\sin\omega t-\frac{1}{2\omega}[\cos\omega t-\cos(-\omega t)]\}=\frac{t}{2\omega}\sin\omega t$$

II. Partial Fraction

If $F(s)=\dfrac{P(s)}{Q(s)}$, where deg[$P(s)$]<deg[$Q(s)$]

1. $Q(s)=0$ with unrepeated factors $s-a_i$

$$\frac{P(s)}{Q(s)}=\frac{A_1}{s-a_1}+\frac{A_2}{s-a_2}+\cdots+\frac{A_n}{s-a_n}$$

$$A_k=\lim_{S\to a_k}[\frac{P(s)}{Q(s)}(s-a_k)]=P(a_k)\lim_{S\to a_k}\frac{s-a_k}{Q(s)}$$

$$=P(a_k)\lim_{s\to a_k}\frac{1}{Q'(s)}=\frac{P(a_k)}{Q'(a_k)}$$

$$\frac{P(s)}{Q(s)}=\frac{P(a_1)/Q'(a_1)}{s-a_1}+\frac{P(a_2)/Q'(a_2)}{s-a_2}+\cdots\cdots+\frac{P(a_n)/Q'(a_n)}{s-a_n}$$

$$\mathcal{L}^{-1}[\frac{P(s)}{Q(s)}]=\frac{P(a_1)}{Q'(a_1)}e^{a_1t}+\frac{P(a_2)}{Q'(a_2)}e^{a_2t}+\cdots\cdots+\frac{P(a_n)}{Q'(a_n)}e^{a_nt}$$

Ex. 7.

$$\mathcal{L}^{-1}[\frac{s+1}{s^3+s^2-6s}].$$

Solution : $\frac{s+1}{s^3+s^2-6s} = \frac{s+1}{s(s-2)(s+3)} = \frac{A_1}{s} + \frac{A_2}{s-2} + \frac{A_3}{s+3}$

$$A_1 = \lim_{s\to 0} \frac{s+1}{(s-2)(s+3)} = -\frac{1}{6}$$

$$A_2 = \lim_{s\to 2} \frac{s+1}{s(s+3)} = \frac{3}{10}$$

$$A_3 = \lim_{s\to -3} \frac{s+1}{s(s-2)} = \frac{-2}{15}$$

$$L^{-1}[\frac{s+1}{s^3+s^2-6s}] = \frac{-\frac{1}{6}}{s} + \frac{\frac{3}{10}}{s-2} + \frac{\frac{-2}{15}}{s+3} = -\frac{1}{6} + \frac{3}{10}e^{2t} - \frac{2}{15}e^{-3t}$$

2. $Q(s)=0$ with repeated factors $(s-a_k)^m$

$$\frac{P(s)}{Q(s)} = \frac{C_m}{(s-a_k)^m} + \frac{C_{m-1}}{(s-a_k)^{m-1}} + \cdots\cdots + \frac{C_1}{s-a_k}$$

$$\frac{P(s)}{Q(s)}(s-a_k)^m = C_m + C_{m-1}(s-a_k) + C_{m-2}(s-a_k)^2 + \cdots + C_1(s-a_k)^{m-1}$$

$$C_m = \lim_{s\to a_k}[\frac{P(s)}{Q(s)}(s-a_k)^m]$$

$$C_{m-1} = \lim_{s\to a_k}\{\frac{d}{ds}[\frac{P(s)}{Q(s)}(s-a_k)^m]\}$$

$$C_{m-2} = \lim_{s\to a_k}\{\frac{d^2}{ds^2}[\frac{P(s)}{Q(s)}(s-a_k)^m]\}\frac{1}{2!}$$

....

$$C_1 = \lim_{s\to a_k}\{\frac{d^{m-1}}{ds^{m-1}}[\frac{P(s)}{Q(s)}(s-a_k)^m]\}\frac{1}{(m-1)!}$$

$$L^{-1}[\frac{P(s)}{Q(s)}] = e^{a_k t}[C_m\frac{t^{m-1}}{(m-1)!} + C_{m-1}\frac{t_{m-2}}{(m-2)!} + \cdots + C_2 t + C_1]$$

Ex. 8.

$$L^{-1}[\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)(s-3)}].$$

Solution : $\dfrac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)(s-3)}=\dfrac{C_2}{s^2}+\dfrac{C_1}{s}+\dfrac{A_1}{s-1}+\dfrac{A_2}{s-2}+\dfrac{A_3}{s-3}$

$$C_2=\lim_{s\to 0}\frac{s^4-7s^3+13s^2+4s-12}{(s-1)(s-2)(s-3)}=\frac{-12}{-6}=2$$

$$C_1=\lim_{s\to 0}\frac{d}{ds}[\frac{s^4-7s^3+13s^2+4s-12}{(s-1)(s-2)(s-3)}]$$

$$=\frac{4(-1)(-2)(-3)-(-12)[(-2)(-3)+(-1)(-3)+(-1)(-2)]}{[(-1)(-2)(-3)]^2}=\frac{-24+12\times 11}{6^2}=3$$

$$A_1=\lim_{s\to 1}\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-2)(s-3)}=\frac{-1}{2}$$

$$A_2=\lim_{s\to 2}\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-3)}=\frac{8}{-4}=-2$$

$$A_3=\lim_{s\to 3}\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)}=\frac{9}{18}=\frac{1}{2}$$

$$\mathcal{L}^{-1}[\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-1)(s-2)(s-3)}]=2t+3-\frac{1}{2}e^{t}-2e^{2t}+\frac{1}{2}e^{3t}$$

3. $Q(s)=0$ with unrepeated factor $(s-\alpha)^2+\beta$, where $\beta>0$

$$\frac{P(s)}{Q(s)}=\frac{As+B}{(s-\alpha)^2+\beta^2}$$

$$\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]=As+B$$

$$\lim_{s\to\alpha+i\beta}\{\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]\}=A(\alpha+i\beta)+B$$

$$R+iI=(A\alpha+\beta)+iA\beta$$

where R and I are the real and imaginary parts of $\lim_{s\to\alpha+i\beta}\{\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]\}$, respectively

then, $\begin{cases} A\alpha+B=R \\ A\beta=I \end{cases}$, where we can get A and B, and

$$\mathcal{L}^{-1}[\frac{P(s)}{Q(s)}]=\mathcal{L}^{-1}[\frac{A(s-\alpha)+(A\alpha+B)}{(s-\alpha)^2+\beta^2}]=e^{\alpha t}\left(A\cos\beta t+\frac{A\alpha+B}{\beta}\sin\beta t\right)$$

Ex. 9.

$\mathcal{L}^{-1}[\dfrac{s^2}{s^4+4}]$.

Solution : $\frac{s^2}{s^4+4}=\frac{s^2}{(s^2)^2+2\cdot s^2\cdot 2+2^2-2\cdot s^2\cdot 2}=\frac{s^2}{(s^2+2)^2-(2s)^2}$

$$=\frac{s^2}{(s^2+2s+2)(s^2-2s+2)}=\frac{A_1s+B_1}{(s+1)^2+1}+\frac{A_2s+B_2}{(s-1)^2+1}$$

$$\lim_{s\to -1+i}\frac{s^2}{(s-1)^2+1}=A_1(-1+i)+B_1\Rightarrow\frac{-2i}{4-4i}=(-A_1+B_1)+iA_1$$

$$\frac{8-8i}{32}=(-A_1+B_1)+iA_1\Rightarrow A_1=-\frac{1}{4},B_1=0$$

$$\lim_{s\to 1+i}\frac{s^2}{(s+1)^2+1}=A_2(1+i)+B_2\Rightarrow\frac{2i}{4+4i}=(A_2+B_2)+iA_2$$

$$\frac{8+8i}{32}=(A_2+B_2)+iA_2\Rightarrow A_2=\frac{1}{4},B_2=0$$

$$\mathcal{L}^{-1}[\frac{s^2}{s^4+4}]=\mathcal{L}^{-1}[\frac{-\frac{1}{4}(s+1)+\frac{1}{4}}{(s+1)^2+1}+\frac{\frac{1}{4}(s-1)+\frac{1}{4}}{(s-1)^2+1}]$$

$$=\frac{e^{-t}}{4}(-\cos t+\sin t)+\frac{e^t}{4}(\cos t+\sin t)$$

4. Q(s)=0 with repeated complex factor $[(s-\alpha)^2+\beta]^2$, where $\beta>0$

$$\frac{P(s)}{Q(s)}=\frac{As+B}{[(s-\alpha)^2+\beta^2]^2}+\frac{Cs+D}{(s-\alpha)^2+\beta^2}$$

$$\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]^2=As+B+(Cs+D)[(s-\alpha)^2+\beta^2]$$

$$\lim_{s\to\alpha+i\beta}\{\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]^2\}=A(\alpha+i\beta)+B$$

$$R_1+iI_1=(A\alpha+B)+iA\beta\Rightarrow\{\begin{matrix}A\alpha+B=R_1\\ A\beta=I_1\end{matrix}$$, where A and B can be obtained

$$\lim_{s\to\alpha+i\beta}\frac{d}{ds}\{\frac{P(s)}{Q(s)}[(s-\alpha)^2+\beta^2]^2\}=A+[C(\alpha+i\beta)+D]\lim_{s\to\alpha+i\beta}\frac{d}{ds}[(s-\alpha)^2+\beta^2]$$

$$R_2+iI_2=A+[C(\alpha+i\beta)+D]2i\beta=(A-2C\beta^2)+i(2\alpha\beta C+2\beta D)$$

$$\Rightarrow\{\begin{matrix}A-2C\beta^2=R_2\\ 2\alpha\beta C+2\beta D=I_2\end{matrix}$$, where we get C and D, hence

$$\mathcal{L}^{-1}[\frac{P(s)}{Q(s)}]=\mathcal{L}^{-1}\{\frac{A(s-\alpha)+(A\alpha+B)}{[(s-\alpha)^2+\beta^2]^2}\}+\mathcal{L}^{-1}[\frac{C(s-\alpha)+(C+D)}{(s-\alpha)^2+\beta^2}]$$

$$=e^{\alpha t}\{[\frac{At}{2\beta}\sin\beta t+(A\alpha+B)\frac{1}{2\beta^3}(\sin\beta t-\beta t\cos\beta t)]+[C\cos\beta t+(C\alpha+D)\frac{1}{\beta}\sin\beta t]\}$$

Ex. 10.

$$\mathcal{L}^{-1}[\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}].$$

Solution : $\dfrac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}=\dfrac{As+B}{[(s-1)^2+1]^2}+\dfrac{cs+D}{(s-1)^2+1}$

$$\lim_{s\to 1+i}(s^3-3s^2+6s+4)=A(1+i)+B$$

$$2i=(A+B)+iA \Rightarrow A=2, B=-2$$

$$\lim_{s\to 1+i}\frac{d}{ds}(s^3-3s^2+6s+4)=A+[c(1+i)+D]\lim_{s\to 1+i}\frac{d}{ds}[(s-1)^2+1]$$

$$0=A+(c+ic+D)2i=(A-2c)+2i(c+D)$$

$$c=1, D=-1$$

$$\mathcal{L}^{-1}[\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}]=\mathcal{L}^{-1}\{\frac{2(s-1)}{[(s-1)^2+1]^2}\}+\mathcal{L}^{-1}[\frac{s-1}{(s-1)^2+1}]$$

$$=e^t(2\cdot\frac{t}{2}\sin t+\cos t)=e^t(t\sin t+\cos t)$$

IV. Differentiation with Respect to a Number

Ex. 11.

$$\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}].$$

Solution : $\dfrac{d}{d\omega}(\dfrac{1}{s^2+\omega^2})=\dfrac{-2\omega}{(s^2+\omega^2)^2}\Rightarrow \mathcal{L}^{-1}[\dfrac{d}{d\omega}(\dfrac{1}{s^2+\omega^2})]=\mathcal{L}^{-1}[\dfrac{-2\omega}{(s^2+\omega^2)^2}]$

$$-2\omega\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]=\frac{d}{d\omega}\mathcal{L}^{-1}[\frac{1}{s^2+\omega^2}]=\frac{d}{d\omega}(\frac{1}{\omega}\sin\omega t)=-\frac{1}{\omega^2}\sin\omega t+\frac{t}{\omega}\cos\omega t$$

$$\mathcal{L}^{-1}[\frac{1}{(s^2+\omega^2)^2}]=\frac{1}{2\omega^3}(\sin\omega t-\omega t\cos\omega t)$$

V. Method of Differential Equation

Ex. 12.

$$\mathcal{L}^{-1}[e^{-\sqrt{s}}].$$

Solution : $\bar{y}=e^{-\sqrt{s}} \Rightarrow \bar{y}'=-\frac{e^{-\sqrt{s}}}{2\sqrt{s}}$, $\bar{y}''=\frac{e^{-\sqrt{s}}}{4s}+\frac{e^{-\sqrt{s}}}{4\sqrt{s^3}}$

we get the equation $4s\bar{y}''+2\bar{y}'-\bar{y}=0 \Rightarrow 4L\ [\frac{d}{dt}(t^2 y)]+2L\ [-ty]-L\ [y]=0$

$4\frac{d}{dt}(t^2 y)-2ty-y=0 \Rightarrow 4t^2 y'+(6t-1)y=0 \Rightarrow \frac{dy}{y}+\frac{6t-1}{4t^2}dt=0$

$\ln y+\frac{3}{2}\ln t+\frac{1}{4t}=c_1 \Rightarrow y=ct^{-\frac{3}{2}}e^{-\frac{1}{4t}}$

$\because L\ [t^{-\frac{1}{2}}]=\frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}}=\frac{\sqrt{\pi}}{\sqrt{s}}$, and $L\ [ty]=L\ [ct^{-\frac{1}{2}}e^{-\frac{1}{4t}}]$

while $L\ [ty]=-\bar{y}'=\frac{e^{-\sqrt{s}}}{2\sqrt{s}} \Rightarrow L\ [ct^{-\frac{1}{2}}e^{-\frac{1}{4t}}]=\frac{e^{-\sqrt{s}}}{2\sqrt{s}}$

Apply general final value theorem $\lim_{t\to\infty}\frac{ct^{-\frac{1}{2}}e^{-\frac{1}{4t}}}{t^{-\frac{1}{2}}}=\lim_{s\to 0}\frac{\frac{e^{-\sqrt{s}}}{2\sqrt{s}}}{\frac{\sqrt{\pi}}{\sqrt{s}}} \Rightarrow c=\frac{1}{2\sqrt{\pi}}$

$\therefore y=\frac{1}{2\sqrt{\pi}t^{3/2}}e^{-\frac{1}{4t}}$

Applied to Solve Differential Equations

I. Ordinary Differential Equations with Constant Coefficients

Ex. 1.

$y''+y'+y=g(x),\ y(0)=1,\ y'(0)=0$, where $g(x)=\begin{cases}1 & 0<x<3\\ 3 & x>3\end{cases}$.

Solution : $g(x)=u(x)+2u(x-3)$

$$[s^2Y-sy(0)-y'(0)]+[sY-y(0)]+Y=\frac{1}{s}+2\frac{e^{-3s}}{s}$$

$$(s^2+s+1)Y=s+1+\frac{1}{s}+2\frac{e^{-3s}}{s}$$

$$Y=\frac{s+1}{s^2+s+1}+\frac{1}{s(s^2+s+1)}+\frac{2e^{-3s}}{s(s^2+s+1)}$$

$$=\frac{s+1}{s^2+s+1}+(\frac{1}{s}-\frac{s+1}{s^2+s+1})+2e^{-3s}(\frac{1}{s}-\frac{s+1}{s^2+s+1})$$

$$\frac{s+1}{s^2+s+1}=\frac{(s+\frac{1}{2})+\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} \Rightarrow L^{-1}[\frac{s+1}{s^2+s+1}]=e^{-\frac{x}{2}}(\cos\frac{\sqrt{3}}{2}x+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}x)$$

$$y(x)=u(x)+2u(x-3)\{1-e^{-\frac{x-3}{2}}[\cos\frac{\sqrt{3}}{2}(x-3)+\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}(x-3)]\}$$

Ex. 2.

$y'''(t) - 2y''(t) + 5y'(t) = 0,\ y(0) = 0,\ y'(0) = 1,\ y(\frac{\pi}{8}) = 1$.

Solution : $[s^3Y - s^2y(0) - sy'(0) - y''(0)] - 2[s^2Y - sy(0) - y'(0)] + 5[sY - y(0)] = 0$

$$y''(0) = c$$

$$Y = \frac{s + c - 2}{s(s^2 - 2s + 5)} = \frac{A}{s} + \frac{Ps + Q}{(s-1)^2 + 2^2}$$

$$A = \lim_{s \to 0} \frac{s + c - 2}{s^2 - 2s + 5} = \frac{c - 2}{5}$$

$$P(1 + 2i) + Q = \lim_{s \to 1+2i} \frac{s + c - 2}{s} = \frac{-1 + c + 2i}{1 + 2i} = \frac{c + 3}{5} + \frac{4 - 2c}{5} i$$

$$P = \frac{2 - c}{5}, \quad Q = \frac{2c + 1}{5}$$

$$y(t) = \frac{c - 2}{5} + e^t \left(\frac{2 - c}{5} \cos 2t + \frac{c + 3}{10} \sin 2t\right)$$

$$y(\frac{\pi}{8}) = 1 \Rightarrow 1 = \frac{c - 2}{5} + e^{\frac{\pi}{8}} \left(\frac{2 - c}{5} \frac{1}{\sqrt{2}} + \frac{c + 3}{10} \frac{1}{\sqrt{2}}\right) \Rightarrow c = 7$$

$$\therefore\ y(t) = 1 + e^t(-\cos 2t + \sin 2t)$$

II. Ordinary Differential Equations with Variable Coefficients

Ex. 3.

$ty'' + (1-2t)y' - 2y = 0,\ \ y(0) = 1,\ y'(0) = 2.$

Solution : $-\frac{d}{ds}[s^2Y - sy(0) - y'(0)] + \{[sY - y(0)] + 2\frac{d}{ds}[sY - y(0)]\} - 2Y = 0$

$$(-s^2Y' - 2sY + 1) + [(sY - 1) + 2(sY' + Y)] - 2Y = 0$$

$$(-s^2 + 2s)Y' + (-2s + s + 2 - 2)Y = 0$$

$$-(s - 2)Y' = Y \Rightarrow \frac{dY}{Y} = -\frac{ds}{s - 2} \Rightarrow \ln Y = -\ln(s - 2) + c_1$$

$$Y = \frac{c}{s - 2} \Rightarrow y(t) = ce^{2t}$$

$$y(0) = 1, \therefore 1 = c,\ y(t) = e^{2t}$$

III. Simultaneous Ordinary Differential Equations

Ex. 4.

$$\begin{cases} \dfrac{dx}{dt} = 2x + y + 2e^{5t} \\ \dfrac{dy}{dt} = x + 2y + 3e^{2t} \end{cases}, \quad x(0) = y(0) = 0.$$

Solution : $\begin{cases} sX - x(0) = 2X + Y + \dfrac{2}{s-5} \\ sY - y(0) = X + 2Y + \dfrac{3}{s-2} \end{cases} \Rightarrow \begin{cases} (s-2)X - Y = \dfrac{2}{s-5} \\ -X + (s-2)Y = \dfrac{3}{s-2} \end{cases}$

$$X = \frac{(s-2)\dfrac{2}{s-5} + \dfrac{3}{s-2}}{(s-2)^2 - 1} = \frac{2s^2 - 5s - 7}{(s-1)(s-2)(s-3)(s-5)}$$

$$Y = \frac{\dfrac{2}{s-5} + (s-2)\dfrac{3}{s-2}}{(s-2)^2 - 1} = \frac{3s - 13}{(s-1)(s-3)(s-5)}$$

$$X = \frac{5/4}{s-1} + \frac{-3}{s-2} + \frac{1}{s-3} + \frac{3/4}{s-5} \Rightarrow x(t) = \frac{5}{4}e^{t} - 3e^{2t} + e^{3t} + \frac{3}{4}e^{5t}$$

$$Y = \frac{-5/4}{s-1} + \frac{1}{s-3} + \frac{1/4}{s-5} \Rightarrow y(t) = -\frac{5}{4}e^{t} + e^{3t} + \frac{1}{4}e^{5t}$$

UNIT-IV
Z TRANSFORM

- Definition of Z-transforms
- Elementary properties
- Inverse Z-transform
- Convolution theorem
- Formation and solution of difference equations.

Introduction

The z-transform is useful for the manipulation of discrete data sequences and has acquired a new significance in the formulation and analysis of discrete-time systems. It is used extensively today in the areas of applied mathematics, digital signal processing, control theory, population science and economics. These discrete models are solved with difference equations in a manner that is analogous to solving continuous models with differential equations. The role played by the z-transform in the solution of difference equations corresponds to that played by the Laplace transforms in the solution of differential equations.

Definition

If the function u_n is defined for discrete value and $u_n = 0$ for n<0 then the Z-transform is defined to be

$$Z(u_n) = U(z) = \sum_{n=1}^{\infty} u_n z^{-n}$$

The inverse Z-transform is written as

$$u_n = Z^{-1}[U(z)]$$

Properties of the z transform

For the following

$$Z\{f[n]\} = \sum_{n=0}^{n=\infty} f[n] z^{-n} = F(z) \; Z\{g_n\} = \sum_{n=0}^{n=\infty} g_n z^{-n} = G(z)$$

Linearity:

$Z\{af_n + bg_n\} = aF(z) + bG(z)$. and ROC is $R_f \cap R_g$
which follows from definition of z-transform.

Time Shifting

If we have $f[n] \Leftrightarrow F(z)$ then $f[n-n_0] \Leftrightarrow z^{-n_0} F(z)$

The ROC of Y(z) is the same as F(z) except that there are possible pole additions or deletions at z = 0 or z = ∞.

Proof: Let $y[n] = f[n-n_0]$ then

$$Y(z) = \sum_{n=-\infty}^{\infty} f[n-n_0] z^{-n}$$

Assume $k = n - n_0$ then $n = k + n_0$, substituting in the above equation we have:

$$Y(z) = \sum_{k=-\infty}^{\infty} f[k] z^{-k-n_0} = z^{-n_0} F[z]$$

Multiplication by an Exponential Sequence

Let $y[n] = z_0^n f[n]$ then $Y(z) = X\left(\frac{z}{z_0}\right)$

The consequence is pole and zero locations are scaled by z_0. If the ROC of FX(z) is $r_R < |z| < r_L$, then the ROC of Y(z) is $r_R < |z/z_0| < r_L$, i.e., $|z_0| r_R < |z| < |z_0| rL$

Proof: $Y(z) = \sum_{n=-\infty}^{\infty} z_0^n x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n]\left(\frac{z}{z_0}\right)^{-n} = X\left(\frac{z}{z_0}\right)$

The consequence is pole and zero locations are scaled by z_0. If the ROC of X(z) is

rR<|z|<rL, then the ROC of Y(z) is
rR< $|z/z_0|$ <rL, i.e., $|z_0|rR < |z| < |z_0|rL$

Differentiation of X(z)

If we have $f[n] \Leftrightarrow F(z)$ then $nf[n] \overset{z}{\longleftrightarrow} -z\dfrac{dF(z)}{z}$ and ROC = R_f

Proof:

$$F(z) = \sum_{n=-\infty}^{\infty} f[n]z^{-n}$$

$$-z\frac{dF(z)}{dz} = -z\sum_{n=-\infty}^{\infty} -n\, f[n]z^{-n-1} = \sum_{n=-\infty}^{\infty} -n\, f[n]z^{-n}$$

$$-z\frac{dF(z)}{dz} \overset{z}{\longleftrightarrow} nf[n]$$

Some Standard Z-Transform

	Sequence	z - transform
1	$\delta[n]$	1
2	$u[n]$	$\frac{z}{z-1}$
3	b^n	$\frac{z}{z-b}$
4	$b^{n-1}u[n-1]$	$\frac{1}{z-b}$
5	e^{an}	$\frac{z}{z-e^a}$
6	n	$\frac{z}{(z-1)^2}$
7	n^2	$\frac{z(z+1)}{(z-1)^3}$
8	$b^n n$	$\frac{bz}{(z-b)^2}$
9	$e^{an}n$	$\frac{ze^a}{(z-e^a)^2}$
10	$\sin(an)$	$\frac{\sin(a)z}{z^2-2\cos(a)z+1}$
11	$b^n\sin(an)$	$\frac{\sin(a)bz}{z^2-2\cos(a)bz+b^2}$
12	$\cos(an)$	$\frac{z(z-\cos(a))}{z^2-2\cos(a)z+1}$
13	$b^n\cos(an)$	$\frac{z(z-b\cos(a))}{z^2-2\cos(a)bz+b^2}$

Problems

1 **Find the *z* transform of 3*n* + 2 × 3^n**

Sol From the linearity property

$Z\{3n + 2 \times 3^n\} = 3Z\{n\} + 2Z\{3^n\}$

and from the Table 1

$$Z\{n\} = \frac{z}{(z-1)^2} \quad \text{and} \quad Z\{3^n\} = \frac{z}{(z-3)}$$

(r^n with $r = 3$). Therefore

$$Z\{3n + 2 \times 3^n\} = \frac{3z}{(z-1)^2} + \frac{2z}{(z-3)}$$

2 **Find the z-transform of each of the following sequences:**

(a) $\boldsymbol{x(n)= 2^n u(n)+3(½)^n u(n)}$

(b) $\boldsymbol{x(n)=\cos(n\omega_0)u(n)}$.

Sol (a) Because $x(n)$ is a sum of two sequences of the form $\alpha^n u(n)$, using the linearity property of the z-transform, and referring to Table 1, the z-transform pair

$$X(z) = \frac{1}{1-2z^{-1}} + \frac{3}{1-\frac{1}{2}z^{-1}} = \frac{4-\frac{13}{2}z^{-1}}{(1-2z)\left(1-\frac{1}{2}z^{-1}\right)}$$

(b) For this sequence we write

$x(n) = \cos(n\omega_0)\, u(n) = ½(e^{jn\omega 0} + e^{-jn\omega 0})\, u(n)$

Therefore, the *z*-transform is

$$X(z) = \frac{1}{2}\frac{1}{1-e^{jn\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1-e^{-jn\omega_0}z^{-1}}$$

with a region of convergence |z| >1. Combining the two terms together, we have

$$X(z) = \frac{1-(\cos\omega_0)\,z^{-1}}{1-2(\cos\omega_0)\,z^{-1}+z^{-2}}$$

3 **Determine f_n by Infinite Series and Partial Fraction Expansion**

$$F(z) = \frac{2z}{(z-2)(z-1)^2}$$

Sol

$$F(z) = \frac{2z}{z^3-4z^2+5z-2}$$

Now divide (long division) with the polynomials written in descending powers of z

```
2z^-2+8z^-3+22z^-4+52z^-5+114z^-6+...
_____________________________________
Z^3-4z^2+5z-2|2z
2z-8+10z^-1-4z^-2
_________________
                8-10z^-1+04z^-2
8-32z^-1+40z^-2-16z^-3
______________________
                   22z^-1-36z^-2+016z^-3
22z^-1-88z^-2+110z^-3-44z^-4
____________________________
                        52z^-2-094z^-3+044z^-4
52z^-2-208z^-3+260z^-4-104z^-5
______________________________
                             114z^-3-216z^-4+104z^-5
```

$$\therefore F(z)=\sum_{n=0}^{\infty} f_n z^{-n} = 2z^{-2} + 8z^{-3} + 22z^{-4} + 52z^{-5} + 114z^{-6} + \dots$$

And the time sequence for f_n is

n	0	1	2	3	4	5	6	…
f_n	0	0	2	8	22	52	114	…

NOTE This method does NOT give a closed form for the answer, but it is a good method for finding the first few sample values or to check out that the closed form given by another method at least starts out correctly.

$$F(z)=\frac{2z}{(z-2)(z-1)^2}=\frac{k_1 z}{z-2}+\frac{k_2 z}{z-1}+\frac{k_3 z}{(z-1)^2}$$

To find k1 multiply both sides of the equation by (z-2), divide by z, and let z→2

$$\frac{2z}{(z-1)^2}=k_1 z+\frac{k_2 z(z-2)}{z-1}+\frac{k_3 z(z-2)}{(z-1)^2}$$

$$\frac{2}{(z-1)^2}=k_1+\frac{k_2(z-2)}{z-1}+\frac{k_3(z-2)}{(z-1)^2}$$

$$\left.\frac{2}{(z-1)^2}\right|_{z=2}=k_1+\left.\frac{k_2(z-2)}{z-1}\right|_{z=2}+\left.\frac{k_3(z-2)}{(z-1)^2}\right|_{z=2}$$

$\mathbf{k_1 = 2}$

Similarly to find k_3 multiply both sides by $(z-1)^2$, divide by z, and let z→1

$$\frac{2}{(z-2)}=\frac{k_1(z-1)^2}{z-2}+k_2(z-1)+k_3 z$$

$\mathbf{k_3 = -2}$

Finding k2 requires going back to Equation A above and taking the derivative of both sides

$$\frac{2}{(z-2)}=\frac{k_1(z-1)^2}{z-2}+k_2(z-1)+k_3 z$$

$$-\frac{2}{(z-2)^2}=k_1\left[\frac{2(z-1)}{z-2}-\frac{2(z-1)^2}{(z-2)^2}\right]+k_2$$

Now again let z→1

$\mathbf{k_2 = -2}$

$$\therefore F(z)=\frac{2z}{z-2}-\frac{2z}{z-1}-\frac{2z}{(z-1)^2}$$

Convolution theorem

If $u_n = Z^{-1}[U(z)]$ and $v_n = Z^{-1}[V(z)]$ then $Z^{-1}[U(z).V(z)] = \sum_{m=0}^{n} u_n v_{n-m} = u_n * v_n$

Where the symbol * denotes the convolution operation

Proof We have

$u_n = Z^{-1}[U(z)]$ and $v_n = Z^{-1}[V(z)]$

$$U(z).V(z) = (u_0 + u_1 z^{-1} + u_2 z^{-2} + ... + u_n z^{-n} + ...) x (v_0 + v_1 z^{-1} + v_2 z^{-2} + ... + v_n z^{-n} + ...)$$

$$U(z).V(z) = \sum_{n=0}^{\infty} (u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + ... + u_n v_0) z^{-n}$$

$$U(z).V(z) = Z(u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + ... + u_n v_0)$$

$$Z^{-1}[U(z).V(z)] = \sum_{m=0}^{n} u_n v_{n-m} = u_n * v_n$$

EX **Use convolution theorem to evaluate** $Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\}$

Sol

$$Z^{-1}\left\{\frac{z}{z-a}\right\} = a^n, Z^{-1}\left\{\frac{z}{z-b}\right\} = b^n$$

$$Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = Z^{-1}\left\{\frac{1}{(z-a)} - \frac{1}{(z-b)}\right\} = a^n * b^n$$

$$Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = \sum_{m=0}^{n} a^n . b^{n-m}$$

$$Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = b^n \frac{(a/b)^{n+1} - 1}{(a/b) - 1}$$

$$Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = \frac{a^{n+1} - b^{n+1}}{a-b}$$

Formation and solution of difference equations.

Take the Z-transform of both sides of the difference equations and the given conditions

Transpose all terms without $U(z)$ to the right

Divide by the coefficient of $U(z)$ getting $U(z)$ as a function of z

Express this function in terms of Z-transforms of known functions and take inverse Z-transform of both sides

This gives u_n as a function of n which is desired solution

Ex **Using Z-transform solve** $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$ **with** $u_0 = 0, u_1 = 1$

Sol

$$Z(u_n) = U(z), Z(u_{n+1}) = z[U(z) - u_0]$$

$$Z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$$

$$Z(3^n) = z/(z-3)$$

$$z^2[U(z) - u_0 - u_1 z^{-1}] + 4z[U(z) - u_0] + 3U(z) = z/(z-3)$$

$$U(z)(z^2 + 4z + 3) = z + z/(z-3)$$

$$\frac{U(z)}{z} = \frac{1}{(z+1)(z+3)} + \frac{1}{(z-3)(z+1)(z+3)}$$

$$U(z) = \frac{3z}{8(z+1)} + \frac{z}{24(z-3)} - \frac{5z}{12(z+3)}$$

$$u_n = \frac{3}{8} Z^{-1}\left[\frac{z}{(z+1)}\right] + \frac{1}{24} Z^{-1}\left[\frac{z}{(z-3)}\right] - \frac{5}{12} Z^{-1}\left[\frac{z}{(z+3)}\right]$$

$$u_n = \frac{3}{8}(-1)^n + \frac{1}{24}(3)^n - \frac{5}{12}(-3)^n$$

UNIT-V

PARTIAL DIFFERENTIAL EQUATION and APPLICATIONS

- **Formation of partial differential equations**
- **Solutions of first order linear equation by Lagrange method**
- **Charpit's method**
- **Method of separation of variables**
- **One dimensional heat equations**
- **One dimensional wave equations**

Introduction

The concept of a differential equation to include equations that involve partial derivatives, not just ordinary ones. Solutions to such equations will involve functions not just of one variable, but of several variables. Such equations arise naturally, for example, when one is working with situations that involve positions in space that vary over time. To model such a situation, one needs to use functions that have several variables to keep track of the spatial dimensions and an additional variable for time.

Examples of some important PDEs:

(1) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ *One-dimensional wave equation*

(2) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ *One-dimensional heat equation*

(3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ *Two-dimensional Laplace equation*

(4) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ *Two-dimensional Poisson equation*

Partial differential equations: An equation involving partial derivatives of one dependent variable with respective more than one independent variables.

Notations which we use in this unit:

$p = \frac{\partial z}{\partial x}$,$q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $= \frac{\partial^2 z}{\partial x\, \partial y}$, t= $\frac{\partial^2 z}{\partial y^2}$,

Formation of partial differential equation:

A partial differential equation of given curve can be formed in two ways

1. By eliminating arbitrary constants
2. By eliminating arbitrary functions

Problems

1 Form a partial differential equation by eliminating a,b,c from

$$\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$$

Sol Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Differentiating partially w.r.to x and y, we have

$\frac{1}{a^2}(2x) + \frac{1}{c^2}(2z)\frac{\partial z}{\partial x}$=o

$\frac{1}{a^2}(x) + \frac{1}{c^2}(z)\,p$ =o _______(1)

And $\frac{1}{b^2}(2x)$ + $\frac{1}{c^2}(2z)\frac{\partial z}{\partial x}$=o

$\frac{1}{b^2}(y) + \frac{1}{c^2}(z)$ q = o _______(2)

Diff (1) partially w.r.to x, we have

$\frac{1}{a^2}+\frac{p}{c^2}\frac{\partial z}{\partial x}+\frac{z}{c^2}\frac{\partial p}{\partial x}$=o ______(3)

$\frac{1}{a^2}+\frac{p^2}{c^2}+\frac{z}{c^2}r$ =O

Multiply this equation by x and then subtracting (1) from it

$$\frac{1}{c^2}(xzr+xp^2-pz)=0$$

2 **Form a partial differential equation by eliminating the constants from $(x-a)^2+(y-b)^2=z^2cot^2$α, where α is a parameter**

Sol Given $(x-a)^2+(y-b)^2=z^2cot^2$α ___________(1)

Differentiating partially w.r.to x and y, we have

2 $(x-a)$+0 = 2 z p$cot^2\alpha$

$(x-a)$ = Zp$cot^2\alpha$

And 0+2(y-b) = 2zq$cot^2\alpha$

(Y-b) = zq$cot^2\alpha$

Substituting the values of (x-a) and (y-b) in (1),we get

$$(zpcot^2\alpha)^2+(zqcot^2\alpha)^2=z^2cot^2\alpha$$

$$(p^2+q^2)(cot^2\alpha)^2=cot^2\alpha$$

$$p^2+q^2=tan^2\alpha$$

3 **Form the partial differential equation by eliminating a and b from log (az-1)=x+ay+b**

Sol Given equation is

Log (az-1)=x+ay+b

Differentiating partially w.r.t. x and y ,we get

$\frac{1}{az-1}(ap)=1\Rightarrow ap=az-1$ --------------- (1)

$\frac{1}{az-1}(aq)=a\Rightarrow aq=a(az-1)$ ------------- (2)

(2)/(1) gives

$\frac{q}{p}=a$ $or\ ap=q$---------------- (3)

Substituting (3) in (1), we get

$$q=\frac{q}{p}.(z-1)$$

i.e. pq=qz-p

$$p(q+1)=qz$$

4 **Find the differential equation of all spheres whose centers lie on z-axis with a given radius r.**

Sol The equation of the family of spheres having their centers on z-axis and having radius r is

$$x^2+y^2+(z-c)^2=r^2$$

Where c and r are arbitrary constants

Differentiating this eqn partially w.r.t. x and y ,we get

$2x+2(z-c)\frac{\partial z}{\partial x}=0\Rightarrow x+(z-c)p=0$ _________(1)

$2y+2(z-c)\frac{\partial z}{\partial y}=0\Rightarrow y+(z-c)q=0$_________(2)

From (1),$(z-c)=-\frac{x}{p}$ ___________(3)

From (2),$(z-c)=-\frac{y}{q}$ ___________(4)

From (3) and (4)

We get $-\frac{x}{p} = -\frac{y}{q}$

i.e. $xq - yp = 0$

Linear partial differential equations of first order :

Lagrange's linear equation: An equation of the form Pp + Qq = R is called Lagrange's linear equation.

To solve Lagrange's linear equation consider auxiliary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Non-linear partial differential equations of first order :

Complete Integral : A solution in which the number of arbitrary constants is equal to the number of independent variables is called complete integral or complete solution of the given equation.

Particular Integral: A solution obtained by giving particular values to the arbitrary constants in the complete integral is called a particular integral.

Singular Integral: let f(x,y,z,p,q) = 0 be a partial differential equation whose complete integral is

To solve non-linear pde we use Charpit's Method :

There are six types of non-linear partial differential equations of first order as given below.

1. f (p,q) = 0
2. f (z,p,q) = 0
3. f1 (x,p) = f_2 (y,q)
4. z = px +qy + f(p,q)
5. $f(x^m p, y^n q) = 0$ and $f(m^y p, y^n q, z) = 0$
6. $f(pz^m, qz^m) = 0$ and $f_1(x, pz^m) = f_2(y, qz^m)$

Charpit's Method:

We present here a general method for solving non-linear partial differential equations. This is known as Charpit's method.

LetF(x,y,u, p.q)=0be a general nonlinear partial differential equation of first-order. Since u depends on x and y, we have

$du = u_x dx + u_y dy = pdx + qdy$ where $p = u_x = \frac{\partial u}{\partial x}$, $q = u_y = \frac{\partial u}{\partial y}$

If we can find another relation between x,y,u,p,q such that f(x,y,u,p,q)=0then we can solve for p and q and substitute them in equation This will give the solution provided is integrable.

To determine f, differentiate w.r.t. x and y so that

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial u} p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0$$

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial u} q + \frac{\partial F}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial y} = 0$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial u} q + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0$$

Eliminating $\frac{\partial p}{\partial x}$ from, equations and $\frac{\partial q}{\partial y}$ from equations we obtain

$$\left(\frac{\partial F}{\partial x}\frac{\partial f}{\partial p}-\frac{\partial f}{\partial x}\frac{\partial F}{\partial p}\right)+\left(\frac{\partial F}{\partial u}\frac{\partial f}{\partial p}-\frac{\partial f}{\partial u}\frac{\partial F}{\partial p}\right)p+\left(\frac{\partial F}{\partial q}\frac{\partial f}{\partial p}-\frac{\partial f}{\partial q}\frac{\partial F}{\partial p}\right)\frac{\partial q}{dx}=0$$

$$\left(\frac{\partial F}{\partial y}\frac{\partial f}{\partial q}-\frac{\partial f}{\partial y}\frac{\partial F}{\partial q}\right)+\left(\frac{\partial F}{\partial u}\frac{\partial f}{\partial q}-\frac{\partial f}{\partial u}\frac{\partial F}{\partial q}\right)q+\left(\frac{\partial F}{\partial p}\frac{\partial f}{\partial q}-\frac{\partial f}{\partial p}\frac{\partial F}{\partial q}\right)\frac{\partial p}{dy}=0$$

Adding these two equations and using

$$\frac{\partial q}{\partial x}=\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial p}{\partial y}$$

and rearranging the terms, we get

$$\left(-\frac{\partial F}{\partial p}\right)\frac{\partial f}{\partial x}+\left(-\frac{\partial F}{\partial q}\right)\frac{\partial f}{\partial y}+\left(-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}\right)\frac{\partial f}{\partial u}+\left(\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial u}\right)\frac{\partial f}{\partial p}$$

$$+\left(\frac{\partial F}{\partial y}+q\frac{\partial f}{\partial u}\right)\frac{\partial f}{\partial q}=0$$

We get the auxiliary system of equations

$$\frac{dx}{-\frac{\partial F}{\partial p}}=\frac{dy}{-\frac{\partial F}{\partial q}}=\frac{du}{-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}}=\frac{dp}{\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial u}}=\frac{dq}{\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial u}}=\frac{df}{0}$$

An Integral of these equations, involving p or q or both, can be taken as the required equation.

Problems

1 **solve** $(x^2-y^2-yz)p+(x^2-y^2-zx)q=z(x-y)$

Sol Here

$P=(x^2-y^2-yz), Q=(x^2-y^2-zx), R=z(x-y)$

The subsidiary equations are $\frac{dx}{(x^2-y^2-yz)}=\frac{dy}{(x^2-y^2-zx)}=\frac{dz}{z(x-y)}$

Using 1,-1,0 and x,-y,0 as multipliers , we have

. $\frac{dz}{z(x-y)}=\frac{dx-dy}{z(x-y)}=\frac{x\,dx-ydy}{(x^2-y^2)(x-y)}$

From the first two rations 0f ,we have

dz= dx-dy

integrating , $z=x-y-c_1$ or $x-y-z=c_1$

now taking first and last ratios in (2) ,we get

$$\frac{dz}{z}=\frac{x\,dx-y\,dy}{x^2-y^2}\quad \text{or}\ \frac{2dz}{z}=\frac{2x\,dx-2y\,dy}{x^2-y^2}$$

Integrating ,2 log z = $\log(x^2-y^2)-\log c_2$

$$\Longrightarrow \frac{x^2-y^2}{z^2}=c_2$$

The required general solution is $f\left(x-y-z,\frac{x^2-y^2}{z^2}\right)=0$

2 **solve** $(\mathbf{mz-ny})\mathbf{p}+(\mathbf{nx-lz})\mathbf{q}=\mathbf{ly-mx}$

Sol The equation is

$$(mz-ny)p+(nx-lz)q=ly-mx$$

Here P= $(mz-ny), Q=(nx-lz)$, $R=ly-mx$

The Auxiliary equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

i.e. $\frac{dx}{(mz-ny)} = \frac{dy}{(nx-lz)} = \frac{dz}{ly-mx}$

Choosing x,y,z as multipliers ,we get

Each fraction $= \frac{x\,dx+ydy+zdz}{0}$, which gives $x\,dx + ydy + zdz = 0$

Integrating, $x^2 + y^2 + z^2 = a$

Again choosing l, m, n as multipliers ,we get

Each fraction $= \frac{l\,dx+mdy+ndz}{0}$, which gives $l\,dx + mdy + ndz = 0$

Integrating, $lx + my + nz = b$

Hence the solution is

$$f(x^2 + y^2 + z^2, lx + my + nz) = 0$$

3 **Solve$(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx.$**

Sol The subsidiary equations are $\frac{dx}{z^2-2yz-y^2} = \frac{dy}{xy+zx} = \frac{dz}{xy-zx}$

Each fraction $= \frac{xdx+ydy+zdz}{0}$

$$\therefore\ xdx + ydy + zdz = 0$$

Integrating, $x^2 + y^2 + z^2 = a$

Taking second and third terms ,we get $(y - z)dy = (y + z)dz$

i.e. $ydy - zdy - ydz - zdz = 0$

$$ydy - (ydz + zdy) - zdz = 0$$

$$d\left(\frac{y^2}{2}\right) - d(yz) - d\left(\frac{z^2}{2}\right) = 0$$

integrating, $\frac{y^2}{2} - yz - \frac{z^2}{2} = b or y^2 - 2yz - z^2 = b$

Hence the general solution is $\varphi(x^2 + y^2 + z^2, y^2 - 2yz - z^2) = b$

4 **Find the integral surface of $x(y^2 + z)p - y(x^2 + z)q = (x^2 + y^2)z$**
Which contains the straight line x+y=0 ,z=1

Sol The subsidiary equations are $\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}$

Each fraction $= \frac{xdx+ydy+zdz}{0}$

And also $= \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$

$$xdx + ydy + zdz = 0\ \ and\ \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating $x^2 + y^2 - 2z = a and xyz = b$

The straight line is x+y =0 , z=1

$$\therefore\ x^2 + y^2 - 2 = a and xy = b$$

Now a + 2b = $x^2 + y^2 - 2 + 2xy = (x + y)^2 - 2 = -2$ $(since x + y = 0)$

a+2b+2=0

Hence the required surface is $x^2 + y^2 - 2z + 2xyz + 2 = 0$

5 **Find the general solution of the first-order linear partial differential equation with the constant coefficients: $4u_x+u_y=x^2y$**

Sol The auxiliary system of equations is

$$\frac{dx}{4} = \frac{dy}{1} = \frac{du}{x^2y}$$

From here we get

$\frac{dx}{4} = \frac{dy}{1}$ or dx-4dy=0. Integrating both sides we get x-4y=c. Also $\frac{dx}{4} = \frac{du}{x^2y}$ or $x^2y\,dx=4du$

or $x^2(\frac{x-c}{4})\,dx=4du$ or

$\frac{1}{16}(x^3 - cx^2)\,dx = du$

Integrating both sides we get

$u=c_1+\frac{3x^4-4cx^3}{192}$

$= f(c)+\frac{3x^4-4cx^3}{192}$

After replacing c by x-4y, we get the general solution

$u=f(x-4y)+\frac{3x^4-4(x-4y)x^3}{192}$

$=f(x-4y)-\frac{x^4}{192}+\frac{x^3y}{12}$

6 Find the general solution of the partial differential equation $y^2up + x^2uq = y^2x$

Sol The auxiliary system of equations is

$$\frac{dx}{y^2u} = \frac{dy}{x^2u} = \frac{du}{xy^2}$$

Taking the first two members we have $x^2dx = y^2dy$ which on integration given $x^3-y^3 = c_1$. Again taking the first and third members,

we have x dx = u du

which on integration given $x^2-u^2 = c_2$

Hence, the general solution is

$F(x^3-y^3, x^2-u^2) = 0$

7 Find the general solution of the partial differential equation.

$$\left(\frac{\partial u}{\partial x}\right)^2 x+\left(\frac{\partial u}{\partial y}\right)^2 y-u=0$$

Sol : Let $p=\frac{\partial u}{\partial x}$, $q=\frac{\partial u}{\partial y}$

The auxiliary system of equations is

$$\frac{dx}{2px}=\frac{dy}{2qy}=\frac{du}{2(p^2x+q^2y)}=\frac{dp}{p-p^2}=\frac{dq}{q-q^2}$$

which we obtain from putting values of

$$\frac{\partial F}{\partial p}=2px\ ,\frac{\partial F}{\partial q}=2qy,\frac{\partial F}{\partial x}=p^2,\frac{\partial F}{\partial u}=-1,\frac{\partial F}{\partial y}=q^2$$

and multiplying by -1 throughout the auxiliary system. From first and 4th expression in (11.38) we get

$dx=\frac{p^2dx+2pxdp}{py}$. From second and 5th expression

$dy=\frac{q^2dy+2qydq}{qy}$

Using these values of dx and dy we get

$$\frac{p^2dx+2pxdp}{p^2x}=\frac{q^2dy+2qydq}{q^2y}$$

or $\frac{dx}{x}+\frac{2}{p}dp=\frac{dy}{y}+\frac{2dq}{q}$

Taking integral of all terms we get

ln|x| + 2ln|p| = ln|y|+2ln|q|+lnc

or ln|x| p^2 = ln|y|q^2c

or $p^2x=cq^2y$, where c is an arbitrary constant.

Solving for p and q we get $cq^2y+q^2y-u=0$

$(c+1)q^2y=u$

$q=\left\{\frac{u}{(c+1)y}\right\}^{1/2}$

$p=\left\{\frac{cu}{(c+1)x}\right\}^{1/2}$

$du=\left\{\frac{cu}{(c+1)x}\right\}^{1/2}dx+\left\{\frac{u}{(c+1)y}\right\}^{1/2}dy$

or $\left(\frac{1+c}{u}\right)^{1/2}du=\left(\frac{c}{x}\right)^{1/2}dx+\left(\frac{i}{y}\right)^{1/2}dy$

By integrating this equation we obtain $((1+c)u)^{1/2}=(cx)^{1/2}+(y)^{1/2}+c_1$

This is a complete solution.

8 Solve $p^2+q^2=1$

Sol The auxiliary system of equation is

$$-\frac{dx}{-2p}=\frac{dy}{2q}=\frac{du}{-2p^2-2q^2}=\frac{dp}{0}=\frac{dq}{0}$$

or $$\frac{dx}{p}=\frac{dy}{q}=\frac{du}{p^2+q^2}=\frac{dp}{0}=\frac{dq}{0}$$

Using dp =0, we get p=c and q=$\sqrt{1-c^2}$, and these two combined with du =pdx+qdy yield

u=cx+y$\sqrt{1-c^2}$ + c_1 which is a complete solution.

Using $\frac{dx}{du}$ = p , we get du = $\frac{dx}{c}$ where p= c

Integrating the equation we get u = $\frac{x}{c}$ + c_1

Also du = $\frac{dy}{q}$, where q = $\sqrt{1-p^2}$ = $\sqrt{1-c^2}$

or du = $\frac{dy}{\sqrt{1-c^2}}$. Integrating this equation we get u = $\frac{1}{\sqrt{1-c^2}}$ y +c_2

This cu = x+cc_1 and $u\sqrt{1-c^2}$ = y + $c_2\sqrt{1-c^2}$

Replacing cc_1 and $c_2\sqrt{1-c^2}$ by - α and -β respectively, and eliminating c, we get

$$u^2 = (x-\alpha)^2 + (y-\beta)^2$$

9 **Solve $u^2+pq - 4 = 0$**

Sol The auxiliary system of equations is

$$\frac{dx}{q} = \frac{dy}{p} = \frac{du}{2pq} = \frac{dp}{-2up} = \frac{dq}{-2uq}$$

The last two equations yield $p = a^2q$.
Substituting in $u^2+pq - 4 = 0$ gives

$q = \pm\frac{1}{a}\sqrt{4-u^2}$ and $p = \pm a\sqrt{4-u^2}$

Then $du = pdx+qdy$ yields

$$du = \pm\sqrt{4-u^2}\left(adx + \frac{1}{a}dy\right)$$

or $\frac{du}{\sqrt{4-u^2}} = \pm adx + \frac{1}{a}dy$

Integrating we get $\sin^{--1}\frac{u}{2} = \pm\left(adx + \frac{1}{a}y + c\right)$

or $u = \pm 2\sin\left(ax + \frac{1}{a}y + c\right)$

10 **Solve $p^2(1-x^2)-q^2(4-y^2) = 0$**

Sol Let $p^2(1-x^2) = q^2(4-y^2) = a^2$

This gives $p = \frac{a}{\sqrt{1-x^2}}$ and $q = \frac{a}{\sqrt{4-y^2}}$

(neglecting the negative sign).
Substituting in $du = pdx + q\,dy$ we have

$$du = \frac{a}{\sqrt{1-x^2}}dx + \frac{a}{\sqrt{4-y^2}}dy$$

Integration gives $u = a\left(\sin' x + \sin'\frac{y}{2}\right) + c$.

Wave Equation

For the rest of this introduction to PDEs we will explore PDEs representing some of the basic types of linear second order PDEs: heat conduction and wave propagation. These represent two entirely different physical processes: the process of diffusion, and the process of oscillation, respectively. The field of PDEs is extremely large, and there is still a considerable amount of undiscovered territory in it, but these two basic types of PDEs represent the ones that are in some sense, the best understood and most developed of all of the PDEs. Although there is no one way to solve all PDEs explicitly, the main technique that we will use to solve these various PDEs represents one of the most important techniques used in the field of PDEs, namely separation of variables (which we saw in a different form while studying ODEs). The essential manner of using separation of variables is to try to break up a differential equation involving several partial derivatives into a series of simpler, ordinary differential equations.

We start with the wave equation. This PDE governs a number of similarly related phenomena, all involving oscillations. Situations described by the wave equation include acoustic waves, such as vibrating guitar or violin strings, the vibrations of drums, waves in fluids, as well as waves generated by electromagnetic fields, or any other physical situations involving oscillations, such as vibrating power lines, or even suspension bridges in certain circumstances. In short, this one type of PDE covers a lot of ground.

We begin by looking at the simplest example of a wave PDE, the one-dimensional wave equation. To get at this PDE, we show how it arises as we try to model a simple vibrating string, one that is held in place between two secure ends. For instance, consider plucking a guitar string and watching (and listening) as it vibrates. As is typically the case with modeling, reality is quite a bit more complex than we can deal with all at once, and so we need to make some simplifying assumptions in order to get started.

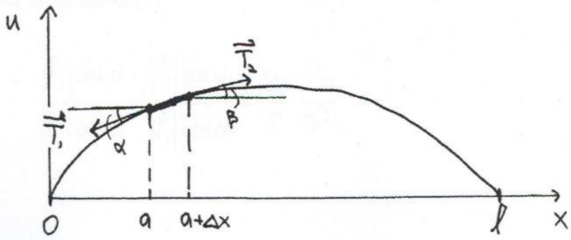
First off, assume that the string is stretched so tightly that the only real force we need to consider is that due to the string's tension. This helps us out as we only have to deal with one force, i.e. we can safely ignore the effects of gravity if the tension force is orders of magnitude greater than that of gravity. Next we assume that the string is as uniform, or homogeneous, as possible, and that it is perfectly elastic. This makes it possible to predict the motion of the string more readily since we don't need to keep track of kinks that might occur if the string wasn't uniform. Finally, we'll assume that the vibrations are pretty minimal in relation to the overall length of the string, i.e. in terms of displacement, the amount that the string bounces up and down is pretty small. The reason this will help us out is that we can concentrate on the simple up and down motion of the string, and not worry about any possible side to side motion that might occur.

Now consider a string of a certain length, l, that's held in place at both ends. First off, what exactly are we trying to do in "modeling the string's vibrations"? What kind of function do we want to solve for to keep track of the motion of string? What will it be a function of? Clearly if the string is vibrating, then its motion changes over time, so *time* is one variable we will want to keep track of. To keep track of the actual motion of the string we will need to have a function that tells us the shape of the string at any particular time. One way we can do this is by looking for a function that tells us the *vertical displacement* (positive up, negative down) that exists at any point along the string – how far away any particular point on the string is from the undisturbed resting position of the string, which is just a straight line. Thus, we would like to find a function $u(x,t)$ of two variables. The variable x can measure distance along the string, measured away from one chosen end of the string (i.e. $x = 0$ is one of the tied down endpoints of the string), and t stands for time. The function $u(x,t)$ then gives the vertical displacement of the string at any point, x, along the string, at any particular time t.

As we have seen time and time again in calculus, a good way to start when we would like to study a surface or a curve or arc is to break it up into a series of very small pieces. At the end of our study of one little segment of the vibrating string, we will think about what happens as the length of the little segment goes to zero, similar to the type of limiting process we've seen as we progress from Riemann Sums to integrals.

Suppose we were to examine a very small length of the vibrating string as shown in figure 1:

Figure 1



Now what? How can we figure out what is happening to the vibrating string? Our best hope is to follow the standard path of modeling physical situations by studying all of the forces involved and then turning to Newton's classic equation $F = ma$. It's not a surprise that this will help us, as we have already pointed out that this equation is itself a differential equation (acceleration being the second derivative of position with respect to time). Ultimately, all we will be doing is substituting in the particulars of our situation into this basic differential equation.

Because of our first assumption, there is only one force to keep track of in our situation, that of the string tension. Because of our second assumption, that the string is perfectly elastic with no kinks, we can assume that the force due to the tension of the string is tangential to the ends of the small string segment, and so we need to keep track of the string tension forces T_1 and T_2 at each end of the string segment. Assuming that the string is only vibrating up and down means that the horizontal components of the tension forces on each end of the small segment must perfectly balance each other out. Thus

(1) $$\left|\vec{T}_1\right| \cos\alpha = \left|\vec{T}_2\right| \cos\beta = T$$

where$T$ is a string tension constant associated with the particular set-up (depending, for instance, on how tightly strung the guitar string is). Then to keep track of all of the forces involved means just summing up the vertical components of T_1 and T_2. This is equal to

(2) $$\left|\vec{T}_2\right| \sin\beta - \left|\vec{T}_1\right| \sin\alpha$$

where we keep track of the fact that the forces are in opposite direction in our diagram with the appropriate use of the minus sign. That's it for "Force," now on to "Mass" and "Acceleration." The mass of the string is simple, just $\delta\Delta x$, where δ is the mass per unit length of the string, and Δx is (approximately) the length of the little segment. Acceleration is the second derivative of position with respect to time. Considering that the position of the string segment at a particular time is just $u(x,t)$, the function we're trying to find, then the acceleration for the little segment is $\frac{\partial^2 u}{\partial t^2}$ (computed at some point between a and $a + \Delta x$). Putting all of this together, we find that:

(3) $$\left|\vec{T}_2\right| \sin\beta - \left|\vec{T}_1\right| \sin\alpha = \delta\Delta x \frac{\partial^2 u}{\partial t^2}$$

Now what? It appears that we've got nowhere to go with this – this looks pretty unwieldy as it stands. However, be sneaky… try dividing both sides by the various respective equal parts written down in equation (1):

(4) $$\frac{\left|\vec{T}_2\right| \sin\beta}{\left|\vec{T}_2\right| \cos\beta} - \frac{\left|\vec{T}_1\right| \sin\alpha}{\left|\vec{T}_1\right| \cos\alpha} = \frac{\delta\Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

or more simply:

(5) $$\tan\beta - \tan\alpha = \frac{\delta\Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

Now, finally, note that $\tan\alpha$ is equal to the slope at the left-hand end of the string segment, which is just $\frac{\partial u}{\partial x}$ evaluated at *a*, i.e. $\frac{\partial u}{\partial x}(a,t)$ and similarly $\tan\beta$ equals $\frac{\partial u}{\partial x}(a+\Delta x,t)$, so (5) becomes…

$$\text{(6)} \qquad \frac{\partial u}{\partial x}(a+\Delta x,t)-\frac{\partial u}{\partial x}(a,t)=\frac{\delta\Delta x}{T}\frac{\partial^2 u}{\partial t^2}$$

or better yet, dividing both sides by Δx …

$$\text{(7)} \qquad \frac{1}{\Delta x}\left(\frac{\partial u}{\partial x}(a+\Delta x,t)-\frac{\partial u}{\partial x}(a,t)\right)=\frac{\delta}{T}\frac{\partial^2 u}{\partial t^2}$$

Now we're ready for the final push. Let's go back to the original idea – start by breaking up the vibrating string into little segments, examine each such segment using Newton's $F = ma$ equation, and finally figure out what happens as we let the length of the little string segment dwindle to zero, i.e. examine the result as Δx goes to 0. Do you see any limit definitions of derivatives kicking around in equation (7)? As Δx goes to 0, the left-hand side of the equation is in fact just equal to $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial^2 u}{\partial x^2}$, so the whole thing boils down to:

$$\text{(8)} \qquad \frac{\partial^2 u}{\partial x^2}=\frac{\delta}{T}\frac{\partial^2 u}{\partial t^2}$$

which is often written as

$$\text{(9)} \qquad \frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$$

by bringing in a new constant $c^2 = \frac{T}{\delta}$ (typically written with c^2, to show that it's a positive constant).

This equation, which governs the motion of the vibrating string over time, is called the ***one-dimensional wave equation***. It is clearly a second order PDE, and it's linear and homogeneous.

Solution of the Wave Equation by Separation of Variables

There are several approaches to solving the wave equation. The first one we will work with, using a technique called separation of variables, again, demonstrates one of the most widely used solution techniques for PDEs. The idea behind it is to split up the original PDE into a series of simpler ODEs, each of which we should be able to solve readily using tricks already learned. The second technique, which we will see in the next section, uses a transformation trick that also reduces the complexity of the original PDE, but in a very different manner. This second solution is due to Jean Le Rond D'Alembert (an 18th century French mathematician), and is called D'Alembert's solution, as a result.

First, note that for a specific wave equation situation, in addition to the actual PDE, we will also have boundary conditions arising from the fact that the endpoints of the string are attached solidly, at the left end of the string, when $x = 0$ and at the other end of the string, which we suppose has overall length l. Let's start the process of solving the PDE by first figuring out what these boundary conditions imply for the solution function, $u(x,t)$.

Answer: for all values of t, the time variable, it must be the case that the vertical displacement at the endpoints is 0, since they don't move up and down at all, so that

(1) $u(0,t) = 0$ and $u(l,t) = 0$ for all values of t

are the ***boundary conditions*** for our wave equation. These will be key when we later on need to sort through possible solution functions for functions that satisfy our particular vibrating string set-up.

You might also note that we probably need to specify what the shape of the string is right when time $t = 0$, and you're right - to come up with a particular solution function, we would need to know $u(x,0)$. In fact we would also need to know the initial velocity of the string, which is just $u_t(x,0)$. These two requirements are called the ***initial conditions*** for the wave equation, and are also necessary to specify a particular vibrating string solution. For instance, as the simplest example of initial conditions, if no one is plucking the string, and it's perfectly flat to start with, then the initial conditions would just be $u(x,0) = 0$ (a perfectly flat string) with initial velocity, $u_t(x,0) = 0$. Here, then, the solution function is pretty unenlightening – it's just $u(x,t) = 0$, i.e. no movement of the string through time.

To start the separation of variables technique we make the key assumption that whatever the solution function is, that it can be written as the product of two independent functions, each one of which depends on just one of the two variables, x or t. Thus, imagine that the solution function, $u(x,t)$ can be written as

(2) $u(x,t) = F(x)G(t)$

where$F$, and G, are single variable functions of x and t respectively. Differentiating this equation for $u(x,t)$ twice with respect to each variable yields

(3) $\dfrac{\partial^2 u}{\partial x^2} = F''(x)G(t)$ and $\dfrac{\partial^2 u}{\partial t^2} = F(x)G''(t)$

Thus when we substitute these two equations back into the original wave equation, which is

(4) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

then we get

(5) $$\frac{\partial^2 u}{\partial t^2} = F(x)G''(t) = c^2 \frac{\partial^2 u}{\partial x^2} = c^2 F''(x)G(t)$$

Here's where our separation of variables assumption pays off, because now if we separate the equation above so that the terms involving *F* and its second derivative are on one side, and likewise the terms involving *G* and its derivative are on the other, then we get

(6) $$\frac{G''(t)}{c^2G(t)} = \frac{F''(x)}{F(x)}$$

Now we have an equality where the left-hand side just depends on the variable *t*, and the right-hand side just depends on *x*. Here comes the critical observation - how can two functions, one just depending on *t*, and one just on *x*, be equal for all possible values of *t* and *x*? The answer is that they must each be constant, for otherwise the equality could not possibly hold for all possible combinations of *t* and *x*. Aha! Thus we have

(7) $$\frac{G''(t)}{c^2G(t)} = \frac{F''(x)}{F(x)} = k$$

where*k* is a constant. First let's examine the possible cases for *k*.

Case One: k = 0

Suppose *k* equals 0. Then the equations in (7) can be rewritten as

(8) $$G''(t) = 0 \cdot c^2 G(t) = 0 \text{ and } F''(x) = 0 \cdot F(x) = 0$$

yielding with very little effort two solution functions for *F* and *G:*

(9) $$G(t) = at + b \text{ and } F(x) = px + r$$

where*a,b, p* and *r*, are constants (note how easy it is to solve such simple ODEs versus trying to deal with two variables at once, hence the power of the separation of variables approach).

Putting these back together to form $u(x,t) = F(x)G(t)$, then the next thing we need to do is to note what the boundary conditions from equation (1) force upon us, namely that

(10) $$u(0,t) = F(0)G(t) = 0 \text{ and } u(l,t) = F(l)G(t) = 0 \quad \text{for all values of } t$$

Unless $G(t) = 0$ (which would then mean that $u(x,t) = 0$, giving us the very dull solution equivalent to a flat, unplucked string) then this implies that

(11) $$F(0) = F(l) = 0 .$$

But how can a linear function have two roots? Only by being identically equal to 0, thus it must be the case that $F(x) = 0$. Sigh, then we still get that $u(x,t) = 0$, and we end up with the dull solution again, the only possible solution if we start with *k* = 0.

So, let's see what happens if…

Case Two: k > 0

So now if *k* is positive, then from equation (7) we again start with

(12) $G''(t) = kc^2 G(t)$

and

(13) $F''(x) = kF(x)$

Try to solve these two ordinary differential equations. You are looking for functions whose second derivatives give back the original function, multiplied by a positive constant. Possible candidate solutions to consider include the exponential and sine and cosine functions. Of course, the sine and cosine functions don't work here, as their second derivatives are *negative* the original function, so we are left with the exponential functions.

Let's take a look at (13) more closely first, as we already know that the boundary conditions imply conditions specifically for $F(x)$, i.e. the conditions in (11). Solutions for $F(x)$ include anything of the form

(14) $F(x) = Ae^{\omega x}$

where $\omega^2 = k$ and *A* is a constant. Since ω could be positive or negative, and since solutions to (13) can be added together to form more solutions (note (13) is an example of a second order linear homogeneous ODE, so that the superposition principle holds), then the general solution for (13) is

(14) $F(x) = Ae^{\omega x} + Be^{-\omega x}$

where now A and B are constants and $\omega = \sqrt{k}$. Knowing that $F(0) = F(l) = 0$, then unfortunately the only possible values of A and B that work are $A = B = 0$, i.e. that $F(x) = 0$. Thus, once again we end up with $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$, i.e. the dull solution once more. Now we place all of our hope on the third and final possibility for *k*, namely…

Case Three: k < 0

So now we go back to equations (12) and (13) again, but now working with *k* as a negative constant. So, again we have

(12) $G''(t) = kc^2 G(t)$

and

(13) $F''(x) = kF(x)$

Exponential functions won't satisfy these two ODEs, but now the sine and cosine functions will. The general solution function for (13) is now

(15) $F(x) = A\cos(\omega x) + B\sin(\omega x)$

where again *A* and *B* are constants and now we have $\omega^2 = -k$. Again, we consider the boundary conditions that specified that $F(0) = F(l) = 0$. Substituting in 0 for *x* in (15) leads to

(16) $F(0) = A\cos(0) + B\sin(0) = A = 0$

so that $F(x) = B\sin(\omega x)$. Next, consider $F(l) = B\sin(\omega l) = 0$. We can assume that *B* isn't equal to 0, otherwise $F(x) = 0$ which would mean that $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$, again, the trivial unplucked string solution. With $B \neq 0$, then it must be the case that $\sin(\omega l) = 0$ in order to have $B\sin(\omega l) = 0$. The only way that this can happen is for ωl to be a multiple of π. This means that

(17) $\omega l = n\pi$ or $\omega = \dfrac{n\pi}{l}$ (where *n* is an integer)

This means that there is an infinite set of solutions to consider (letting the constant *B* be equal to 1 for now), one for each possible integer *n*.

(18) $F(x) = \sin\left(\dfrac{n\pi}{l}x\right)$

Well, we would be done at this point, except that the solution function $u(x,t) = F(x)G(t)$ and we've neglected to figure out what the other function, $G(t)$, equals. So, we return to the ODE in (12):

(12) $G''(t) = kc^2 G(t)$

where, again, we are working with *k*, a negative number. From the solution for $F(x)$ we have determined that the only possible values that end up leading to non-trivial solutions are with

$k = -\omega^2 = -\left(\dfrac{n\pi}{l}\right)^2$ for*n* some integer. Again, we get an infinite set of solutions for (12) that can be written in the form

(19) $G(t) = C\cos(\lambda_n t) + D\sin(\lambda_n t)$

where*C* and *D* are constants and $\lambda_n = c\sqrt{-k} = c\omega = \dfrac{cn\pi}{l}$, where *n* is the same integer that showed up in the solution for $F(x)$ in (18) (we're labeling λ with a subscript "*n*" to identify which value of *n* is used).

Now we really are done, for all we have to do is to drop our solutions for $F(x)$ and $G(t)$ into $u(x,t) = F(x)G(t)$, and the result is

(20) $u_n(x,t) = F(x)G(t) = \left(C\cos(\lambda_n t) + D\sin(\lambda_n t)\right)\sin\left(\dfrac{n\pi}{l}x\right)$

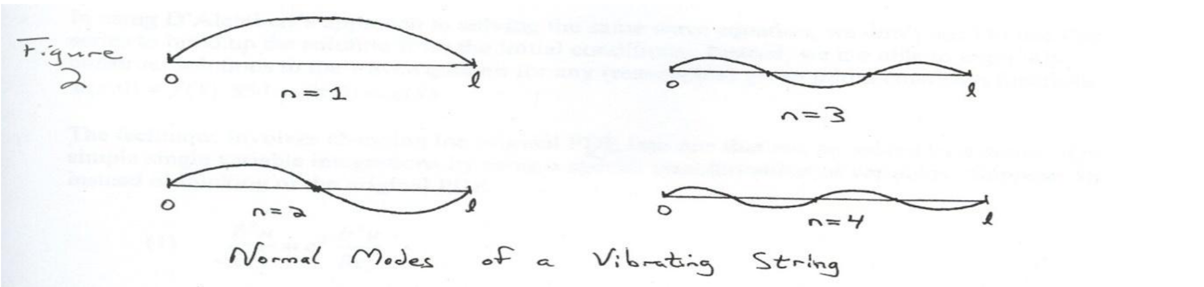
where the integer *n* that was used is identified by the subscript in $u_n(x,t)$ and λ_n, and *C* and *D* are arbitrary constants.

At this point you should be in the habit of immediately checking solutions to differential equations. Is (20) really a solution for the original wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

and does it actually satisfy the boundary conditions $u(0,t) = 0$ and $u(l,t) = 0$ for all values of t

The solution given in the last section really does satisfy the one-dimensional wave equation. To think about what the solutions look like, you could graph a particular solution function for varying values of time, *t*, and then examine how the string vibrates over time for solution functions with different values of *n* and constants *C* and *D*. However, as the functions involved are fairly simple, it's possible to make sense of the solution $u_n(x,t)$ functions with just a little more effort.
For instance, over time, we can see that the $G(t) = \left(C\cos(\lambda_n t) + D\sin(\lambda_n t)\right)$ part of the function is periodic with period equal to $\frac{2\pi}{\lambda_n}$. This means that it has a frequency equal to $\frac{\lambda_n}{2\pi}$ cycles per unit time. In music one cycle per second is referred to as one *hertz*. Middle C on a piano is typically 263 hertz (i.e. when someone presses the middle C key, a piano string is struck that vibrates predominantly at 263 cycles per second), and the A above middle C is 440 hertz. The solution function when *n* is chosen to equal 1 is called the ***fundamental mode*** (for a particular length string under a specific tension). The other ***normal modes*** are represented by different values of *n*. For instance one gets the 2nd and 3rd normal modes when *n* is selected to equal 2 and 3, respectively. The fundamental mode, when *n* equals 1 represents the simplest possible oscillation pattern of the string, when the whole string swings back and forth in one wide swing. In this fundamental mode the widest vibration displacement occurs in the center of the string (see the figures below).



Thus suppose a string of length *l*, and string mass per unit length δ, is tightened so that the values of *T*, the string tension, along the other constants make the value of $\lambda_1 = \frac{\sqrt{T}}{2l\sqrt{\delta}}$ equal to 440. Then if the string is made to vibrate by striking or plucking it, then its fundamental (lowest) tone would be the A above middle C.

Now think about how different values of *n* affect the other part of $u_n(x,t) = F(x)G(t)$, namely $F(x) = \sin\left(\frac{n\pi}{l}x\right)$. Since $\sin\left(\frac{n\pi}{l}x\right)$ function vanishes whenever *x* equals a multiple of $\frac{l}{n}$, then selecting different values of *n* higher than 1 has the effect of identifying which parts of the vibrating string do not move. This has the affect musically of producing *overtones*, which are musically pleasing higher tones relative to the fundamental mode tone. For instance picking $n = 2$ produces a vibrating string that appears to have two separate vibrating sections, with the middle of the string standing still. This mode produces a tone exactly an octave above the fundamental mode. Choosing $n = 3$ produces the 3rd normal mode that sounds like an octave and a fifth above the original fundamental mode tone, then 4th normal mode sounds an octave plus a fifth plus a major third, above the fundamental tone, and so on.

It is this series of fundamental mode tones that gives the basis for much of the tonal scale used in Western music, which is based on the premise that the lower the fundamental mode differences, down to octaves and fifths, the more pleasing the relative sounds. Think about that the next time you listen to some Dave Matthews!

Finally note that in real life, any time a guitar or violin string is caused to vibrate, the result is typically a combination of normal modes, so that the vibrating string produces sounds from many different overtones. The particular combination resulting from a particular set-up, the type of string used, the way the string is plucked or bowed, produces the characteristic tonal quality associated with that instrument. The way in which these different modes are combined makes it possible to produce solutions to the wave equation with different initial shapes and initial velocities of the string. This process of combination involves ***Fourier Series*** which will be covered at the end of Math 21b (come back to see it in action!)

Finally, finally, note that the solutions to the wave equations also show up when one considers acoustic waves associated with columns of air vibrating inside pipes, such as in organ pipes, trombones, saxophones or any other wind instruments (including, although you might not have thought of it in this way, your own voice, which basically consists of a vibrating wind-pipe, i.e. your throat!). Thus the same considerations in terms of fundamental tones, overtones and the characteristic tonal quality of an instrument resulting from solutions to the wave equation also occur for any of these instruments as well. So, the wave equation gets around quite a bit musically!

D'Alembert's Solution of the Wave Equation

As was mentioned previously, there is another way to solve the wave equation, found by Jean Le Rond D'Alembert in the 18th century. In the last section on the solution to the wave equation using the separation of variables technique, you probably noticed that although we made use of the boundary conditions in finding the solutions to the PDE, we glossed over the issue of the initial conditions, until the very end when we claimed that one could make use of something called Fourier Series to build up combinations of solutions. If you recall, being given specific initial conditions meant being given both the shape of the string at time $t = 0$, i.e. the function $u(x,0) = f(x)$, as well as the initial velocity, $u_t(x,0) = g(x)$ (note that these two initial condition functions are functions of x alone, as t is set equal to 0). In the separation of variables solution, we ended up with an infinite set, or family, of solutions, $u_n(x,t)$ that we said could be combined in such a way as to satisfy any reasonable initial conditions.

In using D'Alembert's approach to solving the same wave equation, we don't need to use Fourier series to build up the solution from the initial conditions. Instead, we are able to explicitly construct solutions to the wave equation for any (reasonable) given initial condition functions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.

The technique involves changing the original PDE into one that can be solved by a series of two simple single variable integrations by using a special transformation of variables. Suppose that instead of thinking of the original PDE

$$(1) \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

in terms of the variables *x*, and *t*, we rewrite it to reflect two new variables

(2) $v = x + ct$ and $z = x - ct$

This then means that *u*, originally a function of *x*, and *t*, now becomes a function of *v* and *z*, instead. How does this work? Note that we can solve for *x* and *t* in (2), so that

(3) $x = \frac{1}{2}(v+z)$ and $t = \frac{1}{2c}(v-z)$

Now using the chain rule for multivariable functions, you know that

(4) $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} = c\frac{\partial u}{\partial v} - c\frac{\partial u}{\partial z}$$

since $\frac{\partial v}{\partial t} = c$ and $\frac{\partial z}{\partial t} = -c$, and that similarly

(5) $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial u}{\partial v} + \frac{\partial u}{\partial z}$$

since $\frac{\partial v}{\partial x} = 1$ and $\frac{\partial z}{\partial x} = 1$. Working up to second derivatives, another, more involved application of the chain rule yields that

(6) $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(c\frac{\partial u}{\partial v} - c\frac{\partial u}{\partial z}\right) = c\left(\frac{\partial^2 u}{\partial v^2}\frac{\partial v}{\partial t} + \frac{\partial^2 u}{\partial z\partial v}\frac{\partial z}{\partial t}\right) - c\left(\frac{\partial^2 u}{\partial z^2}\frac{\partial z}{\partial t} + \frac{\partial^2 u}{\partial v\partial z}\frac{\partial v}{\partial t}\right)$$

$$= c^2\left(\frac{\partial^2 u}{\partial v^2} - \frac{\partial^2 u}{\partial z\partial v}\right) + c^2\left(\frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial v\partial z}\right) = c^2\left(\frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial z\partial v} + \frac{\partial^2 u}{\partial z^2}\right)$$

Another almost identical computation using the chain rule results in the fact that

(7) $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial z}\right) = \left(\frac{\partial^2 u}{\partial v^2}\frac{\partial v}{\partial x} + \frac{\partial^2 u}{\partial z\partial v}\frac{\partial z}{\partial x}\right) + \left(\frac{\partial^2 u}{\partial z^2}\frac{\partial z}{\partial x} + \frac{\partial^2 u}{\partial v\partial z}\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial z\partial v} + \frac{\partial^2 u}{\partial z^2}$$

Now we revisit the original wave equation

(8) $$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

and substitute in what we have calculated for $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^2 u}{\partial x^2}$ in terms of $\frac{\partial^2 u}{\partial v^2}$, $\frac{\partial^2 u}{\partial z^2}$ and $\frac{\partial^2 u}{\partial z\partial v}$. Doing this gives the following equation, ripe with cancellations:

(9) $$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial v^2} - 2\frac{\partial^2 u}{\partial z\partial v} + \frac{\partial^2 u}{\partial z^2}\right) = c^2\frac{\partial^2 u}{\partial x^2} = c^2\left(\frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial z\partial v} + \frac{\partial^2 u}{\partial z^2}\right)$$

Dividing by c^2 and canceling the terms involving $\frac{\partial^2 u}{\partial v^2}$ and $\frac{\partial^2 u}{\partial z^2}$ reduces this series of equations to

$$(10) \qquad -2\frac{\partial^2 u}{\partial z \partial v} = +2\frac{\partial^2 u}{\partial z \partial v}$$

which means that

$$(11) \qquad \frac{\partial^2 u}{\partial z \partial v} = 0$$

So what, you might well ask, after all, we still have a second order PDE, and there are still several variables involved. But wait, think about what (11) implies. Picture (11) as it gives you information about the partial derivative of a partial derivative:

$$(12) \qquad \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial v}\right) = 0$$

In this form, this implies that $\frac{\partial u}{\partial v}$ considered as a function of *z* and *v* is a constant in terms of the variable *z*, so that $\frac{\partial u}{\partial v}$ can only depend on *v*, i.e.

$$(13) \qquad \frac{\partial u}{\partial v} = M(v)$$

Now, integrating this equation with respect to *v* yields that

$$(14) \qquad u(v,z) = \int M(v)dv$$

This, as an indefinite integral, results in a constant of integration, which in this case is just constant from the standpoint of the variable *v*. Thus, it can be any arbitrary function of *z* alone, so that actually

$$(15) \qquad u(v,z) = \int M(v)dv + N(z) = P(v) + N(z)$$

where $P(v)$ is a function of *v* alone, and $N(z)$ is a function of *z* alone, as the notation indicates.

Substituting back the original change of variable equations for *v* and *z* in (2) yields that

$$(16) \qquad u(x,t) = P(x+ct) + N(x-ct)$$

where *P* and *N* are arbitrary single variable functions. This is called D'Alembert's solution to the wave equation. Except for the somewhat annoying but easy enough chain rule computations, this was a pretty straightforward solution technique. The reason it worked so well in this case was the fact that the change of variables used in (2) were carefully selected so as to turn the original PDE into one in which the variables basically had no interaction, so that the original second order PDE could be solved by a series of two single variable integrations, which was easy to do.

Check out that D'Alembert's solution really works. According to this solution, you can pick any functions for P and N such as $P(v) = v^2$ and $N(v) = v + 2$. Then

(17) $$u(x,t) = (x+ct)^2 + (x-ct) + 2 = x^2 + x + ct + c^2t^2 + 2$$

Now check that

(18) $$\frac{\partial^2 u}{\partial t^2} = 2c^2$$

and that

(19) $$\frac{\partial^2 u}{\partial x^2} = 2$$

so that indeed

(20) $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

and so this is in fact a solution of the original wave equation.

This same transformation trick can be used to solve a fairly wide range of PDEs. For instance one can solve the equation

(21) $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y^2}$$

by using the transformation of variables

(22) $v = x$ and $z = x + y$

(Try it out! You should get that $u(x, y) = P(x) + N(x + y)$ with arbitrary functions P and N)

Note that in our solution (16) to the wave equation, nothing has been specified about the initial and boundary conditions yet, and we said we would take care of this time around. So now we take a look at what these conditions imply for our choices for the two functions P and N.

If we were given an initial function $u(x,0) = f(x)$ along with initial velocity function $u_t(x,0) = g(x)$ then we can match up these conditions with our solution by simply substituting in $t = 0$ into (16) and follow along. We start first with a simplified set-up, where we assume that we are given the initial displacement function $u(x,0) = f(x)$, and that the initial velocity function $g(x)$ is equal to 0 (i.e. as if someone stretched the string and simply released it without imparting any extra velocity over the string tension alone).

Now the first initial condition implies that

(23) $u(x,0) = P(x + c \cdot 0) + N(x - c \cdot 0) = P(x) + N(x) = f(x)$

We next figure out what choosing the second initial condition implies. By working with an initial condition that $u_t(x,0) = g(x) = 0$, we see that by using the chain rule again on the functions *P* and *N*

(24) $u_t(x,0) = \frac{\partial}{\partial t}\left(P(x+ct) + N(x-ct)\right) = cP'(x+ct) - cN'(x-ct)$

(remember that *P* and *N* are just single variable functions, so the derivative indicated is just a simple single variable derivative with respect to their input). Thus in the case where $u_t(x,0) = g(x) = 0$, then

(25) $cP'(x+ct) - cN'(x+ct) = 0$

Dividing out the constant factor *c* and substituting in $t = 0$

(26) $P'(x) = N'(x)$

and so $P(x) + k = N(x)$ for some constant *k*. Combining this with the fact that $P(x) + N(x) = f(x)$, means that $2P(x) + k = f(x)$, so that $P(x) = \left(f(x) - k\right)/2$ and likewise $N(x) = \left(f(x) + k\right)/2$. Combining these leads to the solution

(27) $u(x,t) = P(x+ct) + N(x-ct) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right)$

To make sure that the boundary conditions are met, we need

(28) $u(0,t) = 0$ and $u(l,t) = 0$ for all values of *t*

The first boundary condition implies that

(29) $u(0,t) = \frac{1}{2}\left(f(ct) + f(-ct)\right) = 0$

or

(30) $f(-ct) = -f(ct)$

so that to meet this condition, then the initial condition function *f* must be selected to be an odd function. The second boundary condition that $u(l,t) = 0$ implies

(31) $u(l,t) = \frac{1}{2}\left(f(l+ct) + f(l-ct)\right) = 0$

so that $f(l+ct) = -f(l-ct)$. Next, since we've seen that *f* has to be an odd function, then $-f(l-ct) = f(-l+ct)$. Putting this all together this means that

(32) $f(l+ct) = f(-l+ct)$ for all values of *t*

which means that *f* must have period 2*l*, since the inputs vary by that amount. Remember that this just means the function repeats itself every time 2*l* is added to the input, the same way that the sine and cosine functions have period 2π.

What happens if the initial velocity isn't equal to 0? Thus suppose $u_t(x,0) = g(x) \neq 0$. Tracing through the same types of arguments as the above leads to the solution function

(33) $$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

In the next installment of this introduction to PDEs we will turn to the ***Heat Equation***.

Heat Equation

For this next PDE, we create a mathematical model of how heat spreads, or diffuses through an object, such as a metal rod, or a body of water. To do this we take advantage of our knowledge of vector calculus and the divergence theorem to set up a PDE that models such a situation. Knowledge of this particular PDE can be used to model situations involving many sorts of diffusion processes, not just heat. For instance the PDE that we will derive can be used to model the spread of a drug in an organism, of the diffusion of pollutants in a water supply.

The key to this approach will be the observation that heat tends to flow in the direction of decreasing temperature. The bigger the difference in temperature, the faster the heat flow, or heat loss (remember Newton's heating and cooling differential equation). Thus if you leave a hot drink outside on a freezing cold day, then after ten minutes the drink will be a lot colder than if you'd kept the drink inside in a warm room - this seems pretty obvious!

If the function $u(x, y, z, t)$ gives the temperature at time *t* at any point (*x, y, z*) in an object, then in mathematical terms the direction of fastest decreasing temperature away from a specific point (*x, y, z*), is just the gradient of *u* (calculated at the point (*x, y, z*) and a particular time *t*). Note that here we are considering the gradient of *u* as just being with respect to the spatial coordinates *x, y* and *z*, so that we write

(1) $$grad(u) = \nabla u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}$$

Thus the rate at which heat flows away (or toward) the point is proportional to this gradient, so that if **F** is the vector field that gives the velocity of the heat flow, then

(2) $$\mathbf{F} = -k(grad(u))$$

(negative as the flow is in the direction of fastest *decreasing* temperature).

The constant, k, is called the *thermal conductivity* of the object, and it determines the rate at which heat is passed through the material that the object is made of. Some metals, for instance, conduct heat quite rapidly, and so have high values for k, while other materials act more like insulators, with a much lower value of k as a result.

Now suppose we know the temperature function, $u(x, y, z, t)$, for an object, but just at an initial time, when $t = 0$, i.e. we just know $u(x, y, z, 0)$. Suppose we also know the thermal conductivity of the material. What we would like to do is to figure out how the temperature of the object, $u(x, y, z, t)$, changes over time. The goal is to use the observation about the rate of heat flow to set up a PDE involving the function $u(x, y, z, t)$ (i.e. the Heat Equation), and then solve the PDE to find $u(x, y, z, t)$.

Deriving the Heat Equation

To get to a PDE, the easiest route to take is to invoke something called the Divergence Theorem. As this is a multivariable calculus topic that we haven't even gotten to at this point in the semester, don't worry! (It will be covered in the vector calculus section at the end of the course in Chapter 13 of Stewart). It's such a neat application of the use of the Divergence Theorem, however, that at this point you should just skip to the end of this short section and take it on faith that we will get a PDE in this situation (i.e. skip to equation (10) below. Then be sure to come back and read through this section once you've learned about the divergence theorem.

First notice if E is a region in the body of interest (the metal bar, the pool of water, etc.) then the amount of heat that leaves E per unit time is simply a surface integral. More exactly, it is the flux integral over the surface of E of the heat flow vector field, **F**. Recall that **F** is the vector field that gives the velocity of the heat flow - it's the one we wrote down as $\mathbf{F} = -k\nabla u$ in the previous section. Thus the amount of heat leaving E per unit time is just

(1) $$\iint_{S} \mathbf{F} \cdot d\mathbf{S}$$

where$S$ is the surface of E. But wait, we have the highly convenient divergence theorem that tells us that

(2) $$\iint_{S} \mathbf{F} \cdot d\mathbf{S} = -k \iiint_{E} div(grad(u))dV$$

Okay, now what is *div(grad(u))*? Given that

(3) $$grad(u) = \nabla u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}$$

then*div(grad(u))* is just equal to

(4) $$div(grad(u)) = \nabla \cdot \left(\nabla u\right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Incidentally, this combination of divergence and gradient is used so often that it's given a name, the *Laplacian*. The notation $div(grad(u) = \nabla \cdot \left(\nabla u\right)$ is usually shortened up to simply $\nabla^2 u$. So we could rewrite (2), the heat leaving region E per unit time as

(5) $$\iint_{\mathbf{S}} \mathbf{F} \cdot d\mathbf{S} = -k \iiint_E (\nabla^2 u) dV$$

On the other hand, we can calculate the total amount of heat, *H*, in the region, *E*, at a particular time, *t*, by computing the triple integral over *E*:

(6) $$H = \iiint_E (\sigma\delta) u(x, y, z, t) dV$$

where δ is the *density* of the material and the constant σ is the *specific heat* of the material (don't worry about all these extra constants for now - we will lump them all together in one place in the end). How does this relate to the earlier integral? On one hand (5) gives the rate of heat leaving *E* per unit time. This is just the same as $-\frac{\partial H}{\partial t}$, where *H* gives the total amount of heat in *E*. This means we actually have two ways to calculate the same thing, because we can calculate $\frac{\partial H}{\partial t}$ by differentiating equation (6) giving *H*, i.e.

(7) $$-\frac{\partial H}{\partial t} = -\iiint_E (\sigma\delta) \frac{\partial u}{\partial t} dV$$

Now since both (5) and (7) give the rate of heat leaving *E* per unit time, then these two equations must equal each other, so…

(8) $$-\frac{\partial H}{\partial t} = -\iiint_E (\sigma\delta) \frac{\partial u}{\partial t} dV = -k \iiint_E (\nabla^2 u) dV$$

For these two integrals to be equal means that their two integrands must equal each other (since this integral holds over any arbitrary region *E* in the object being studied), so…

(9) $$(\sigma\delta) \frac{\partial u}{\partial t} = k(\nabla^2 u)$$

or, if we let $c^2 = \frac{k}{\sigma\delta}$, and write out the Laplacian, $\nabla^2 u$, then this works out simply as

(10) $$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

This, then, is the PDE that models the diffusion of heat in an object, i.e. the Heat Equation! This particular version (10) is the ***three-dimensional heat equation.***

Solving the Heat Equation in the one-dimensional case

We simplify our heat diffusion modeling by considering the specific case of heat flowing in a long thin bar or wire, where the cross-section is very small, and constant, and insulated in such a way that the heat flow is just along the length of the bar or wire. In this slightly contrived situation, we can model the heat flow by keeping track of the temperature at any point along the bar using just one spatial dimension, measuring the position along the bar.

This means that the function, *u*, that keeps track of the temperature, just depends on *x*, the position along the bar, and *t*, time, and so the heat equation from the previous section becomes the so-called ***one-dimensional heat equation***:

(1) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

One of the interesting things to note at this point is how similar this PDE appears to the wave equation PDE. However, the resulting solution functions are remarkably different in nature. Remember that the solutions to the wave equation had to do with oscillations, dealing with vibrating strings and all that. Here the solutions to the heat equation deal with temperature flow, not oscillation, so that means the solution functions will likely look quite different. If you're familiar with the solution to Newton's heating and cooling differential equations, then you might expect to see some type of exponential decay function as part of the solution function.

Before we start to solve this equation, let's mention a few more conditions that we will need to know to nail down a specific solution. If the metal bar that we're studying has a specific length, *l*, then we need to know the temperatures at the ends of the bars. These temperatures will give us boundary conditions similar to the ones we worked with for the wave equation. To make life a bit simpler for us as we solve the heat equation, let's start with the case when the ends of the bar, at $x = 0$ and $x = l$ both have temperature equal to 0 for all time (you can picture this situation as a metal bar with the ends stuck against blocks of ice, or some other cooling apparatus keeping the ends exactly at 0 degrees). Thus we will be working with the same boundary conditions as before, namely

(2) $$u(0,t) = 0 \text{ and } u(l,t) = 0 \quad \text{for all values of } t$$

Finally, to pick out a particular solution, we also need to know the initial starting temperature of the entire bar, namely we need to know the function $u(x,0)$. Interestingly, that's all we would need for an initial condition this time around (recall that to specify a particular solution in the wave equation we needed to know two initial conditions, $u(x,0)$ and $u_t(x,0)$).

The nice thing now is that since we have already solved a PDE, then we can try following the same basic approach as the one we used to solve the last PDE, namely separation of variables. With any luck, we will end up solving this new PDE. So, remembering back to what we did in that case, let's start by writing

(3) $$u(x,t) = F(x)G(t)$$

where*F*, and *G*, are single variable functions. Differentiating this equation for $u(x,t)$ with respect to each variable yields

(4) $$\frac{\partial^2 u}{\partial x^2} = F''(x)G(t) \text{ and } \frac{\partial u}{\partial t} = F(x)G'(t)$$

When we substitute these two equations back into the original heat equation

(5) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

we get

$$(6)\qquad \frac{\partial u}{\partial t} = F(x)G'(t) = c^2\frac{\partial^2 u}{\partial x^2} = c^2 F''(x)G(t)$$

If we now separate the two functions *F* and *G* by dividing through both sides, then we get

$$(7)\qquad \frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)}$$

Just as before, the left-hand side only depends on the variable *t*, and the right-hand side just depends on *x*. As a result, to have these two be equal can only mean one thing, that they are both equal to the same constant, *k*:

$$(8)\qquad \frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k$$

As before, let's first take a look at the implications for $F(x)$ as the boundary conditions will again limit the possible solution functions. From (8) we get that $F(x)$ has to satisfy

$$(9)\qquad F''(x) - kF(x) = 0$$

Just as before, one can consider the various cases with *k* being positive, zero, or negative. Just as before, to meet the boundary conditions, it turns out that *k* must in fact be negative (otherwise $F(x)$ ends up being identically equal to 0, and we end up with the trivial solution $u(x,t) = 0$). So skipping ahead a bit, let's assume we have figured out that *k* must be negative (you should check the other two cases just as before to see that what we've just written is true!). To indicate this, we write, as before, that $k = -\omega^2$, so that we now need to look for solutions to

$$(10)\qquad F''(x) + \omega^2 F(x) = 0$$

These solutions are just the same as before, namely the general solution is:

$$(11)\qquad F(x) = A\cos(\omega x) + B\sin(\omega x)$$

where again *A* and *B* are constants and now we have $\omega = \sqrt{-k}$. Next, let's consider the boundary conditions $u(0,t) = 0$ and $u(l,t) = 0$. These are equivalent to stating that $F(0) = F(l) = 0$. Substituting in 0 for *x* in (11) leads to

$$(12)\qquad F(0) = A\cos(0) + B\sin(0) = A = 0$$

so that $F(x) = B\sin(\omega x)$. Next, consider $F(l) = B\sin(\omega l) = 0$. As before, we check that *B* can't equal 0, otherwise $F(x) = 0$ which would then mean that $u(x,t) = F(x)G(t) = 0 \cdot G(t) = 0$, the trivial solution, again. With $B \neq 0$, then it must be the case that $\sin(\omega l) = 0$ in order to have $B\sin(\omega l) = 0$. Again, the only way that this can happen is for ωl to be a multiple of π. This means that once again

(13) $\omega l = n\pi$ or $\omega = \frac{n\pi}{l}$ (where *n* is an integer)

and so

(14) $F(x) = \sin\left(\frac{n\pi}{l}x\right)$

where*n* is an integer. Next we solve for $G(t)$, using equation (8) again. So, rewriting (8), we see that this time

(15) $G'(t) + \lambda_n{}^2 G(t) = 0$

where $\lambda_n = \frac{cn\pi}{l}$, since we had originally written $k = -\omega^2$, and we just determined that $\omega = \frac{n\pi}{l}$ during the solution for $F(x)$. The general solution to this first order differential equation is just

(16) $G(t) = Ce^{-\lambda_n{}^2 t}$

So, now we can put it all together to find out that

(17) $u(x,t) = F(x)G(t) = C\sin\left(\frac{n\pi}{l}x\right)e^{-\lambda_n{}^2 t}$

Where*n* is an integer, *C* is an arbitrary constant, and $\lambda_n = \frac{cn\pi}{l}$. As is always the case, given a supposed solution to a differential equation, you should check to see that this indeed is a solution to the original heat equation, and that it satisfies the two boundary conditions we started with.

The next question is how to get from the general solution to the heat equation

(1) $u(x,t) = C\sin\left(\frac{n\pi}{l}x\right)e^{-\lambda_n{}^2 t}$

that we found in the last section, to a specific solution for a particular situation. How can one figure out which values of *n* and *C*are needed for a specific problem? The answer lies not in choosing one such solution function, but more typically it requires setting up an infinite series of such solutions. Such an infinite series, because of the principle of superposition, will still be a solution function to the equation, because the original heat equation PDE was linear and homogeneous. Using the superposition principle, and by summing together various solutions with carefully chosen values of *C*, then it is possible to create a specific solution function that will match any (reasonable) given starting temperature function $u(x,0)$.