INSTITUTE OF AERONAUTICAL ENGINEERING

(Autonomous)

Dundigal, Hyderabad -500 043

AERONAUTICAL ENGINEERING

COURSE HANDOUT

Course Name	**THEORY OF STRUCTURES**
Course Code	AAE002
Programme	B TECH
Regulation	IARE_R16
Semester	III SEMESTER
Course Coordinator	Dr. SUDHIR SASTRY.Y.B, Professor, AE.
Course Faculty	MR. T MAHESH KUMAR, Assistant Professor, AE

UNIT-I

INTRODUCTION

1. Mechanical properties of engineering materials

Often materials are subject to forces (loads) when they are used. Mechanical engineers calculate those forces and material scientists how materials deform (elongate, compress, and twist) or break as a function of applied load, time, temperature, and other conditions. Materials scientists learn about these mechanical properties by testing materials. Results from the tests depend on the size and shape of material to be tested (specimen), how it is held, and the way of performing the test. That is why we use common procedures, or *standards*.

The engineering tension test is widely used to provide basic design information on the strength of materials and as an acceptance test for the specification of materials. In the tension test a specimen is subjected to a continually increasing uniaxial tensile force while simultaneous observations are made of the elongation of the specimen. The parameters, which are used to describe the stress-strain curve of a metal, are the tensile strength, yield strength or yield point, percent elongation, and reduction of area**.** The first two are strength parameters; the last two indicate ductility.

In the tension test a specimen is subjected to a continually increasing uniaxial tensile force while simultaneous observations are made of the elongation of the specimen. An engineering stress-strain curve is constructed from the load elongation measurements. The tensile test is probably the simplest and most widely used test to characterize the mechanical properties of a material. The test is performed using a loading apparatus such as the Tinius Olsen machine. The capacity of this machine is 10,000 pounds (tension and compression). The specimen of a given material (i.e. steel, aluminum, cast iron) takes a cylindrical shape that is 2.0 in. long and 0.5 in. in diameter in its undeformed (with no permanent strain or residual stress), or original shape.

The results from the tensile test have direct design implications. Many common engineering structural components are designed to perform under tension. The truss is probably the most common example of a structure whose members are designed to be in tension (and compression).

2. Concepts of Stress and Strain

Stress can be defined by ratio of the perpendicular force applied to a specimen divided by its original cross sectional area, formally called engineering stress. To compare specimens of different sizes, the load is calculated per unit area, also called normalization to the area. Force divided by area is called stress. In tension and compression tests, the relevant area is that perpendicular to the force. In shear or torsion tests, the area is perpendicular to the axis of rotation. The stress is obtained by dividing the load (F) by the original area of the cross section of the specimen (A_O).

$$\sigma = \frac{F}{A_0}$$

The unit is the Megapascal = 10^6 Newtons/m^2.

There is a change in dimensions, or deformation elongation, *L* as a result of a tensile or compressive stress. To enable comparison with specimens of different length, the elongation is also normalized, this time to the length l_o. This is called strain. So, Strain is the ratio of change in length due to deformation to the original length of the specimen, formally called engineering strain. Strain is unit less, but often units of m/m (or mm/mm) are used

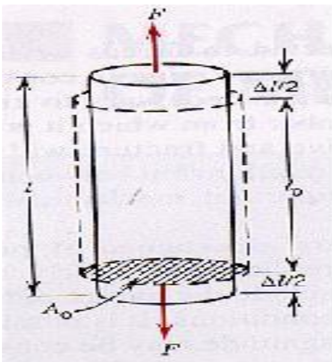
The strain used for the engineering stress-strain curve is the average linear strain, which is obtained by dividing the elongation of the gage length of the specimen, by its original length.

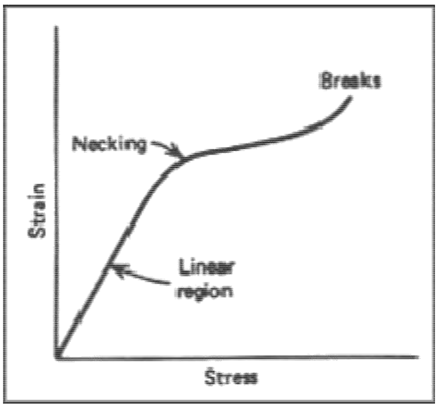
$$\varepsilon = \frac{l_i - l_o}{l_o} = \frac{\Delta l}{l_o}$$

Since both the stress and the strain are obtained by dividing the load and elongation by constant factors, the load-elongation curve will have the same shape as the engineering stress-strain curve. The two curves are frequently used interchangeably. The shape and magnitude of the stress-strain curve of a metal will depend on its composition, heat treatment, prior history of plastic deformation, and the strain rate, temperature, and state of stress imposed during the testing. The parameters used to describe stress-strain curve are tensile strength, yield strength or yield point, percent elongation, and reduction of area**.** The **f**irst two are strength parameters; the last two indicate ductility.

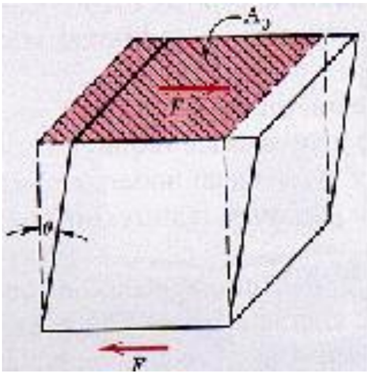
The general shape of the engineering stress-strain curve requires further explanation. In the elastic region stress is linearly proportional to strain. When the load exceeds a value corresponding to the yield strength, the specimen undergoes gross plastic deformation. It is permanently deformed if the load is released to zero. The stress to produce continued plastic deformation increases with increasing plastic strain, i.e., the metal strain-hardens. The volume of the specimen remains constant during plastic deformation, $A \cdot L = A_0 \cdot L_0$ and as the specimen elongates, it decreases uniformly along the gage length in cross-sectional area.

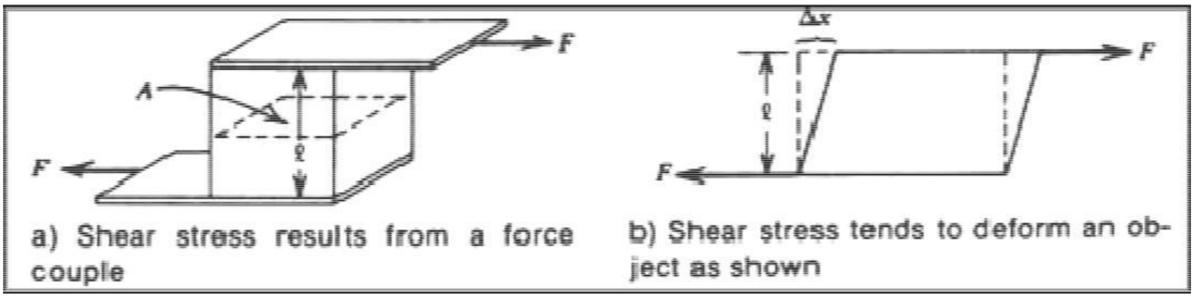
Initially the strain hardening more than compensates for this decrease in area and the engineering stress (proportional to load P) continues to rise with increasing strain. Eventually a point is reached where the decrease in specimen cross-sectional area is greater than the increase in deformation load arising from strain hardening. This condition will be reached first at some point in the specimen that is slightly weaker than the rest. All further plastic deformation is concentrated in this region, and the specimen begins to neck or thin down locally. Because the cross-sectional area now is decreasing far more rapidly than strain hardening increases the deformation load, the actual *load* required to deform the specimen falls off and the engineering stress likewise continues to decrease until fracture occurs.





Tensile and compression stress can be defined in terms of forces applied to a uniform rod.





Shear stress is defined in terms of a couple that tends to deform a joining member

A typical stress-strain curve showing the linear region, necking and eventual breaks.

Shear strain is defined as the tangent of the angle theta, and, in essence, determines to what extent the plane was displaced. In this case, the force is applied as a *couple* (that is, *not* along the same line), tending to shear off the solid object that separates the force arms.

In this case, the force is applied as a *couple* (that is, *not* along the same line), tending to shear off the solid object that separates the force arms. In this case, the stress is again the strain in this case is denned as the fractional change in dimension of the sheared member.

3. **Stress—Strain Behavior**
   3.1. **Hooke's Law**
   - for materials stressed in tension, at relatively low levels, stress and strain are proportional through:

$$\sigma = \mathbf{E}\varepsilon$$

- Constant E is known as the modulus of elasticity, or Young's modulus. Measured in MPa and can range in values from ~4.5x104 - 40x107 MPa

The engineering stress strain graph shows that the relationship between stress and strain is linear over some range of stress. If the stress is kept within the linear region, the material is essentially *elastic* in that if the stress is removed, the deformation is also gone. But if the elastic limit is exceeded, permanent deformation results. The material may begin to "neck" at some location and finally break. Within the linear region, a specific type of material will always follow the same curves despite different physical dimensions. Thus, it can say that the linearity and slope are a constant of the type of material only. In tensile and compression stress, this constant is called the *modulus of elasticity* or *Young's modulus (E).*

$$E = \frac{F/A}{\Delta l/l}$$

Where

Stress = F/A in N/m^2

E = Modulus of elasticity in N/m^2

The modulus of elasticity has units of stress, that is, N/m^2. The following table gives the modulus of elasticity for several materials. In an exactly similar fashion, the shear modulus is defined for shear stress-strain as modulus of elasticity.

Material	**Modulus (N/m^2)**
Aluminum	6.89 x 10^{10}
Copper	11.73 X 10'° 20.70 X 10^{10}
Steel	2.1 x 10^{8}

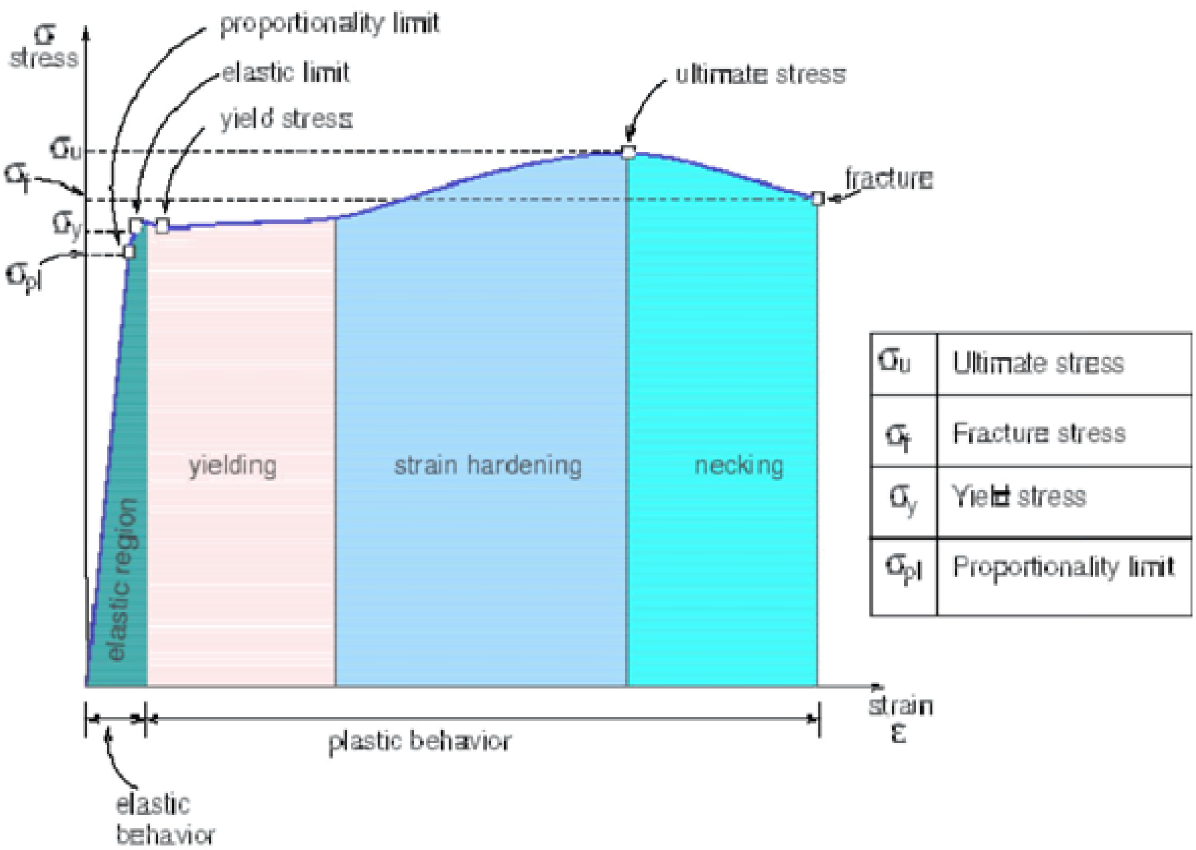
3.2. Stress-strain curve

The stress-strain curve characterizes the behavior of the material tested. It is most often plotted using engineering stress and strain measures, because the reference length and cross-sectional area are easily measured. Stress-strain curves generated from tensile test results help engineers gain insight into the constitutive relationship between stress and strain for a particular material. The constitutive relationship can be thought of as providing an answer to the following question: Given a strain history for a specimen, what is the state of stress? As we shall see, even for the simplest of materials, this relationship can be very complicated.

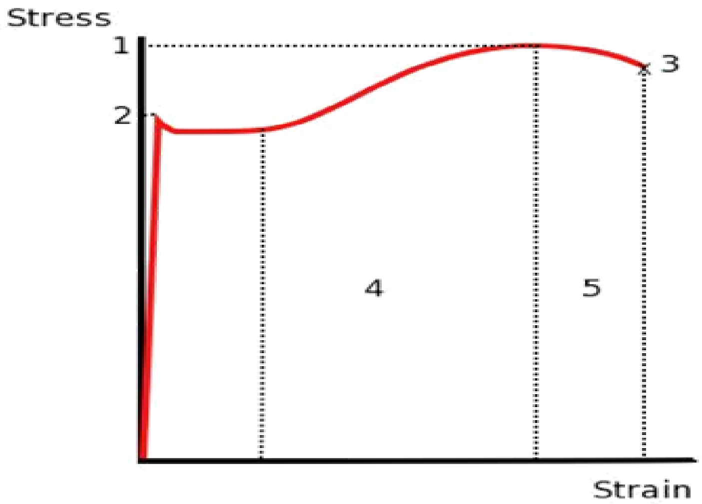
In addition to providing quantitative information that is useful for the constitutive relationship, the stress-strain curve can also be used to qualitatively describe and classify the material. Typical regions that can be observed in a stress-strain curve are:

1. Elastic region
2. Yielding
3. Strain Hardening
4. Necking and Failure

A stress-strain curve with each region identified is shown below. The curve has been sketched using the assumption that the strain in the specimen is monotonically increasing - no unloading occurs. It should also be emphasized that a lot of variation from what's shown is possible with real materials, and each of the above regions will not always be so clearly delineated. It should be emphasized that the extent of each region in stress-strain space is material dependent, and that not all materials exhibit all of the above regions. A stress-strain curve is a graph derived from measuring load (stress - σ) versus extension (strain - ε) for a sample of a material. The nature of the curve varies from material to material. The following diagrams illustrate the stress-strain behavior of typical materials in terms of the engineering stress and engineering strain where the stress and strain are calculated based on the original dimensions of the sample and not the instantaneous values. In each case the samples are loaded in tension although in many cases similar behavior is observed in compression.



Various regions and points on the stress-strain curve.



Stress vs. Strain curve for mild steel steel (Ductile material).

Reference numbers are:

1- Ultimate strength   2- Yield Strength   3- Rupture   4- Strain hardening region

5- Necking region

3.3. Brittle and Ductile Behavior

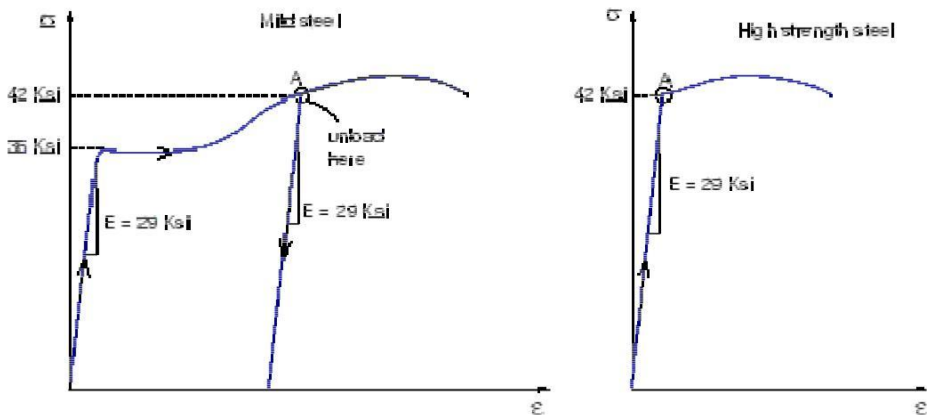
The behavior of materials can be broadly classified into two categories; brittle and ductile. Steel and aluminum usually fall in the class of ductile materials. Glass, ceramics, plain concrete and cast iron fall in the class of brittle materials. The two categories can be distinguished by comparing the stress-strain curves, such as the ones shown in Figure.

Ductile and brittle material behavior

The material response for ductile and brittle materials is exhibited by both qualitative and quantitative differences in their respective stress-strain curves. Ductile materials will withstand large strains before the specimen ruptures; brittle materials fracture at much lower strains. The yielding region for ductile materials often takes up the majority of the stress-strain curve, whereas for brittle materials it is nearly nonexistent. Brittle materials often have relatively large Young's moduli and ultimate stresses in comparison to ductile materials.

These differences are a major consideration for design. Ductile materials exhibit large strains and yielding before they fail. On the contrary, brittle materials fail suddenly and without much warning. Thus ductile materials such as steel are a natural choice for structural members in buildings as we desire considerable warning to be provided before a building fails. The energy absorbed (per unit volume) in the tensile test is simply the area under the stress strain curve. Clearly, by comparing the curves in Figure, It can be observed that ductile materials are capable of absorbing much larger quantities of energy before failure.

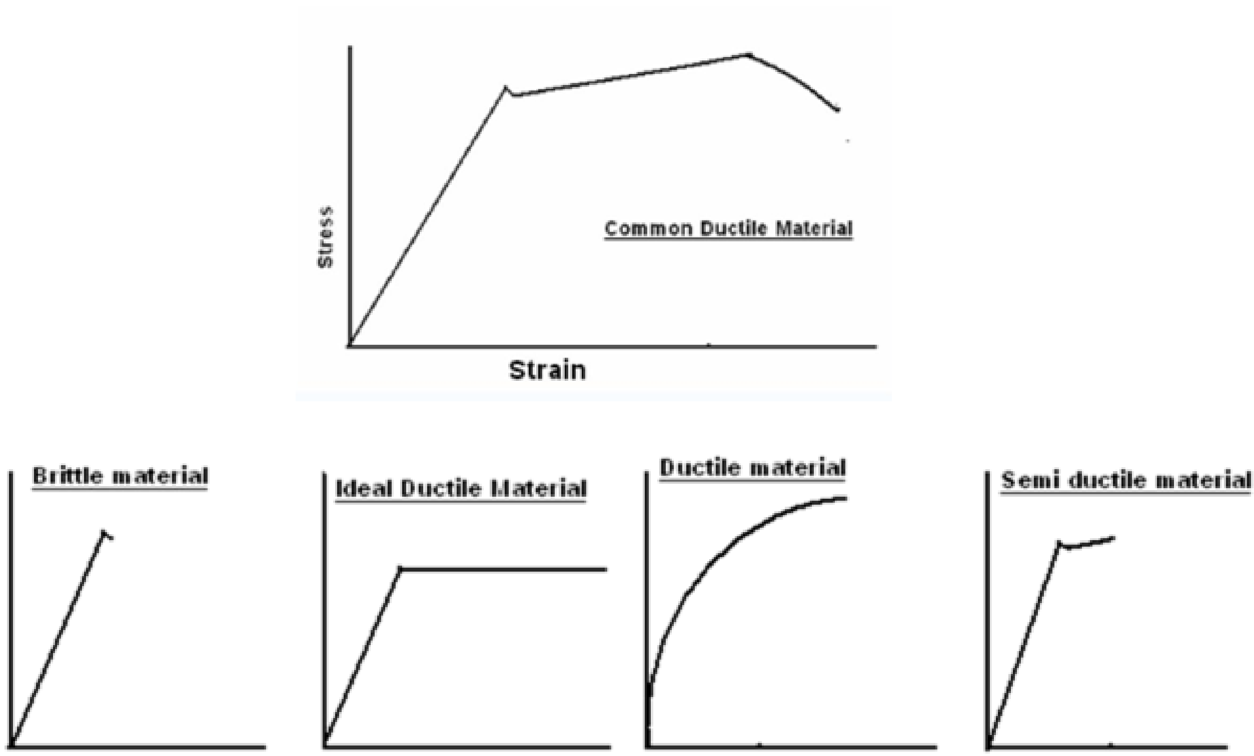
Finally, it should be emphasized that not all materials can be easily classified as either ductile or brittle. Material response also depends on the operating environment; many ductile materials become brittle as the temperature is decreased. With advances in metallurgy and composite technology, other materials are advanced combinations of ductile and brittle constituents. Often in structural design; structural members are designed to be in service below the yield stress. The reason being that once the load exceeds the yield limit, the structural members will exhibit large deformations (imagine for instance a roof sagging) that are undesirable. Thus materials with larger yield strength are preferable.



After work hardening, the stress-strain curve of mild steel (left) resembles that of high-strength steel (right).

We will for now concentrate on steel, a commonly used structural material. Mild steels have yield strength somewhere between 240 and 360 N/mm^2. When work-hardened, the yield strength of this steel increases. Work hardening is the process of loading mild steel beyond its yield point and unloading as shown in Figure. When the material is loaded again, the linear elastic behavior now extends up to point A as shown. The negative aspect of work hardening is some loss in ductility of the material. It is noteworthy that mild steel is usually recycled. Because of this, the yield strength may be a little higher than expected for the mild steel specimens tested in the laboratory.

Often in structural design, structural members are designed to be in service below the yield stress. The reason being that once the load exceeds the yield limit, the structural members will exhibit large deformations (imagine for instance a roof sagging) that are undesirable. Thus materials with larger yield strength are preferable. Generally, the stress strain distribution varies from a material to another and could be in different forms as follows. Consequently, the type of material and fracture pattern can be defined and determined according to its stress-strain distribution diagram.



Various stress-strain diagrams for different engineering materials

3.3. Yield strength

The yield point is defined in engineering and materials science as the stress at which a material begins to plastically deform. Prior to the yield point the material will deform elastically and will return to its original shape when the applied stress is removed. Once the yield point is passed some fraction of the deformation will be permanent and non-reversible. Knowledge of the yield point is vital when designing a component since it generally represents an upper limit to the load that can be applied. It is also important for the control of many materials production techniques such as forging, rolling, or pressing.

In structural engineering, **yield** is the permanent plastic deformation of a structural member under stress. This is a soft failure mode which does not normally cause catastrophic failure unless it accelerates buckling. It is often difficult to precisely define yield due to the wide variety of stress-strain behaviors exhibited by real materials. In addition there are several possible ways to define the yield point in a given material. Yield occurs when dislocations first begin to move. Given that dislocations begin to move at very low stresses, and the difficulty in detecting such movement, this definition is rarely used.

Elastic Limit: The lowest stress at which permanent deformation can be measured. This requires a complex iterative load-unload procedure and is critically dependent on the accuracy of the equipment and the skill of the operator.

Proportional Limit: The point at which the stress-strain curve becomes non-linear. In most metallic materials the elastic limit and proportional limit are essentially the same.

Offset Yield Point (proof stress): Due to the lack of a clear border between the elastic and plastic regions in many materials, the yield point is often defined as the stress at some arbitrary plastic strain (typically 0.2%). This is determined by the intersection of a line offset from the linear region by the required strain. In some materials there is essentially no linear region and so a certain value of plastic strain is defined instead. Although somewhat arbitrary this method does allow for a consistent comparison of materials and is the most common.

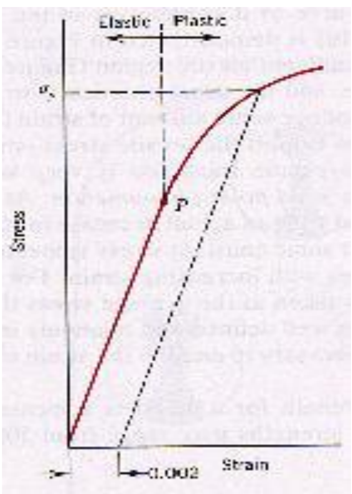
Yield point: If the stress is too large, the strain deviates from being proportional to the stress. The point at which this happens is the *yield point* because there the material yields, deforming permanently (plastically).

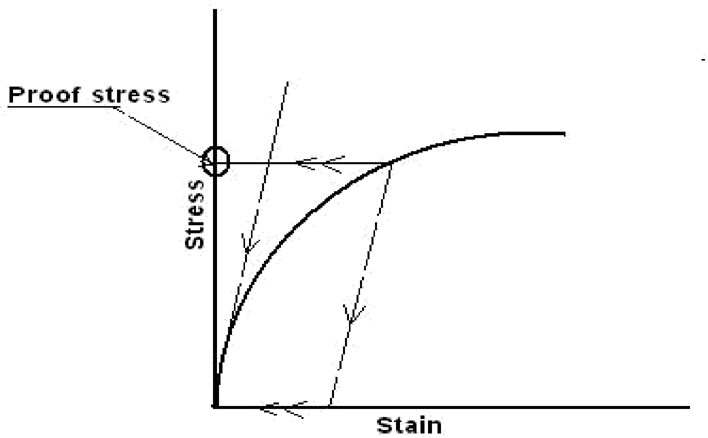
Yield stress: Hooke's law is not valid beyond the yield point. The stress at the yield point is called *yield stress*, and is an important measure of the mechanical properties of materials. In practice, the yield stress is chosen as that causing a permanent strain of 0.002, which called as **proof stress**. The yield stress measures the resistance to plastic deformation.

The ***yield strength*** is the stress required to produce a small-specified amount of plastic deformation. The usual definition of this property is the ***offset yield strength*** determined by the stress corresponding to the intersection of the stress-strain curve and a line parallel to the elastic part of the curve offset by a specified strain. In the United States the offset is usually specified as a strain of 0.2 or 0.1 percent (**e** = 0.002 or 0.001).

$$R_{p0.2} = \frac{P_{(strain offset = 0.002)}}{A_0}$$

A good way of looking at offset yield strength is that after a specimen has been loaded to its 0.2 percent offset yield strength and then unloaded it will be 0.2 percent longer than before the test. The offset yield strength is often referred to in Great Britain as the ***proof stress***, where offset values are either 0.1 or 0.5 percent. The yield strength obtained by an offset method is commonly used for design and specification purposes because it avoids the practical difficulties of measuring the elastic limit or proportional limit.



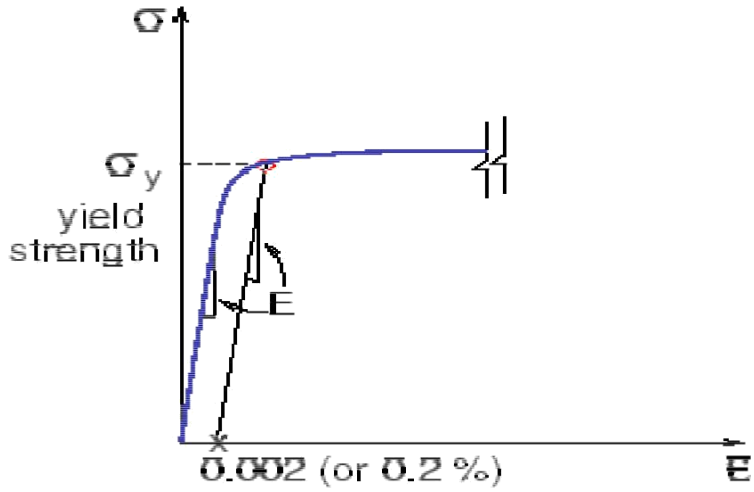


Determination of proof stress

Some materials have essentially no linear portion to their stress-strain curve, for example, soft copper or gray cast iron. For these materials the offset method cannot be used and the usual practice is to define the yield strength as the stress to produce some total strain, for example, **e** = 0.005.

Determination of Yield Strength in Ductile Materials

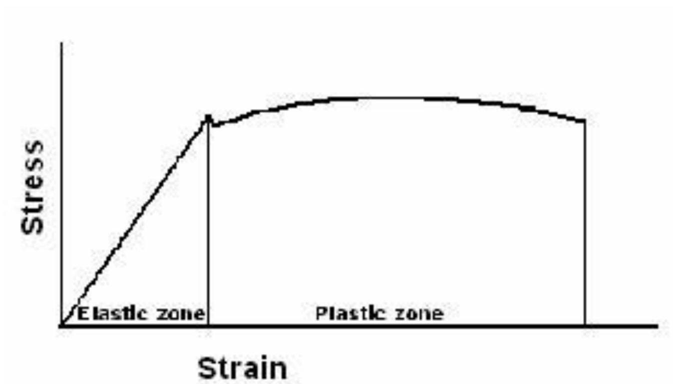
In many materials, the yield stress is not very well defined and for this reason a standard has been developed to determine its value. The standard procedure is to project a line parallel to the initial elastic region starting at 0.002 strains. The 0.002 strain point is often referred to as the 0.2% offset strain point. The intersection of this new line with the stress-strain curve then defines the *yield strength* as shown in Figure.



4. Elastic Properties of Materials

When the stress is removed, the material returns to the dimension it had before the load was applied. Valid for small strains (except the case of rubbers).

Deformation is *reversible, non permanent* Materials subject to tension shrink laterally. Those subject to compression, bulge. The ratio of lateral and axial strains is called the *Poisson's ratio*. When a material is placed under a tensile stress, an accompanying strain is created in the same direction.



Poisson's ratio is the ratio of the lateral to axial strains.

$$\nu = -\frac{\varepsilon_x}{\varepsilon_z} = \frac{\varepsilon_y}{\varepsilon_z}$$

The elastic modulus, shear modulus and Poisson's ratio are related by $E = 2G\ (1+ \nu)$

$$\mathbf{E} = 2\mathbf{G}(1+\nu)$$

- Theoretically, isotropic materials will have a value for Poisson's ratio of 0.25.
- The maximum value of ν is 0.5
- Most metals exhibit values between 0.25 and 0.35

1. **Plastic deformation.**

When the stress is removed, the material does not return to its previous dimension but there is a *permanent*, irreversible deformation. For metallic materials, elastic deformation only occurs to strains of about 0.005. After this point, plastic (non-recoverable) deformation occurs, and Hooke's Law is no longer valid. On an atomic level, plastic deformation is caused by *slip*, where atomic bonds are broken by dislocation motion, and new bonds are formed.

2. **Anelasticity**

Here the behavior is elastic but not the stress- strain curve is not immediately reversible. It takes a while for the strain to return to zero. The effect is normally small for metals but can be significant for polymers.

3. **Tensile strength**.

When stress continues in the plastic regime, the stress-strain passes through a maximum, called the *tensile strength* (s_{TS}) , and then falls as the material starts to develop a *neck* and it finally breaks at the *fracture point*.

Note that it is called strength, not stress, but the units are the same, MPa.

For structural applications, the yield stress is usually a more important property than the tensile strength, since once it is passed, the structure has deformed beyond acceptable limits. The tensile strength, or ultimate tensile strength (UTS), is the maximum load divided by the original cross-sectional area of the specimen.

The tensile strength is the value most often quoted from the results of a tension test; yet in reality it is a value of little fundamental significance with regard to the strength of a metal. For ductile metals the tensile strength should be regarded as a measure of the maximum load, which a metal can withstand under the very restrictive conditions of uniaxial loading. It will be shown that this value bears little relation to the useful strength of the metal under the more complex conditions of stress, which are usually encountered.

For many years it was customary to base the strength of members on the tensile strength, suitably reduced by a factor of safety. The current trend is to the more rational approach of basing the static design of ductile metals on the yield strength.

However, because of the long practice of using the tensile strength to determine the strength of materials, it has become a very familiar property, and as such it is a very useful identification of a material in the same sense that the chemical composition serves to identify a metal or alloy.

Further, because the tensile strength is easy to determine and is a quite reproducible property, it is useful for the purposes of specifications and for quality control of a product. Extensive empirical correlations between tensile strength and properties such as hardness and fatigue strength are often quite useful. For brittle materials, the tensile strength is a valid criterion for design.

4. Ductility

The ability to deform before braking. It is the opposite of **brittleness**. Ductility can be given either as percent maximum elongation e_{max} or maximum area reduction. At our present degree of understanding, ductility is a qualitative, subjective property of a material. In general, measurements of ductility are of interest in three ways:

1. To indicate the extent to which a metal can be deformed without fracture in metal working operations such as rolling and extrusion.
2. To indicate to the designer, in a general way, the ability of the metal to flow plastically before fracture. A high ductility indicates that the material is "forgiving" and likely to deform locally without fracture should the designer err in the stress calculation or the prediction of severe loads.
3. To serve as an indicator of changes in impurity level or processing conditions. Ductility measurements may be specified to assess material quality even though no direct relationship exists between the ductility measurement and performance in service.

The conventional measures of ductility that are obtained from the tension test are the engineering strain at fracture ef (usually called the *elongation*) and the *reduction of area* at fracture q. Both of these properties are obtained after fracture by putting the specimen back together and taking measurements of L_f and A_f .

Because an appreciable fraction of the plastic deformation will be concentrated in the necked region of the tension specimen, the value of $\mathbf{e}_f$ will depend on the gage length L_0 over which the measurement was taken. The smaller the gage length the greater will be the contribution to the overall elongation from the necked region and the higher will be the value of $\mathbf{e}_f$. Therefore, when reporting values of percentage elongation**,** the gage length L_0 always should be given.

The reduction of area does not suffer from this difficulty. Reduction of area values can be converted into an equivalent **zero-gage-length elongation e_0**. From the constancy of volume relationship for plastic deformation $A*L = A_0*L_0$, we obtain

$$\frac{L}{L_0} = \frac{A_0}{A} = \frac{1}{1-q}, \quad e_0 = \frac{L-L_0}{L_0} = \frac{A_0}{A} - 1 = \frac{1}{1-q} - 1 = \frac{q}{1-q}$$

This represents the elongation based on a very short gage length near the fracture.

Another way to avoid the complication from necking is to base the percentage elongation on the uniform strain out to the point at which necking begins. The uniform elongation $\mathbf{e}_u$ correlates well with stretch-forming operations. Since the engineering stress-strain curve often is quite flat in the vicinity of necking, it may be difficult to establish the strain at maximum load without ambiguity. In this case the method suggested by Nelson and Winlock is useful.

5. **Resilience**

The resilience of the material is the triangular area underneath the elastic region of the curve. **Resilience** generally means the ability to recover from (or to resist being affected by) some shock, insult, or disturbance. However, it is used quite differently in different fields. In physics and engineering, **resilience** is defined as the capacity of a material to absorb energy when it is deformed elastically and then, upon unloading to have this energy recovered. In other words, it is the maximum energy per volume that can be elastically stored. It is represented by the area under the curve in the elastic region in the Stress-Strain diagram.

Modulus of Resilience, U_r, can be calculated using the following formula:

$$U_r = \frac{\sigma^2}{2E} = 0.5\sigma\epsilon = 0.5\sigma\left(\frac{\sigma}{E}\right)$$

Where σ is yield stress, E is Young's modulus, and ε is strain.

The ability of a material to absorb energy when deformed elastically and to return it when unloaded is called resilience. This is usually measured by the modulus of resilience, which is the strain energy per unit volume required to stress the material from, zero stress to the yield stress. The ability of a material to absorb energy when deformed elastically and to return it when unloaded is called **resilience**. This is usually measured by the **modulus of resilience**, which is the strain energy per unit volume required to stress the material from, zero stress to the yield stress s. The strain energy per unit volume for uniaxial tension is

$$U_0 = \frac{}{2}\sigma_x e_x$$

Table 1 gives some values of modulus of resilience for different materials.

Table 1. Modulus of resilience for various materials

Material	E, psi	s_0, psi	Modulus of resilience, Ur
Medium-carbon steel	30×10^6	45000	33,7
High-carbon spring steel	30×10^6	140000	320
Duraluminium	$10{,}5\times10^6$	18000	17,0
Cooper	16×10^6	4000	5,3
Rubber	150	300	300
Acrylic polymer	$0{,}5\times10^6$	2000	4,0

6. Toughness

The area underneath the stress-strain curve is the toughness of the material- i.e. the energy the material can absorb prior to rupture.. It also can be defined as the resistance of a material to crack propagation. In materials science and metallurgy, **toughness** is the resistance to fracture of a material when stressed. It is defined as the amount of energy that a material can absorb before rupturing, and can be found by finding the area (i.e., by taking the integral) underneath the stress-strain curve.

The ability of a metal to deform plastically and to absorb energy in the process before fracture is termed toughness. The emphasis of this definition should be placed on the ability to absorb energy before fracture. Recall that ductility is a measure of how much something deforms plastically before fracture, but just because a material is ductile does not make it tough. The key to called “material toughness” and it has units of energy per volume. Material toughness equates to a slow absorption of energy by the material.

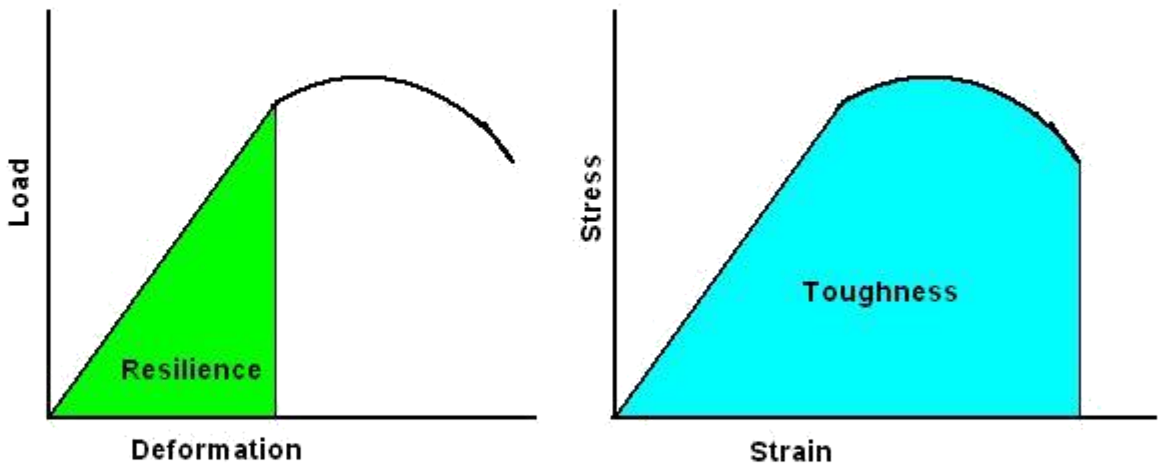
The toughness of a material is its ability to absorb energy in the plastic range. The ability to withstand occasional, stresses above the yield stress without fracturing is particularly desirable in parts such as freight-car couplings, gears, chains, and crane hooks. Toughness is a commonly used concept, which is difficult to pin down and define. One way of looking at toughness is to consider that it is the total area under the stress-strain curve. This area is an indication of the amount of work per unit volume, which can be done, on the material without causing it to rupture. The following Figure shows the stress-strain curves for high- and low-toughness materials. The high-carbon spring steel has a higher yield strength and tensile strength than the medium-carbon structural steel. However, the structural steel is more ductile and has a greater total elongation. The total area under the stress train curve is greater for the structural steel, and therefore it is a tougher material. This illustrates that toughness is a parameter that comprises **both** strength and ductility. The crosshatched regions in Figure indicate the modulus of resilience for each steel. Because of its higher yield strength, the spring steel has the greater resilience.

Several mathematical approximations for the area under the stress-strain curve have been suggested. For ductile metals that have a stress-strain curve like that of the structural steel, the area under the curve can be approximated by either of the following equations:

For brittle materials the stress-strain curve is sometimes assumed to be a parabola, and the area under the curve is given by

$$U_T \approx \frac{\;}{3} s_u e_f$$

All these relations are only approximations to the area under the stress-strain curves. Further, the curves do not represent the true behavior in the plastic range, since they are all based on the original area of the specimen.



Comparison between resilience and toughness of metals

Impact Toughness

Three of the toughness properties that will be discussed in more detail are 1) impact toughness, 2) notch toughness and 3) fracture toughness.

The impact toughness (AKA Impact strength) of a material can be determined with a Charpy or Izod test. These tests are named after their inventors and were developed in the early 1900's before fracture mechanics theory was available. Impact properties are not directly used in fracture mechanics calculations, but the economical impact tests continue to be used as a quality control method to assess

The two tests use different specimens and methods of holding the specimens, but both tests make use of a pendulum-testing machine. For both tests, the specimen is broken by a single overload event due to the impact of the pendulum. A stop pointer is used to record how far the pendulum swings back up after fracturing the specimen. The impact toughness of a metal is determined by measuring the energy absorbed in the fracture of the specimen. This is simply obtained by noting the height at which the pendulum is released and the height to which the pendulum swings after it has struck the specimen. The height of the pendulum times the weight of the pendulum produces the potential energy and the difference in potential energy of the pendulum at the start and the end of the test is equal to the absorbed energy.

Since toughness is greatly affected by temperature, a Charpy or Izod test is often repeated numerous times with each specimen tested at a different temperature. This produces a graph of impact toughness for the material as a function of temperature. Impact toughness versus temperature graph for steel is shown in the image. It can be seen that at low temperatures the material is more brittle and impact toughness is low. At high temperatures the material is more ductile and impact toughness is higher. The transition temperature is the boundary between brittle and ductile behavior and this temperature is often an extremely important consideration in the selection of a material.

6.1 Notch-Toughness

Notch toughness is the ability that a material possesses to absorb energy in the presence of a flaw. As mentioned previously, in the presence of a flaw, such as a notch or crack, a material will likely exhibit a lower level of toughness. When a flaw is present in a material, loading induces a triaxial tension stress state adjacent to the flaw. The material develops plastic strains as the yield stress is exceeded in the region near the crack tip. However, the amount of plastic deformation is restricted by the surrounding material, which remains elastic. When a material is prevented from deforming plastically, it fails in a brittle manner.

Notch-toughness is measured with a variety of specimens such as the Charpy V-notch impact specimen or the dynamic tear test specimen. As with regular impact testing the tests are often repeated numerous times with specimens tested at a different temperature. With these specimens and by varying the loading speed and the temperature, it is possible to generate curves such as those shown in the graph. Typically only static and impact testing is conducted but it should be recognized that many components in service see intermediate loading rates in the range of the dashed red line.

6.3. Fracture Toughness

In materials science, **fracture toughness** is a property which describes the ability of a material containing a crack to resist fracture, and is one of the most important properties of any material for virtually all design applications. It is denoted K_{1c} and has the units of $MPa\sqrt{m}$. The subscript '1c' denotes mode 1 crack opening or plain strain, the material has to be too thick to shear, mode 2, or tear, mode 3. Fracture toughness is a quantitative way of expressing a material's resistance to brittle fracture when a crack is present. If a material has a large value of fracture toughness it will probably undergo ductile fracture. Brittle fracture is very characteristic of materials with a low fracture toughness value. Fracture mechanics, which leads to the concept of fracture toughness, was largely based on the work of A. A. Griffith who, amongst other things, studied the behavior of cracks in brittle materials.

Fracture toughness is an indication of the amount of stress required to propagate a preexisting flaw. It is a very important material property since the occurrence of flaws is not completely avoidable in the processing, fabrication, or service of a material/component. Flaws may appear as cracks, voids, metallurgical inclusions, weld defects, design discontinuities, or some combination thereof. Since engineers can never be totally sure that a material is flaw free, it is common practice to assume that a flaw of some chosen size will be present in some number of components. This approach uses the flaw size and features, component geometry, loading conditions and the material property called fracture toughness to evaluate the ability of a component containing a flaw to resist fracture.

A parameter called the stress-intensity factor (K) is used to determine the fracture toughness of most materials. A Roman numeral subscript indicates the mode of fracture and the three modes of fracture are illustrated in the image to the right. Mode I fracture is the condition in which the crack plane is normal to the direction of largest tensile loading. This is the most commonly encountered mode and, therefore, for the remainder of the material we will consider K_I the stress intensity factor is a function of loading, crack size, and structural geometry. The stress intensity factor may be represented by the following equation:

$$K_I = \sigma\sqrt{\pi a \beta}$$

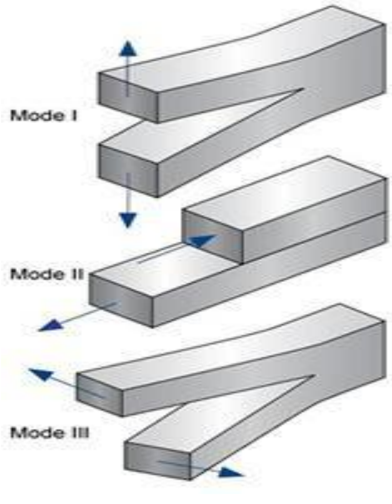
Where: K_I is the fracture toughness in $MPa\sqrt{m}$ $(psi\sqrt{in})$

σ is the applied stress in MPa or psi

a is the crack length in meters or inches

is a crack length and component geometry

B factor that is different for each specimen and is imensionless



There are several variables that have a profound influence on the toughness of a material. These variables are:

- Strain rate (rate of loading)
- Temperature
- Notch effect

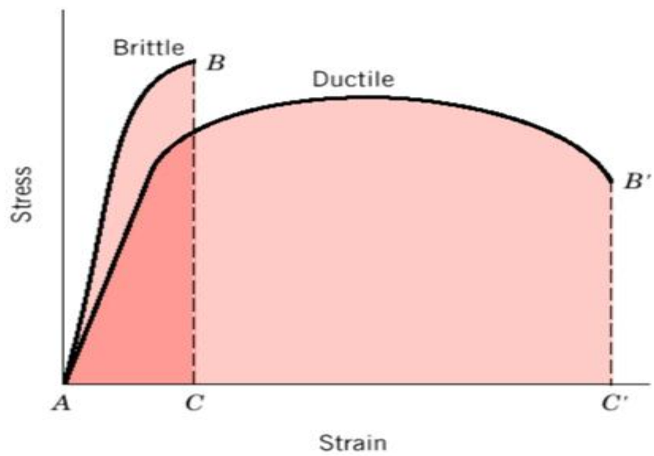
A metal may possess satisfactory toughness under static loads but may fail under dynamic loads or impact. As a rule ductility and, therefore, toughness decrease as the rate of loading increases. Temperature is the second variable to have a major influence on its toughness. As temperature is lowered, the ductility and toughness also decrease. The third variable is termed notch effect, has to do with the distribution of stress. A material might display good toughness when the applied stress is uniaxial; but when a multi axial stress state is produced due to the presence of a notch, the material might not withstand the simultaneous elastic and plastic deformation in the various directions.

There are several standard types of toughness test that generate data for specific loading conditions and/or component design approaches.

6.4 Material Types

Brittle materials such as concrete or ceramics do not have a yield point. For these materials the rupture strength and the ultimate strength are the same.

Ductile material (such as steel) generally exhibits a very linear stress-strain relationship up to a well defined yield point . The linear portion of the curve is the elastic region and the slope is the modulus of elasticity or Young's Modulus. After the yield point the curve typically decreases slightly due to dislocations escaping from Cottrell atmospheres. As deformation continues the stress increases due to strain hardening until it reaches the ultimate strength. Until this point the cross-sectional area decreases uniformly due to Poisson contractions. However, beyond this point a *neck* forms where the local cross-sectional area decreases more quickly than the rest of the sample resulting in an increase in the true stress. On an engineering stress-strain curve this is seen as a decrease in the stress. Conversely, if the curve is plotted in terms of *true stress* and *true strain* the stress will continue to rise until failure. Eventually the neck becomes unstable and the specimen ruptures (fractures).



Most ductile metals other than steel do not have a well-defined yield point. For these materials the yield strength is typically determined by the "offset yield method", by which a line is drawn parallel to the linear elastic portion of the curve and intersecting the abscissa at some arbitrary value (most commonly .2%). The intersection of this line and the stress-strain curve is reported as the yield point.

a- **Ductile materials** - extensive plastic deformation and energy absorption “toughness”) before fracture. Ductile materials can be classified into various classifications; 1- Very ductile, soft metals (e.g. Pb, Au) at room temperature, other metals, polymers, glasses at high temperature., 2- Moderately ductile fracture, typical for ductile metals, 3- Brittle fracture, cold metals, ceramics.

b- **Brittle materials** – has a little plastic deformation and low energy absorption before fracture

7. Elastic Recovery During Plastic Deformation

If a material is taken beyond the yield point (it is deformed plastically) and the stress is then released, the material ends up with a permanent strain. If the stress is reapplied, the material again responds elastically at the beginning up to a new yield point *that is higher than the original yield point* (strain hardening). The amount of elastic strain that it will take before reaching the yield point is called *elastic strain recovery*.

8. Compressive, Shear, and Torsional Deformation

Compressive and shear stresses give similar behavior to tensile stresses, but in the case of compressive stresses there is no maximum in the s-e curve, since no necking occurs.

9. Hardness

Hardness is the resistance to plastic deformation (e.g., a local dent or scratch). Thus, it is a measure of *plastic* deformation, as is the tensile strength, so they are well correlated. Historically, it was measured on an empirically scale, determined by the ability of a material to scratch another, diamond being the hardest and talc the softer. Now we use standard tests, where a ball or point is pressed into a material and the size of the dent is measured. There are a few different hardness tests: Rockwell, Brinell, Vickers, etc. They are popular because they are easy and non-destructive (except for the small dent).

Hardness is the resistance of a material to localized deformation. The term can apply to deformation from indentation, scratching, cutting or bending. In metals, ceramics and most polymers, the deformation considered is plastic deformation of the surface. For elastomers and some polymers, hardness is defined at the resistance to elastic deformation of the surface. The lack of a fundamental definition indicates that hardness is not be a basic property of a material, but rather a composite one with contributions from the yield strength, work hardening, true tensile strength, modulus, and others factors. Hardness measurements are widely used for the quality control of materials because they are quick and considered to be nondestructive tests when the marks or indentations produced by the test are in low stress areas.

There are a large variety of methods used for determining the hardness of a substance. A few of the more common methods are introduced below.

Mohs Hardness Test

One of the oldest ways of measuring hardness was devised by the German mineralogist Friedrich Mohs in 1812. The Mohs hardness test involves observing whether a materials surface is scratched by a substance of known or defined hardness. To give numerical values to this physical property, minerals are ranked along the Mohs scale, which is composed of 10 minerals that have been given arbitrary hardness values. Mohs hardness test, while greatly facilitating the identification of minerals in the field, is not suitable for accurately gauging the hardness of industrial materials such as steel or ceramics. For engineering materials, a variety of instruments have been developed over the years to provide a precise measure of hardness. Many apply a load and measure the depth or size of the resulting indentation. Hardness can be measured on the macro-, micro- or nano- scale.

Brinell Hardness Test

The oldest of the hardness test methods in common use on engineering materials today is the Brinell hardness test. Dr. J. A. Brinell invented the Brinell test in Sweden in 1900. The Brinell test uses a desktop machine to applying a specified load to a hardened sphere of a specified diameter. The Brinell hardness number, or simply the Brinell number, is obtained by dividing the load used, in kilograms, by the measured surface area of the indentation, in square millimeters, left on the test surface. The Brinell test is frequently used to determine the hardness metal forgings and castings that have a large grain structures. The Brinell test provides a measurement over a fairly large area that is less affected by the course grain structure of these materials than are Rockwell or Vickers tests.

A wide range of materials can be tested using a Brinell test simply by varying the test load and indenter ball size. In the USA, Brinell testing is typically done on iron and steel castings using a 3000Kg test force and a 10mm diameter ball. A 1500 kilogram load is usually used for aluminum castings. Copper, brass and thin stock are frequently tested using a 500Kg test force and a 10 or 5mm ball. In Europe Brinell testing is done using a much wider range of forces and ball sizes and it is common to perform Brinell tests on small parts using a 1mm carbide ball and a test force as low as 1kg. These low load tests are commonly referred to as baby Brinell tests. The test conditions should be reported along with the Brinell hardness number. A value reported as "60 HB 10/1500/30" means that a Brinell Hardness of 60 was obtained using a 10mm diameter ball with a 1500 kilogram load applied for 30 seconds.

Rockwell Hardness Test

The Rockwell Hardness test also uses a machine to apply a specific load and then measure the depth of the resulting impression. The indenter may either be a steel ball of some specified diameter or a spherical diamond-tipped cone of 120° angle and 0.2 mm tip radius, called a brale. A minor load of 10 kg is first applied, which causes a small initial penetration to seat the indenter and remove the effects of any surface irregularities. Then, the dial is set to zero and the major load is applied. Upon removal of the major load, the depth reading is taken while the minor load is still on. The hardness number may then be read directly from the scale. The indenter and the test load used determine the hardness scale that is used (A, B, C, etc).

For soft materials such as copper alloys, soft steel, and aluminum alloys a 1/16" diameter steel ball is used with a 100-kilogram load and the hardness is read on the "B" scale. In testing harder materials, hard cast iron and many steel alloys, a 120 degrees diamond cone is used with up to a 150 kilogram load and the hardness is read on the "C" scale. There are several Rockwell scales other than the "B" & "C" scales, (which are called the common scales). A properly reported Rockwell value will have the hardness number followed by "HR" (Hardness Rockwell) and the scale letter. For example, 50 HRB indicates that the material has a hardness reading of 50 on the B scale.

Rockwell Superficial Hardness Test

The Rockwell Superficial Hardness Tester is used to test thin materials, lightly carburized steel surfaces, or parts that might bend or crush under the conditions of the regular test. This tester uses the same indenters as the standard Rockwell tester but the loads are reduced. A minor load of 3 kilograms is used and the major load is either 15 or 45 kilograms depending on the indenter used. Using the 1/16" diameter, steel ball indenter, a "T" is added (meaning thin sheet testing) to the superficial hardness designation. An example of a superficial Rockwell hardness is 23 HR15T, which indicates the superficial hardness as 23, with a load of 15 kilograms using the steel ball.

10. Variability of Material Properties

Tests do not produce exactly the same result because of variations in the test equipment, procedures, operator bias, specimen fabrication, etc. But, even if all those parameters are controlled within strict limits, a variation remains in the materials, due to uncontrolled variations during fabrication, non homogenous composition and structure, etc. The measured mechanical properties will show scatter, which is often distributed in a Gaussian curve (bell-shaped), that is characterized by the mean value and the standard deviation (width).

11. Design/Safety Factors

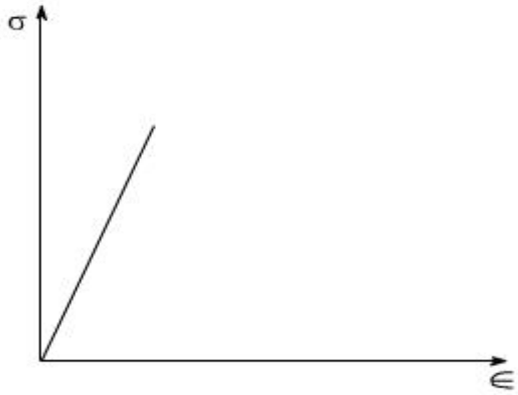
To take into account variability of properties, designers use, instead of an average value of, say, the tensile strength, the probability that the yield strength is above the minimum value tolerable. This leads to the use of a *safety factor* $N > 1$ (varies from 1.2 to 4). Thus, a working value for the tensile strength would be $s_W = s_{TS} / N$.

STRESS - STRAIN RELATIONS

Stress – Strain Relations: The Hook's law, states that within the elastic limits the stress is proportional to the strain since for most materials it is impossible to describe the entire stress – strain curve with simple mathematical expression, in any given problem the behavior of the materials is represented by an idealized stress – strain curve, which emphasizes those aspects of the behaviors which are most important is that particular problem.

(i) Linear elastic material:

A linear elastic material is one in which the strain is proportional to stress as shown below:

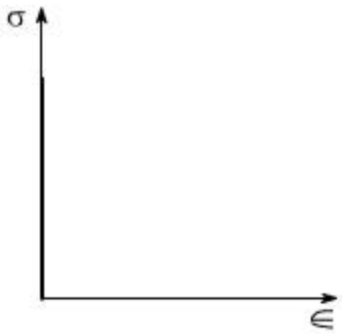


Linearly elastic material

There are also other types of idealized models of material behavior.

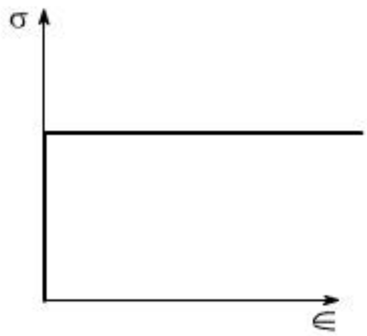
(ii) Rigid Materials:

It is the one which do not experience any strain regardless of the applied stress.



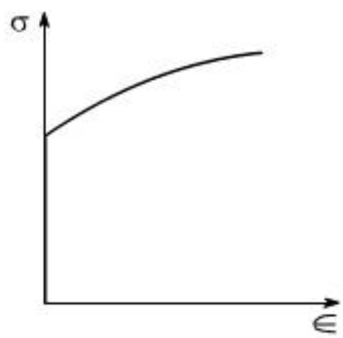
(iii) Perfectly plastic (non-strain hardening):

A perfectly plastic i.e. non-strain hardening material is shown below:



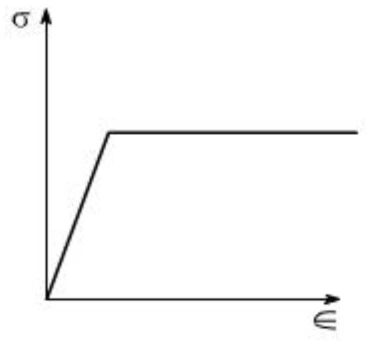
(iv) Rigid Plastic material (strain hardening):

A rigid plastic material i.e. strain hardening is depicted in the figure below:



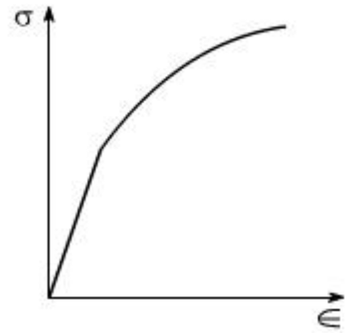
(v) Elastic Perfectly Plastic material:

The elastic perfectly plastic material is having the characteristics as shown below:



(vi) Elastic – Plastic material:

The elastic plastic material exhibits a stress Vs strain diagram as depicted in the figure below:



Elastic Stress – strain Relations:

Previously stress – strain relations were considered for the special case of a uniaxial loading i.e. only one component of stress i.e. the axial or normal component of stress was coming into picture. In this section we shall generalize the elastic behavior, so as to arrive at the relations which connect all the six components of stress with the six components of elastic stress. Further, we would restrict over selves to linearly elastic material. Before writing down the relations let us introduce a term ISOTROPY

ISOTROPIC: If the response of the material is independent of the orientation of the load axis of the sample, then we say that the material is isotropic or in other words we can say that isotropy of a material in a characteristics, which gives us the information that the properties are the same in the three orthogonal directions x y z, on the other hand if the response is dependent on orientation it is known as anisotropic.

Examples of anisotropic materials, whose properties are different in different directions, are

(i) Wood

(ii) Fiber reinforced plastic

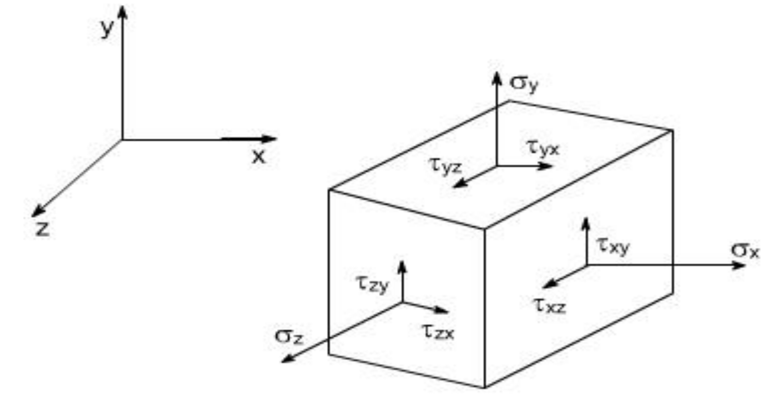
(iii) Reinforced concrete

HOMOGENIUS: A material is homogenous if it has the same composition through our body. Hence the elastic properties are the same at every point in the body. However, the properties need not to be the same in all the direction for the material to be homogenous. Isotropic materials have the same elastic properties in all the directions. Therefore, the material must be both homogenous and isotropic in order to have the lateral strains to be same at every point in a particular component.

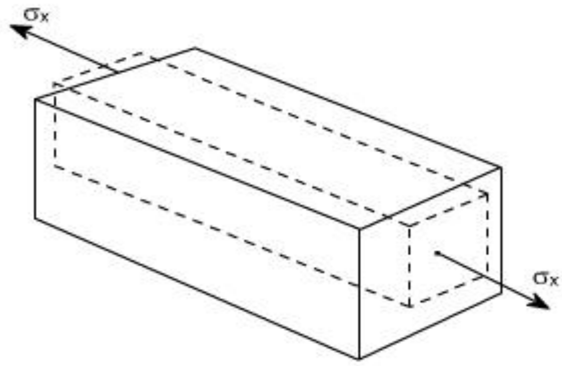
Generalized Hook's Law: We know that for stresses not greater than the proportional limit.

$$\epsilon = \frac{\sigma}{E} \text{ or } \mu = -\frac{|\epsilon_{lateral}|}{|\epsilon_{axial}|}$$

This equation expresses the relationship between stress and strain (Hook's law) for uniaxial state of stress only when the stress is not greater than the proportional limit. In order to analyze the deformational effects produced by all the stresses, we shall consider the effects of one axial stress at a time. Since we presumably are dealing with strains of the order of one percent or less. These effects can be superimposed arbitrarily. The figure below shows the general triaxial state of stress.



Let us consider a case when σ_x alone is acting. It will cause an increase in dimension in X-direction whereas the dimensions in y and z direction will be decreased.

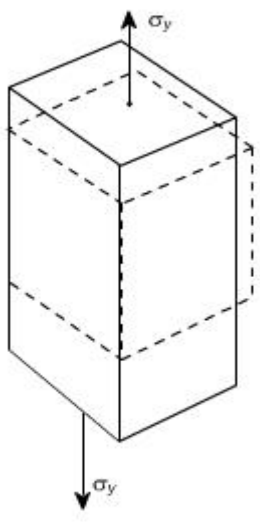


$$\epsilon_x = \frac{\sigma_x}{E}, \epsilon_y = -\mu\epsilon_x ; \epsilon_z = -\mu\epsilon_x$$

$$\epsilon_x = \frac{\sigma_x}{E} ; \epsilon_y = -\mu\frac{\sigma_x}{E} ; \epsilon_z = -\mu\frac{\sigma_x}{E}$$

Therefore the resulting strains in three directions are

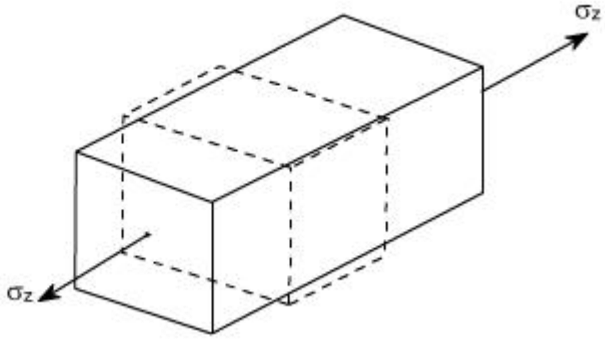
Similarly let us consider that normal stress $_y$ alone is acting and the resulting strains are



$$\epsilon_y = \frac{\sigma_y}{E}, \epsilon_x = -\mu\epsilon_y ; \epsilon_z = -\mu\epsilon_y$$

$$\epsilon_y = \frac{\sigma_y}{E} ; \epsilon_x = -\mu\frac{\sigma_y}{E} ; \epsilon_z = -\mu\frac{\sigma_y}{E}$$

Now let us consider the stress σ_z acting alone, thus the strains produced are



$$\epsilon_z = \frac{\sigma_z}{E}, \epsilon_y = -\mu\epsilon_z; \epsilon_x = -\mu\epsilon_z$$

$$\epsilon_z = \frac{\sigma_z}{E}; \epsilon_y = -\mu\frac{\sigma_z}{E}; \epsilon_x = -\mu\frac{\sigma_z}{E}$$

Thus the total strain in any one direction is

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\mu}{E}(\sigma_y + \sigma_z) \qquad (1)$$

In a similar manner, the total strain in the y and z directions become

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\mu}{E}(\sigma_x + \sigma_z) \qquad (2)$$

$$\epsilon_z = \frac{\sigma_z}{E} - \frac{\mu}{E}(\sigma_x + \sigma_y) \qquad (3)$$

In the following analysis shear stresses were not considered. It can be shown that for an isotropic material's a shear stress will produce only its corresponding shear strain and will not influence the axial strain. Thus, we can write Hook's law for the individual shear strains and shear stresses in the following manner.

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad (4)$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} \qquad (5)$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} \qquad (6)$$

The Equations (1) through (6) are known as Generalized Hook's law and are the constitutive equations for the linear elastic isotropic materials. When these equations isotropic materials. When these equations are used as written, the strains can be completely determined from known values of the stresses. Thus then the above set of equations reduces to

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\mu\sigma_x}{E}$$

$$\epsilon_z = -\mu\frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E} \text{ and } \tau_{xy} = \frac{\gamma_{xy}}{G}$$

Their inverse relations can be also determined and are given as

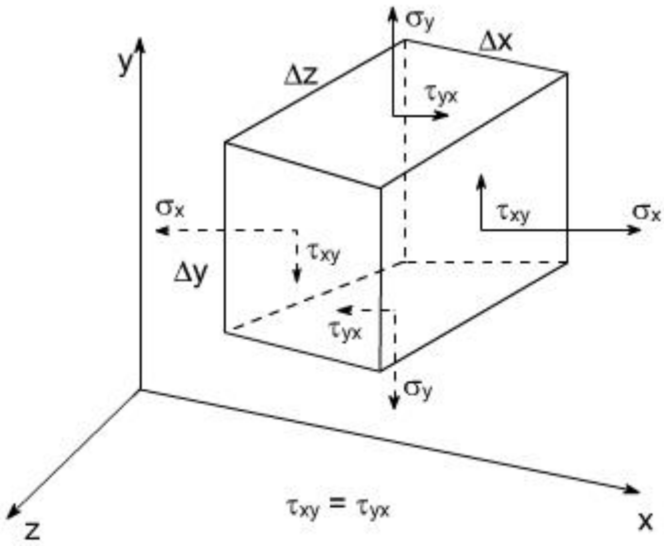
$$\sigma_x = \frac{E}{(1-\mu^2)}(\epsilon_x + \mu\epsilon_y)$$

$$\sigma_y = \frac{E}{(1-\mu^2)}(\epsilon_y + \mu\epsilon_x)$$

$$\tau_{xy} = G.\gamma_{xy}$$

Hook's law is probably the most well known and widely used constitutive equations for engineering materials." However, we cannot say that all the engineering materials are linear elastic isotropic ones. Because now in the present times, the new materials are being developed every day. Many useful materials exhibit nonlinear response and are not elastic too.

Plane Stress: In many instances the stress situation is less complicated for example if we pull one long thin wire of uniform section and examine – small parallepiped where x – axis coincides with the axis of the wire



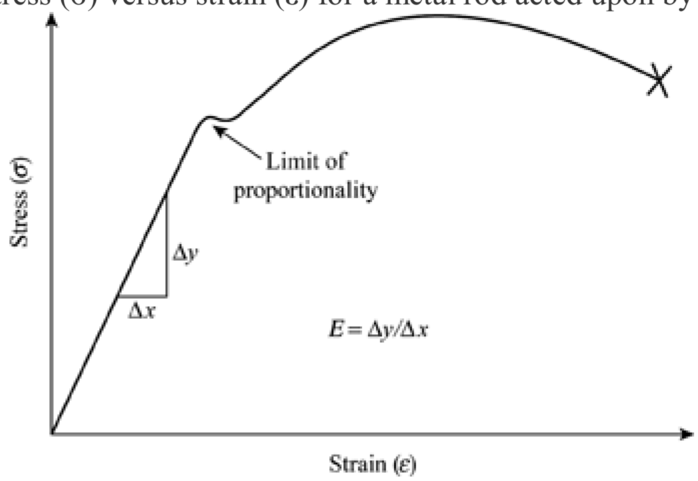
$\sigma_z = \tau_{zx} = \tau_{zy} = 0$

Hooke's Law

It states that within the limit of elasticity, the stress induced (σ) in the solid due to some external force is always in proportion with the strain (ε). In other words the force causing stress in a solid is directly proportional to the solid's deformation. Consider a spring with a spring constant *k* that is stretched with a force *F* extends to a distance *x* with reference to the initial position.

Force required for deformation *x* is given by $F = -k \cdot x$

Plot the graph for the stress (σ) versus strain (ε) for a metal rod acted upon by trending tensile force.



The determination of elastic modulus *E* from the tensile experiment results is depicted in the figure. It can be seen from the graph that the curve of stress versus strain is linear within the limit of elasticity of the material. It is inferred that for the load below the limit of elasticity, the stress induced is in proportion with the strain in the solid.

$$\sigma \propto \varepsilon$$
$$\sigma = E\varepsilon$$
$$E = \frac{\sigma}{\varepsilon}$$

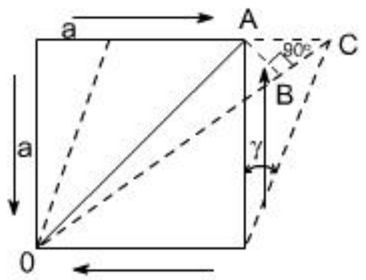
Here young's modulus or proportionality limit is *E*

RELATION AMONG ELASTIC CONSTANTS

Relation between E, G and μ

Let us establish a relation among the elastic constants E,G and μ. Consider a cube of material of side ‘a' subjected to the action of the shear and complementary shear stresses as shown in the figure and producing the strained shape as shown in the figure below.

Assuming that the strains are small and the angle A C B may be taken as 45^0.



Therefore strain on the diagonal OA

= Change in length / original length

Since angle between OA and OB is very small hence OA = OB therefore BC, is the change in the length of the diagonal OA

Thus, strain on diagonal OA $= \frac{BC}{OA}$

$$= \frac{AC\cos 45^0}{OA}$$

$$OA = \frac{a}{\sin 45^0} = a.\sqrt{2}$$

hence strain $= \frac{AC}{a\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$

$$= \frac{AC}{2a}$$

but AC = aγ

where γ = shear strain

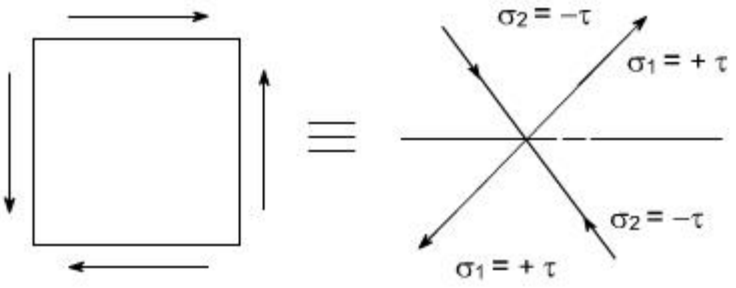
Thus, the strain on diagonal $= \frac{a\gamma}{2a} = \frac{\gamma}{2}$

From the definition

$$G = \frac{\tau}{\gamma} \text{ or } \gamma = \frac{\tau}{G}$$

thus, the strain on diagonal $= \frac{\gamma}{2} = \frac{\tau}{2G}$

Now this shear stress system is equivalent or can be replaced by a system of direct stresses at 45^0 as shown below. One set will be compressive, the other tensile, and both will be equal in value to the applied shear strain.



Thus, for the direct state of stress system this applies along the diagonals:

$$\text{strain on diagonal} = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E}$$
$$= \frac{\tau}{E} - \mu\frac{(-\tau)}{E}$$
$$= \frac{\tau}{E}(1+\mu)$$

equating the two strains one may get

$$\frac{\tau}{2G} = \frac{\tau}{E}(1+\mu)$$

or $$\boxed{E = 2G(1+\mu)}$$

We have introduced a total of four elastic constants, i.e E, G, K and □ . It turns out that not all of these are independent of the others. Infact given any two of then, the other two can be found.

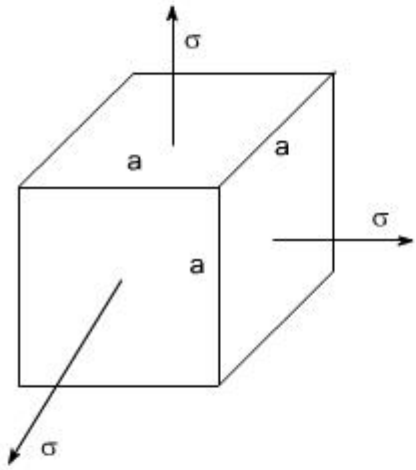
$$\text{Again} \quad E = 3K(1-2\gamma)$$
$$\Rightarrow \frac{E}{3(1-2\gamma)} = K$$
$$\text{if } \gamma = 0.5 \;\; K = \infty$$
$$\epsilon_v = \frac{(1-2\gamma)}{E}(\epsilon_x + \epsilon_y + \epsilon_z) = 3\frac{\sigma}{E}(1-2\gamma)$$
$$(\text{for } \epsilon_x = \epsilon_y = \epsilon_z \text{ hydrostatic state of stress})$$
$$\epsilon_v = 0 \text{ if } \gamma = 0.5$$

Irrespective of the stresses i.e, the material is incompressible.

When γ = 0.5 Value of k is infinite, rather than a zero value of E and volumetric strain is zero, or in other words, the material is incompressible.

Relation between E, K and γ:

Consider a cube subjected to three equal stresses σ as shown in the figure below



The total strain in one direction or along one edge due to the application of hydrostatic stress or

$$= \frac{\sigma}{E} - \gamma\frac{\sigma}{E} - \gamma\frac{\sigma}{E}$$

$$= \frac{\sigma}{E}(1-2\gamma)$$

volumetre strain = 3.linear strain

volumetre strain = $\epsilon_x + \epsilon_y + \epsilon_z$

or thus, $\epsilon_x = \epsilon_y = \epsilon_z$

volumetric strain = $3\frac{\sigma}{E}(1-2\gamma)$

By definition

$$\text{Bulk Modulus of Elasticity (K)} = \frac{\text{Volumetric stress}(\sigma)}{\text{Volumetric strain}}$$

or

$$\text{Volumetric strain} = \frac{\sigma}{k}$$

Equating the two strains we get

$$\frac{\sigma}{k} = 3.\frac{\sigma}{E}(1-2\gamma)$$

volumetric stress (σ) is given as $\boxed{E = 3K(1-2\gamma)}$

Relation between E, G and K:

The relationship between E, G and K can be easily determined by eliminating from the already derived relations

E = 2 G (1 + μ) and E = 3 K (1-2γ)

Thus, the following relationship may be obtained

$$\boxed{E = \frac{9\,GK}{(3K+G)}}$$

Relation between E, K and G:

From the already derived relations, E can be eliminated

$$E = 2G(1+\gamma)$$
$$E = 3K(1-2\gamma)$$

Thus,we get

$$3k(1-2\gamma) = 2G(1+\gamma)$$

therefore

$$\gamma = \frac{(3K-2G)}{2(G+3K)}$$

or

$$\boxed{\gamma = 0.5(3K-2G)(G+3K)}$$

Engineering Brief about the elastic constants:

We have introduced a total of four elastic constants i.e. E, G, K and γ. It may be seen that not all of these are independent of the others. In fact given any two of them, the other two can be determined. Further, it may be noted that

$$E = 3K(1-2\gamma)$$

or

$$K = \frac{E}{(1-2\gamma)}$$

if $\gamma = 0.5$; $K = \infty$

Also $\epsilon_v = \frac{(1-2\gamma)}{E}(\sigma_x + \sigma_y + \sigma_z)$

$$= \frac{(1-2\gamma)}{E}.3\sigma \text{ (for hydrostatic state of stress i.e } \sigma_x = \sigma_y = \sigma_z = \sigma \text{)}$$

Hence if γ = 0.5, the value of K becomes infinite, rather than a zero value of E and the volumetric strain is zero or in other words, the material becomes incompressible

Further, it may be noted that under condition of simple tension and simple shear, all real materials tend to experience displacements in the directions of the applied forces and under hydrostatic loading they tend to increase in volume. In other words the value of the elastic constants E, G and K cannot be negative

Therefore, the relations

$$E = 2\,G\,(1 + \mu)$$

$$E = 3\,K\,(1-2\gamma)$$

Yields $\boxed{-1 \leq \nu \leq 0.5}$

In actual practice no real material has value of Poisson's ratio negative. Thus, the value of γ cannot be greater than 0.5.

Definition of factor of SAFETY: the ratio of the ultimate strength of a member or piece of material (as in an airplane) to the actual working stress or the maximum permissible stress when in use

Ultimate strength or tensile strength or ultimate tensile strength is the capacity of the material to withstand tensile loads. Ultimate tensile strength is measured by the maximum stress that a material can withstand while being stretched or pulled before breaking. In the study of strength of materials, tensile strength, compressive strength, and shear strength can be analyzed independently. Ultimate tensile strength is measured by Universal Testing Machine. A specimen of material with standard dimensions is fastened in the machine and is subjected to tensile load i.e. elongated or pulled by hydraulic power. Once the machine is started it begins to apply an increasing load on specimen. Throughout the tests the control system and its associated software record the load and extension or compression of the specimen.

In a stress strain curve for a ductile material, the Ultimate tensile strength is the highest point at curve in stress axis. If the pulling force is increased beyond the ultimate stress, the strain increases i.e. the specimen starts expanding in plastic phase resulting in decrease in stress and on removal of the pulling force the specimen would not be able to regain its shape. In this phase the specimen undergoes Necking and fractures on further increment in pulling force.

Ductile material makes cup and cone formation on fracture.

Factor of safety can be defined as the ratio of ultimate strength to the design strength. It is a constant factor that is considered for designing of machine components or structure beyond its working strength.

F.O.S= Ultimate strength/Design load

F.O.S. is taken generally around 1.5 to 3 for every industrial machine or equipment. Buildings commonly use a factor of safety of 2.0 for each structural member. The value for buildings is relatively low because the loads are well understood and most structures are redundant. Pressure vessels use 3.5 to 4.0, automobiles use 3.0, and aircraft and spacecraft use 1.2 to 3.0 depending on the application and materials. Ductile, metallic materials tend to use the lower value while brittle materials use the higher values. Cost of an element depends directly on factor of safety.

Definition of Poisson's ratio:

Poisson's ratio is the ratio of transverse contraction strain to longitudinal extension strain in the direction of stretching force. Tensile deformation is considered positive and compressive deformation is considered negative. The definition of Poisson's ratio contains a minus sign so that normal materials have a positive ratio. Poisson's ratio, also called Poisson ratio or the Poisson coefficient, or coefficient de Poisson, is usually represented as a lower case Greek nu, ν.

Thermal Stress

One of the properties of metals is that they transfer heat. Physical changes that occur with this transfer include that expansion when the temperature increases and shrinkage when the temperature decreases. This happens in all three dimensions.

Thermal stress occurs as a result of thermal expansion of metallic structural members when the temperature changes. Changes in temperature cause thermal deformation to the structural members. The values of these deformations can be described using the following formula, or relationship:

$$\delta_t = \alpha * L * (T - T_0)$$

where:

- *delta* is the deformation of the structural member due to a change in temperature
- *alpha* is the temperature coefficient of expansion, a material property measured in units per Kelvin (K)
- *L* is original length of the structural member, measured in feet or meters
- *T* is the final temperature measured in units of Kelvin or Celsius^0 for the international system and Fahrenheit^0 for the English system

T_0 is the initial temperature, again measured in units of K or C^o for the international system and F^o for the English system

Formulas

When a structural member is free to move and expand, there is no stress exerted on it. However, when movement and expansion are restricted, then thermal stress occurs. When motion is restricted in the direction of expansion, the value of the reaction force is equal to the value of the force necessary to compress a beam in the opposite direction, and by the same amount of deformation. We can use the following formula to describe this relationship:

$$F = \frac{\delta * E * A}{L}$$

Here,

- *delta* is the deflection of the beam due to the reaction force, which is equal to the deflection of the beam due to thermal expansion but in the opposite direction, shown in meters for the S.I. (or international system) and feet for the English system.
- *A* is the area of the cross sectional section in f^2 or m^2.
- *E* is the modulus of elasticity of the material from which the beam is manufactured in Pascal (Pa) for the international units system or a or lb/ft^2 for the English system.
- *L* is the length of the beam in feet or meters.

Putting together our understanding of thermal expansion and the forces involved, we can now solve for thermal stress, represented by the Greek letter *sigma*, measured in Newtons per square meter or Pascals (Pa).

Since:

$$\delta_T = \delta_F$$

$$\alpha * L * (T - T_0) = \frac{F * L}{E * A}$$

And since,

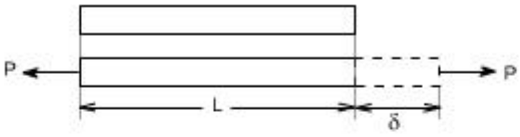
$$\frac{F}{A} = \sigma$$

Members Subjected to Uniaxial Stress

Members in Uni – axial state of stress

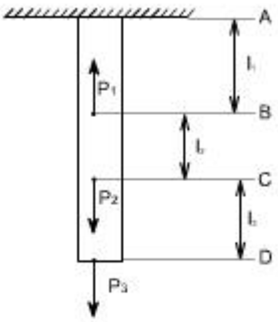
Introduction: [For members subjected to uniaxial state of stress]

For a prismatic bar loaded in tension by an axial force P, the elongation of the bar can be determined as



$$\delta = \frac{PL}{AE} \quad \text{——— (1)}$$

Suppose the bar is loaded at one or more intermediate positions, then equation (1) can be readily adapted to handle this situation, i.e. we can determine the axial force in each part of the bar i.e. parts AB, BC, CD, and calculate the elongation or shortening of each part separately, finally, these changes in lengths can be added algebraically to obtain the total charge in length of the entire bar.



When either the axial force or the cross – sectional area varies continuously along the axis of the bar, then equation (1) is no longer suitable. Instead, the elongation can be found by considering a deferential element of a bar and then the equation (1) becomes

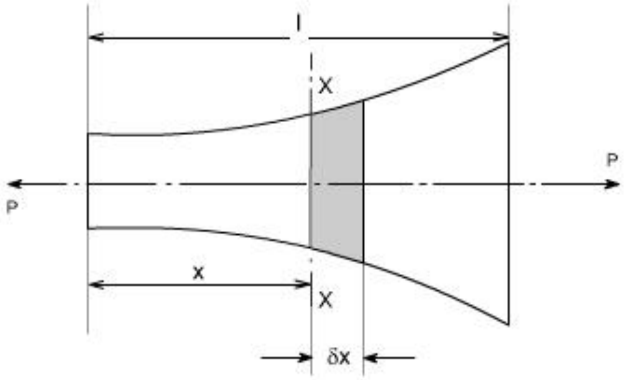
$$d\delta = \frac{P_x dx}{E.A_x}$$

$$\delta = \int_0^l \frac{P_x dx}{E.A_x}$$

i.e. the axial force P_x and area of the cross – section A_x must be expressed as functions of x. If the expressions for P_x and A_x are not too complicated, the integral can be evaluated analytically, otherwise Numerical methods or techniques can be used to evaluate these integrals.

Stresses in Non – Uniform bars

Consider a bar of varying cross section subjected to a tensile force P as shown below.



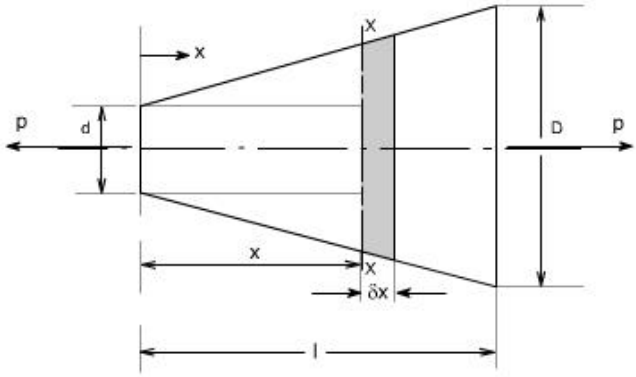
$$= \frac{P}{E} \frac{\delta x}{a}$$

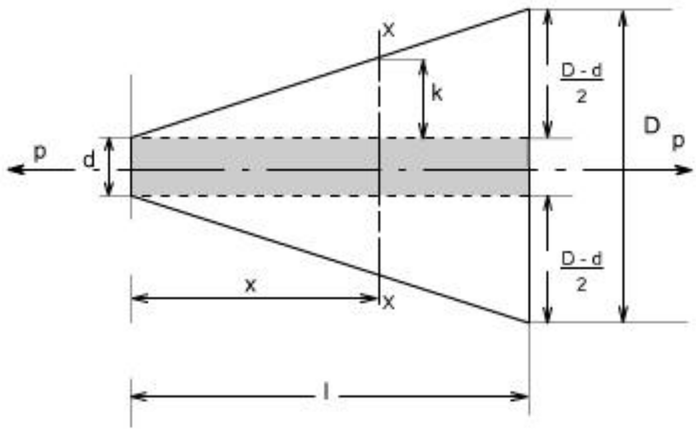
Thus, the extension for the entire bar is

$$\delta = \int_0^l \frac{P}{E} \frac{\delta x}{a}$$

$$\text{or total extension} = \frac{P}{E} \int_0^l \frac{\delta x}{a}$$

Now let us for example take a case when the bar tapers uniformly from d at x = 0 to D at x = l





In order to compute the value of diameter of a bar at a chosen location let us determine the value of dimension k, from similar triangles

$$\frac{(D - d)/2}{l} = \frac{k}{x}$$

$$\text{Thus, } k = \frac{(D - d)x}{2l}$$

Therefore, the diameter 'y' at the X-section is or = d + 2k

$$y = d + \frac{(D - d)x}{l}$$

Hence the cross –section area at section X- X will be

$$A_x \text{ or } a = \frac{\pi}{4}y^2$$

$$= \frac{\pi}{4}\left[d + (D - d)\frac{x}{l}\right]^2$$

hence the total extension of the bar will be given by expression

$$= \frac{P}{E}\int_0^l \frac{\delta x}{a}$$

substitutingthe value of 'a' to get the total extention of the bar

$$= \frac{\pi P}{4E}\int_0^l \frac{\delta x}{\left[d + (D - d)\frac{x}{l}\right]^2}$$

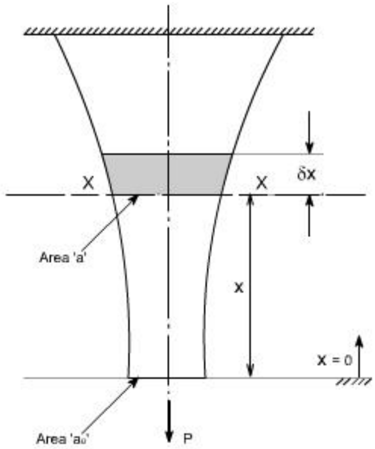
after carrying out the intergration we get

$$= -\frac{4.P.l}{\pi E}\left[\frac{1}{D} - \frac{1}{d}\right]$$

$$= \frac{4.P.l}{\pi E D.d}$$

hence the total strain in the bar $= \frac{\mathbf{4.P.l}}{\mathbf{\pi E\, D.d}}$

An interesting problem is to determine the shape of a bar which would have a uniform stress in it under the action of its own weight and a load P.



Let us consider such a bar as shown in the figure below: The weight of the bar being supported under section XX is

$$= \int_0^x \rho\, g\, a\, dx$$

where ρ is density of the bar.

thus the stress at XX is

$$\sigma = \frac{P + \int_0^x \rho\, g\, a\, dx}{a}$$

$$\text{or } \sigma.a = P + \int_0^x \rho.\, g.a\, dx$$

Differentiating the above equation with respect to x we get

$$\sigma.\frac{da}{dx} = \rho.\, g.a$$

$$\frac{da}{a} = \frac{\rho.\, g}{\sigma}.dx$$

intergrating the above equation we get

$$\int \frac{da}{a} = \int \frac{\rho.\, g}{\sigma} dx$$

$$\log_e{}^{a} = \frac{\rho.\, g.\, x}{\sigma} + \text{constant}$$

In order to determine the constant of integration

let us apply the boundary conditions

at $x = 0;\ a = a_0$

thus, constant $= \log_e{}^{a_0}$

or

$$\log_e{}^{a} = \frac{\rho.\, g.\, x}{\sigma} + \log_e{}^{a_0}$$

$$\log_e\left(\frac{a}{a_0}\right) = \frac{\rho.\, g.\, x}{\sigma}$$

$$\text{or } \boxed{e^{\frac{\rho.g.x}{\sigma}} = \frac{a}{a_0}}$$

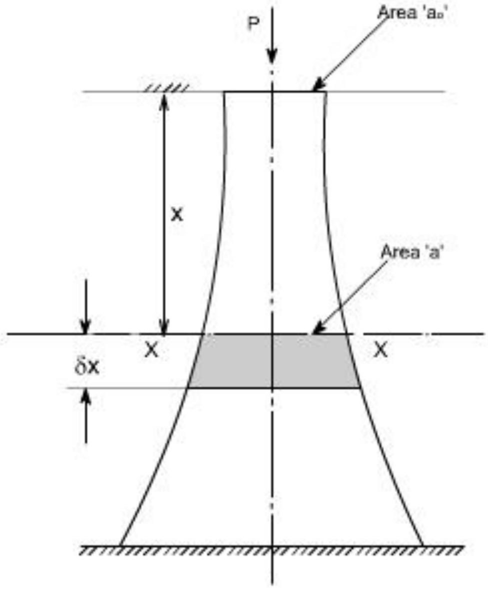
also at $x = 0$

$$\sigma = \frac{P}{a_0}$$

Thus,

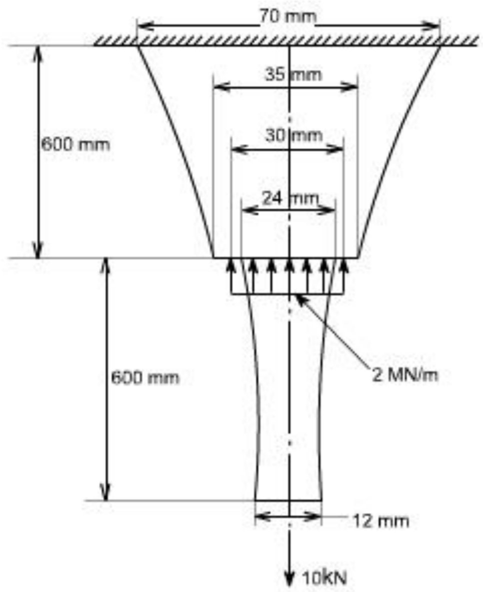
$$\frac{a}{a_0} = e^{\frac{\rho.g.x\, a_0}{P}}$$

The same results are obtained if the bar is turned upside down and loaded as a column as shown in the figure below:



IIIustrative Problem 1: Calculate the overall change in length of the tapered rod as shown in figure below. It carries a tensile load of 10kN at the free end and at the step change in section a compressive load of 2 MN/m evenly distributed around a circle of 30 mm diameter take the value of E = 208 GN / m^2.

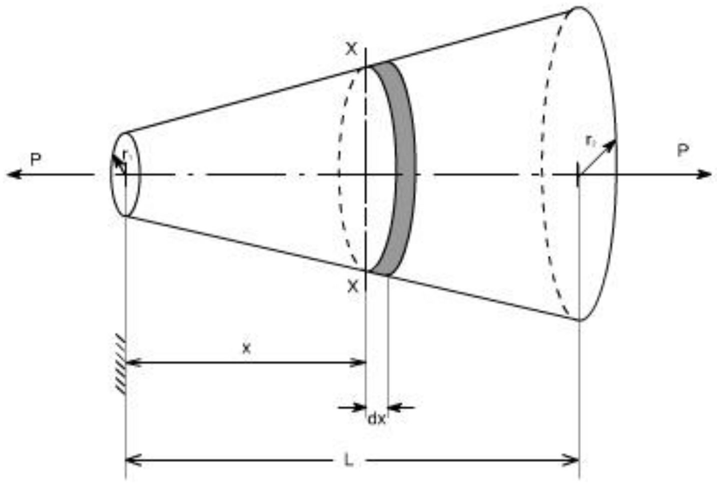
This problem may be solved using the procedure as discussed earlier in this section



IIIustrative Problem 2: A round bar, of length L, tapers uniformly from radius r_1 at one end to radius $r_2$at the other. Show that the extension produced by a tensile axial load P is $\frac{PL}{2\pi E r_1^2}$

If $r_2 = 2r_1$, compare this extension with that of a uniform cylindrical bar having a radius equal to the mean radius of the tapered bar.

Solution:



Consider the above figure let r_1 be the radius at the smaller end. Then at a X cross-

$$= r_1 + \frac{x}{L}(r_2 - r_1)$$

$$= r_1(1 + kx)$$

$$\text{where } k = \left(\frac{r_2 - r_1}{L}\right).\frac{1}{r_1}$$

$$\text{stress at section XX} = \frac{\text{load}}{\text{area}}$$

$$= \frac{P}{\pi r_1^2 (1 + kx)^2}$$

$$\text{hence strain at this section} = \frac{\text{stress}}{E}$$

$$= \frac{P}{E.\pi r_1^2 (1 + kx)^2}$$

Thus, for a small length dx of the bar at this section the extention is $\frac{P.dx}{E\pi r_1^2 (1 + kx)^2}$

Total extension of the bar can be found by integrating the above expression within the limits from x=0 to x=L

$$\text{Extension} = \int_0^L \frac{P.dx}{E.\pi r_1^2 (1 + k.x)^2}$$

$$= \frac{P}{E.\pi r_1^2}\int_0^L (1 + k.x)^{-2} dx$$

$$= \frac{P}{E.\pi r_1^2}\left[\frac{(1+kx)^{-1}}{-k}\right]_0^L$$

$$= \frac{P}{E.\pi r_1^2}\left[\frac{(1 + kx)^{-1}}{-k} - \frac{1}{-k}\right]$$

$$= \frac{P}{E.\pi r_1^2 .k}\left[1 - \frac{1}{1 + kL}\right]$$

$$= \frac{PL}{E.\pi r_1^2 (1 + kL)}$$

$$\text{since } k = \frac{(r_2 - r_1)}{r_1 L}$$

$$\text{Thus, } 1 + kL = {}^{r_2}\!/_{r_1}$$

$$\text{Therefore, the extension} = \frac{PL}{\pi E r_1 r_2}$$

Comparing of extensions

For the case when $r_2 = 2.r_1$, the value of computed extension as above becomes equal to $\frac{PL}{2\pi E r_1^2}$

The mean radius of taper bar

$= 1 / 2(r_1 + r_2)$

$= 1 / 2(r_1 +2 r_2)$

$= 3 / 2 .r_1$

Therefore, the extension of uniform bar

= Original length. Strain

$$= L.\frac{\sigma}{E}$$

$$= \frac{L}{E}.\frac{P}{\pi\left(\frac{3}{2}r_1\right)^2}$$

$$= \frac{4PL}{9\pi E \pi r_1^2}$$

hence the

$$\frac{\text{Extension of uniform}}{\text{Extension of tapered}} = \left(\frac{4PL}{9\pi E \pi r_1^2}\right) \Big/ \frac{PL}{2\pi E r_1^2}$$

$$= \frac{8}{9}$$

Distribution of shear stresses in circular Shafts subjected to torsion:

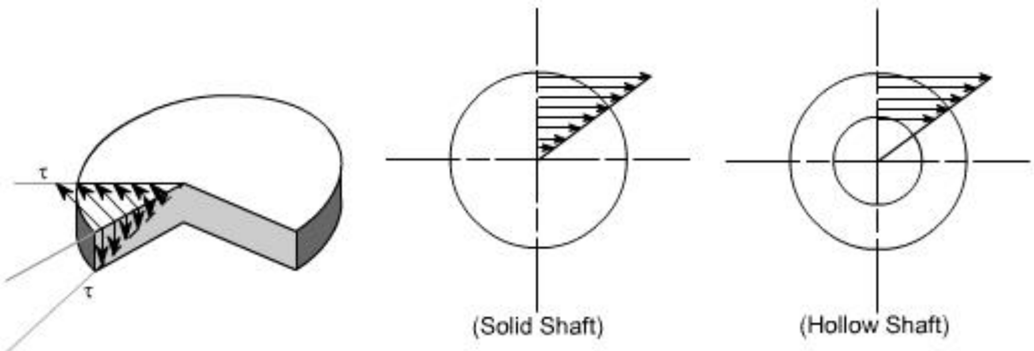
The simple torsion equation is written as

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

or

$$\tau = \frac{G\theta.r}{L}$$

This states that the shearing stress varies directly as the distance ‘r' from the axis of the shaft and the following is the stress distribution in the plane of cross section and also the complementary shearing stresses in an axial plane.



Hence the maximum streak stress occurs on the outer surface of the shaft where r = R

The value of maximum shearing stress in the solid circular shaft can be determined as

$$\frac{\tau}{r} = \frac{T}{J}$$

$$\tau_{max}\Big|_{r=d/2} = \frac{T.R}{J} = \frac{T}{\frac{\pi d^4}{32}} \cdot \frac{d}{2}$$

where d=diameter of solid shaft

$$\text{or} \quad \tau_{max^m} = \frac{16T}{\pi d^3}$$

From the above relation, following conclusion can be drawn

Power Transmitted by a shaft:

In practical application, the diameter of the shaft must sometimes be calculated from the power which it is required to transmit.

Given the power required to be transmitted, speed in rpm ‘N' Torque T, the formula connecting These quantities can be derived as follows

$$P = T.\omega$$

$$= \frac{T.2\pi N}{60} \text{watts}$$

$$= \frac{2\pi NT}{60 \times 10^3} (kw)$$

Torsional stiffness: The torsional stiffness k is defined as the torque per radian twist .

$$k = \frac{T}{\theta}$$

$$\text{i.e} \ = \frac{GJ}{L}$$

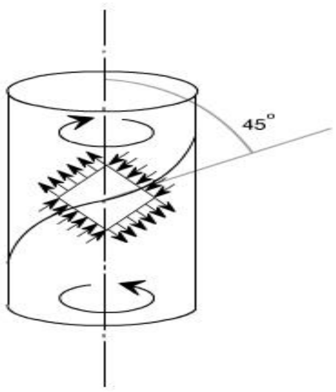
$$\text{or} \ k = \frac{G.J}{L}$$

For a ductile material, the plastic flow begins first in the outer surface. For a material which is weaker in shear longitudinally than transversely – for instance a wooden shaft, with the fibers parallel to axis the first cracks will be produced by the shearing stresses acting in the axial section and they will upper on the surface of the shaft in the longitudinal direction.

In the case of a material which is weaker in tension than in shear. For instance a, circular shaft of cast iron or a cylindrical piece of chalk a crack along a helix inclined at 45^0 to the axis of shaft often occurs.

Explanation: This is because of the fact that the state of pure shear is equivalent to a state of stress tension in one direction and equal compression in perpendicular direction.

A rectangular element cut from the outer layer of a twisted shaft with sides at 45^0 to the axis will be subjected to such stresses; the tensile stresses shown will produce a helical crack mentioned.



TORSION OF HOLLOW SHAFTS:

From the torsion of solid shafts of circular x – section, it is seen that only the material at the outer surface of the shaft can be stressed to the limit assigned as an allowable working stresses. All of the material within the shaft will work at a lower stress and is not being used to full capacity. Thus, in these cases where the weight reduction is important, it is advantageous to use hollow shafts. In discussing the torsion of hollow shafts the same assumptions will be made as in the case of a solid shaft. The general torsion equation as we have applied in the case of torsion of solid shaft will hold good

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

For the hollow shaft

$$J = \frac{\pi(D_0^4 - d_i^4)}{32} \quad \text{where } D_0 = \text{Outside diameter}$$

$$d_i = \text{Inside diameter}$$

$$\text{Let } d_i = \frac{1}{2}.D_0$$

$$\tau_{max^m}\Big|_{solid} = \frac{16T}{\pi D_0^3} \qquad (1)$$

$$\tau_{max^m}\Big|_{hollow} = \frac{T.D_0/2}{\frac{\pi}{32}(D_0^4 - d_i^4)}$$

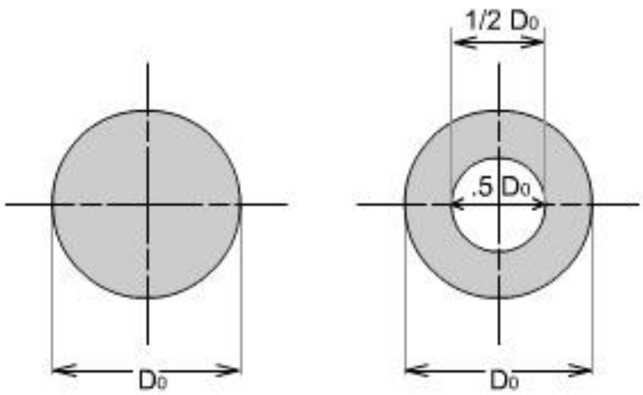
$$= \frac{16T.D_0}{\pi D_0^4\left[1 - (d_i/D_0)^4\right]}$$

$$= \frac{16T}{\pi D_0^3\left[1 - (1/2)^4\right]} = 1.066.\frac{16T}{\pi D_0^3} \qquad (2)$$

Hence by examining the equation (1) and (2) it may be seen that the in the case of hollow shaft is 6.6% larger than in the case of a solid shaft having the same outside diameter.

Reduction in weight:

Considering a solid and hollow shafts of the same length 'l' and density with $d_i = 1/2\ D_o$



Weight of hollow shaft

$$= \left[\frac{\pi D_0^2}{4} - \frac{\pi (D_0/2)^2}{4}\right] l \times \rho$$

$$= \left[\frac{\pi D_0^2}{4} - \frac{\pi D_0^2}{16}\right] l \times \rho$$

$$= \frac{\pi D_0^2}{4}\left[1 - 1/4\right] l \times \rho$$

$$= 0.75 \frac{\pi D_0^2}{4} l \times \rho$$

$$\text{Weight of solid shaft} = \frac{\pi D_0^2}{4} . l . \rho$$

$$\text{Reduction in weight} = (1 - 0.75) \frac{\pi D_0^2}{4} l \times \rho$$

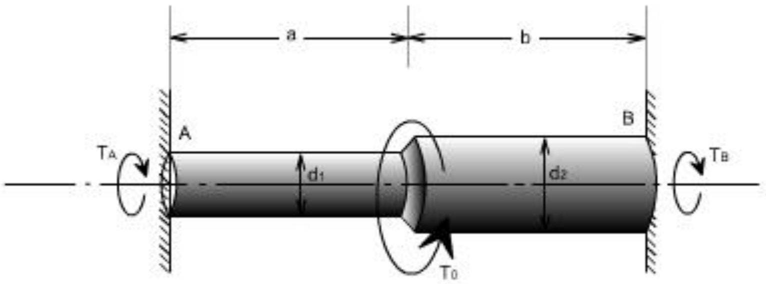
$$= 0.25 \frac{\pi D_0^2}{4} l \times \rho$$

Hence the reduction in weight would be just 25%.

Illustrative Examples:

Problem 1

A stepped solid circular shaft is built in at its ends and subjected to an externally applied torque. T_0 at the shoulder as shown in the figure. Determine the angle of rotation □$_0$ of the shoulder section where T_0 is applied?



Solution: This is a statically indeterminate system because the shaft is built in at both ends. All that we can find from the statics is that the sum of two reactive torque T_A and T_B at the built – in ends of the shafts must be equal to the applied torque T_0

Thus $T_A + T_B = T_0$ ------ (1)

[From static principles]

Where T_A, TB are the reactive torque at the built in ends A and B. whereas T_0 is the applied torque

From consideration of consistent deformation, we see that the angle of twist in each portion of the shaft must be same.

$$\frac{T}{J} = \frac{G.\theta}{l}$$

$$\text{or } \theta_A = \frac{T_A a}{J_A G}$$

$$\theta_B = \frac{T_B a}{J_B G}$$

Using the relation for angle of twist $\Rightarrow \frac{T_A a}{J_A G} = \frac{T_B b}{J_B G} = \theta_0 \quad \text{or} \quad \frac{T_A}{T_B} = \frac{J_A}{J_B} . \frac{b}{a} \quad (2)$

N.B: Assuming modulus of rigidity G to be same for the two portions

So the defines the ratio of T_A and T_B

So by solving (1) & (2) we get

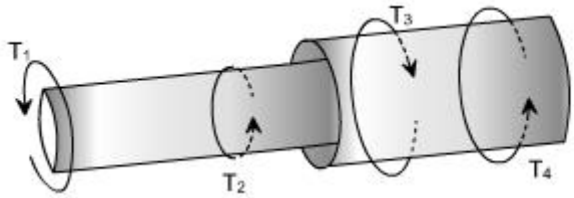
$$T_A = \frac{T_0}{1 + \frac{J_B a}{J_A b}}$$

$$T_b = \frac{T_0}{1 + \frac{J_a b}{J_b a}}$$

Using either of these values in (2) we have the angle of rotation θ_0 at the junction

$$\theta_0 = \frac{T_0 . a.b}{[J_A . b + J_B . a]G}$$

Non Uniform Torsion: The pure torsion refers to a torsion of a prismatic bar subjected to torques acting only at the ends. While the non uniform torsion differs from pure torsion in a sense that the bar / shaft need not to be prismatic and the applied torques may vary along the length.



Here the shaft is made up of two different segments of different diameters and having torques applied at several cross sections. Each region of the bar between the applied loads between changes in cross section is in pure torsion, hence the formula's derived earlier may be applied. Then form the internal torque, maximum shear stress and angle of rotation for each region can be calculated from the relation

$$\frac{T}{J} = \frac{\tau}{r} \text{ and } \frac{T}{J} = \frac{G\theta}{L}$$

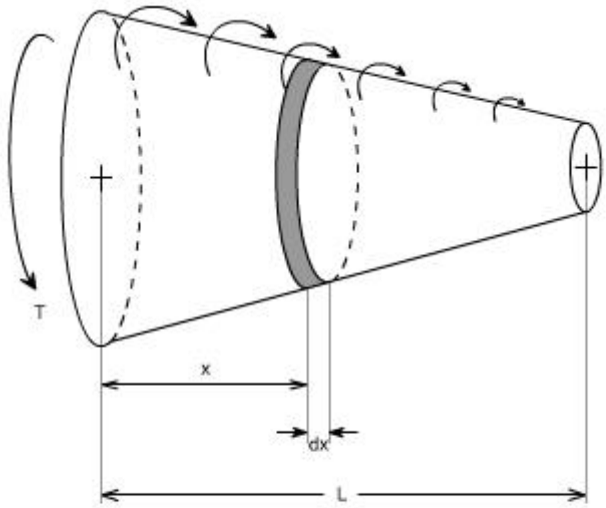
The total angle to twist of one end of the bar with respect to the other is obtained by summation using the formula

$$\theta = \sum_{i=1}^{n} \frac{T_i L_i}{G_i J_i}$$

i = index for no. of parts

n = total number of parts

If either the torque or the cross section changes continuously along the axis of the bar, then the (summation can be replaced by an integral sign (∫). i.e we will have to consider a differential element.



After considering the differential element, we can write $d\theta = \frac{T_x dx}{GI_x}$

Substituting the expressions for T_x and J_x at a distance x from the end of the bar, and then integrating between the limits 0 to L, find the value of angle of twist may be determined.

$$\theta = \int_0^L d\theta = \int_0^L \frac{T_x dx}{GI_x}$$

Concept of Shear Force and Bending moment in beams:

When the beam is loaded in some arbitrarily manner, the internal forces and moments are developed and the terms shear force and bending moments come into pictures which are helpful to analyze the

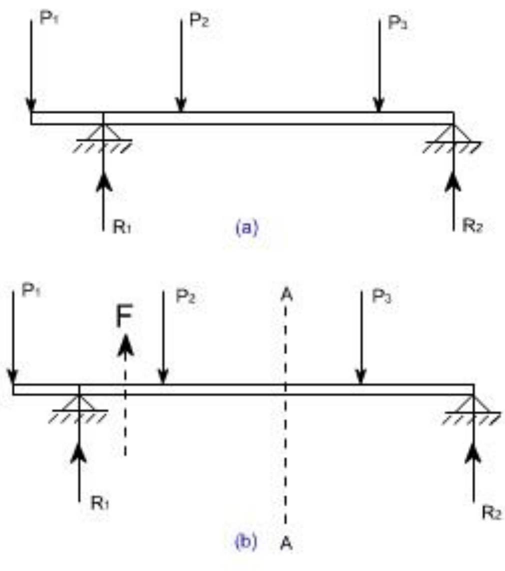


Fig 1

Now let us consider the beam as shown in fig 1(a) which is supporting the loads P_1, P_2, P_3 and is simply supported at two points creating the reactions R_1 and R_2 respectively. Now let us assume that the beam is to divided into or imagined to be cut into two portions at a section AA. Now let us assume that the resultant of loads and reactions to the left of AA is 'F' vertically upwards, and since the entire beam is to remain in equilibrium, thus the resultant of forces to the right of AA must also be F, acting downwards. This forces 'F' is as a shear force. The shearing force at any x-section of a beam represents the tendency for the portion of the beam to one side of the section to slide or shear laterally relative to the other portion. Therefore, now we are in a position to define the shear force 'F' to as follows: At any x-section of a beam, the shear force 'F' is the algebraic sum of all the lateral components of the forces acting on either side of the x-section.

Sign Convention for Shear Force:

The usual sign conventions to be followed for the shear forces have been illustrated in figures 2 and 3.

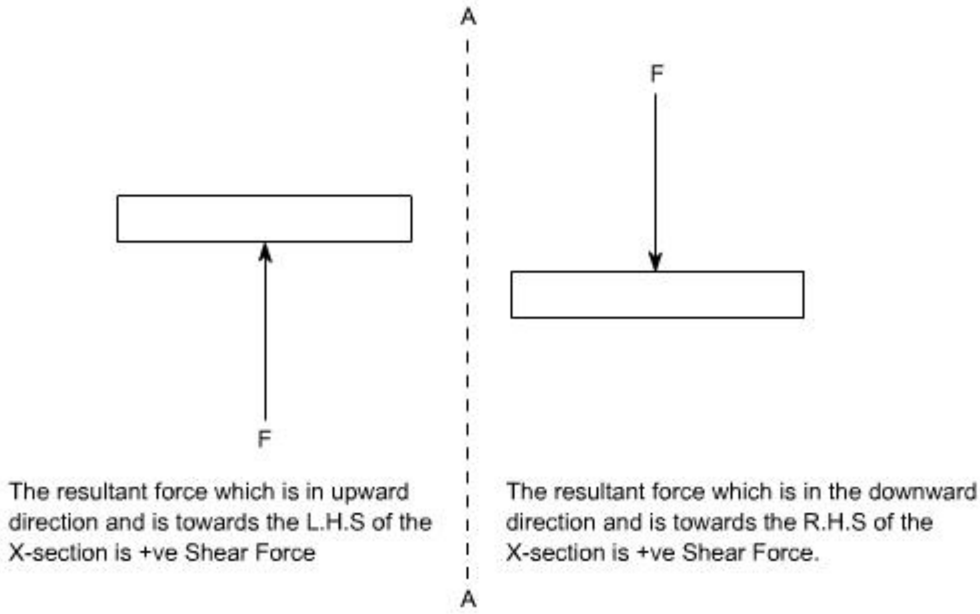


Fig 2: Positive Shear Force

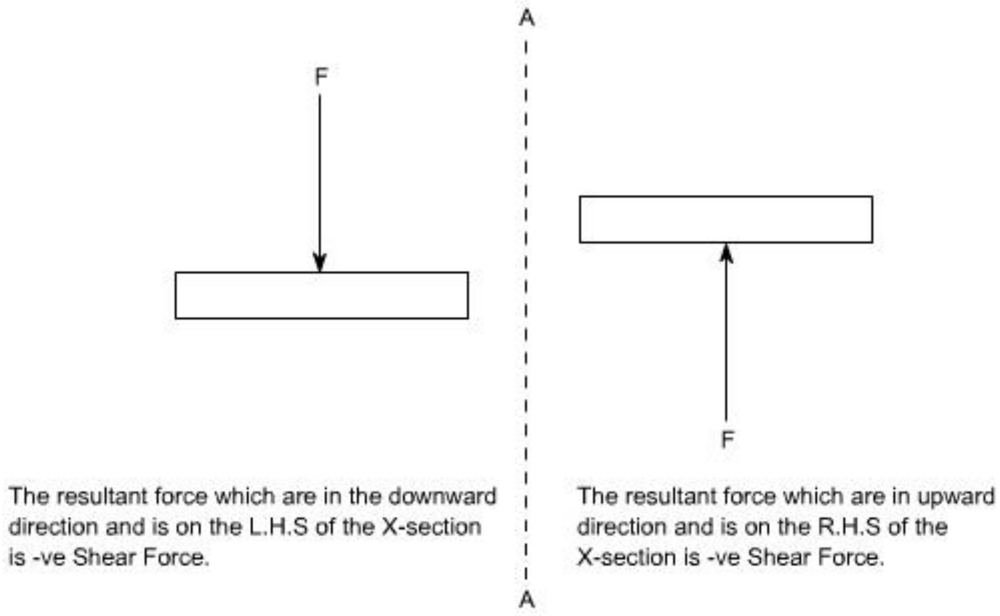
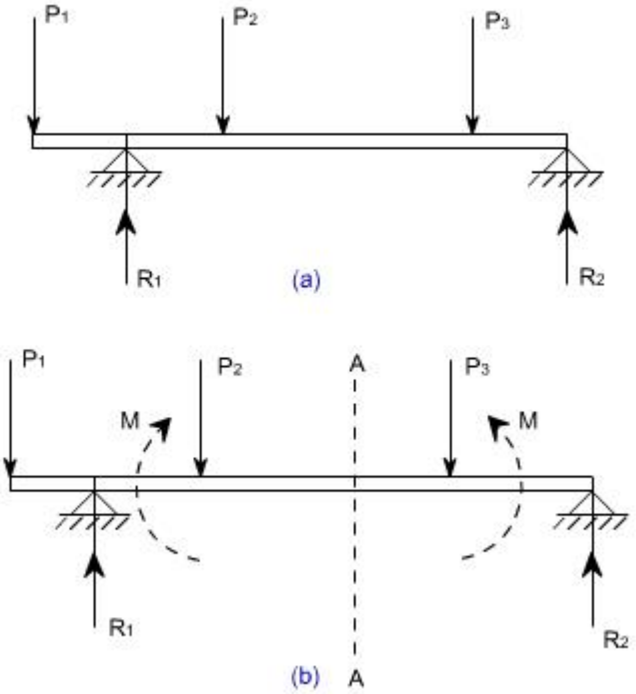


Fig 3: Negative Shear Force



Bending Moment:

Fig 4

Let us again consider the beam which is simply supported at the two prints, carrying loads P_1, P_2 and P_3 and having the reactions R_1 and R_2 at the supports Fig 4. Now, let us imagine that the beam is cut into two potions at the x-section AA. In a similar manner, as done for the case of shear force, if we say that the resultant moment about the section AA of all the loads and reactions to the left of the x-section at AA is M in C.W direction, then moment of forces to the right of x-section AA must be 'M' in C.C.W. Then 'M' is called as the Bending moment and is abbreviated as B.M. Now one can define the bending moment to be simply as the algebraic sum of the moments about an x-section of all the forces acting on either side of the section

Sign Conventions for the Bending Moment:

For the bending moment, following sign conventions may be adopted as indicated in Fig 5 and Fig 6.

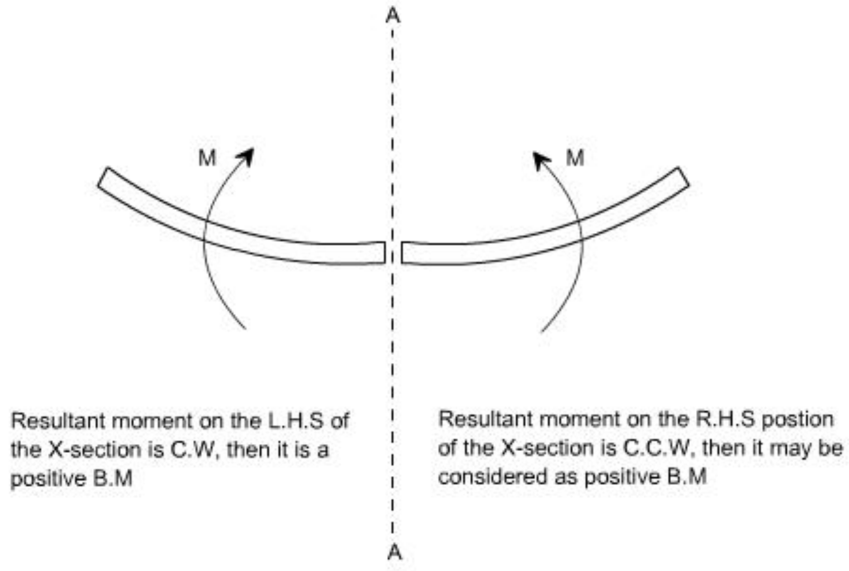


Fig 5: Positive Bending Moment

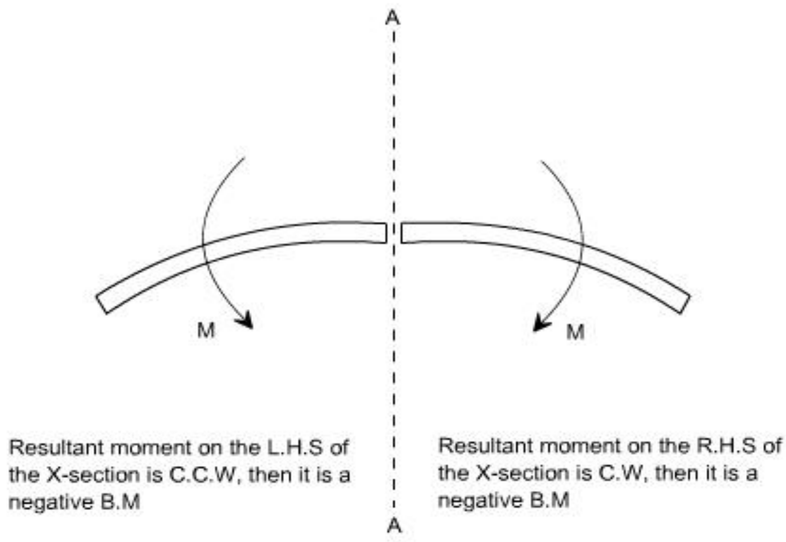


Fig 6: Negative Bending Moment

Sometimes, the terms 'Sagging' and Hogging are generally used for the positive and negative bending moments respectively.

Bending Moment and Shear Force Diagrams:

The diagrams which illustrate the variations in B.M and S.F values along the length of the beam for any fixed loading conditions would be helpful to analyze the beam further.

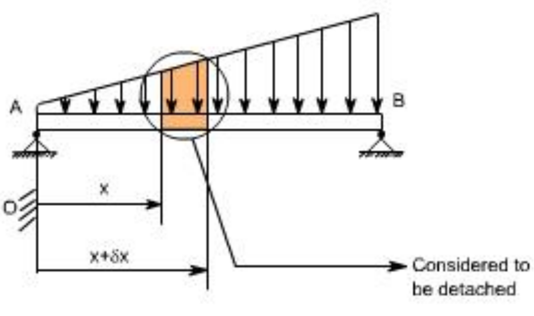
Thus, a shear force diagram is a graphical plot, which depicts how the internal shear force 'F' varies along the length of beam. If x denotes the length of the beam, then F is function x i.e. F(x).

Similarly a bending moment diagram is a graphical plot which depicts how the internal bending moment 'M' varies along the length of the beam. Again M is a function x i.e. M(x).

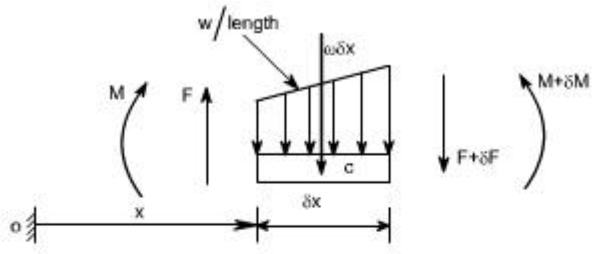
Basic Relationship between the Rate of Loading, Shear Force and Bending Moment:

The construction of the shear force diagram and bending moment diagrams is greatly simplified if the relationship among load, shear force and bending moment is established.

Let us consider a simply supported beam AB carrying a uniformly distributed load w/length. Let us imagine cutting a short slice of length dx cut out from this loaded beam at distance 'x' from the origin '0'.



Let us detach this portion of the beam and draw its free body diagram.



The forces acting on the free body diagram of the detached portion of this loaded beam are the following

• The shearing force F and F+ δF at the section x and x + δx respectively.

• The bending moment at the sections x and x + δx be M and M + δM respectively.

• Force due to external loading, if 'w' is the mean rate of loading per unit length then the total loading on this slice of length δx is w. δx, which is approximately acting through the centre 'c'. If the loading is assumed to be uniformly distributed then it would pass exactly through the centre 'c'.

This small element must be in equilibrium under the action of these forces and couples.

Now let us take the moments at the point 'c'. Such that

$$M + F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = M + \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + F.\frac{\delta x}{2} + \delta F.\frac{\delta x}{2} = \delta M \text{ [Neglecting the product of } \delta F \text{ and } \delta x \text{ being small quantities]}$$

$$\Rightarrow F.\delta x = \delta M$$

$$\Rightarrow F = \frac{\delta M}{\delta x}$$

Under the limits $\delta x \to 0$

$$\boxed{F = \frac{dM}{dx}} \quad \ldots\ldots\ldots\ldots \quad (1)$$

Resolving the forces vertically we get

$$w.\delta x + (F + \delta F) = F$$

$$\Rightarrow w = -\frac{\delta F}{\delta x}$$

Under the limits $\delta x \to 0$

$$\Rightarrow w = -\frac{dF}{dx} \text{ or } -\frac{d}{dx}\left(\frac{dM}{dx}\right)$$

$$\boxed{w = -\frac{dF}{dx} = -\frac{d^2M}{dx^2}} \quad \ldots\ldots\ldots\ldots (2)$$

Conclusions: From the above relations, the following important conclusions may be drawn

• From Equation (1), the area of the shear force diagram between any two points, from the basic calculus is the bending moment diagram

$$M = \int F.dx$$

• The slope of bending moment diagram is the shear force, thus

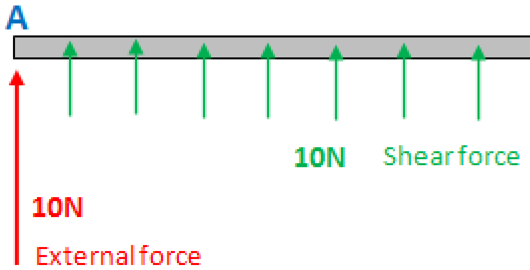
$$F = \frac{dM}{dx}$$

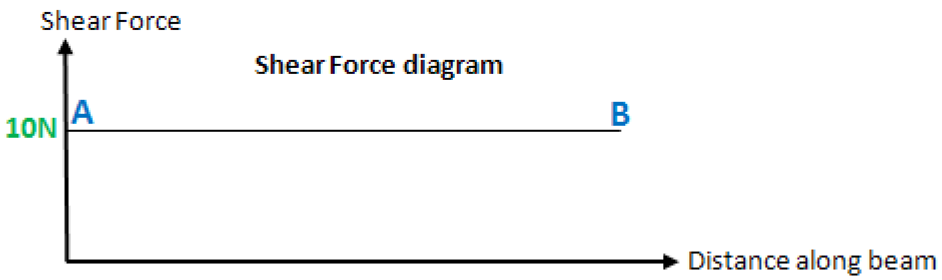
Thus, if F=0; the slope of the bending moment diagram is zero and the bending moment is therefore constant.'

• The maximum or minimum Bending moment occurs where $\frac{dM}{dx}=0.$

The slope of the shear force diagram is equal to the magnitude of the intensity of the distributed loading at any position along the beam. The –ve sign is as a consequence of our particular choice of sign conventions

Below a force of 10N is exerted at point A on a beam. This is an external force. However because the beam is a rigid structure,the force will be internally transferred all along the beam. This internal force is known as shear force. The shear force between point A and B is usually plotted on a shear force diagram. As the shear force is 10N all along the beam, the plot is just a straight line, in this example.

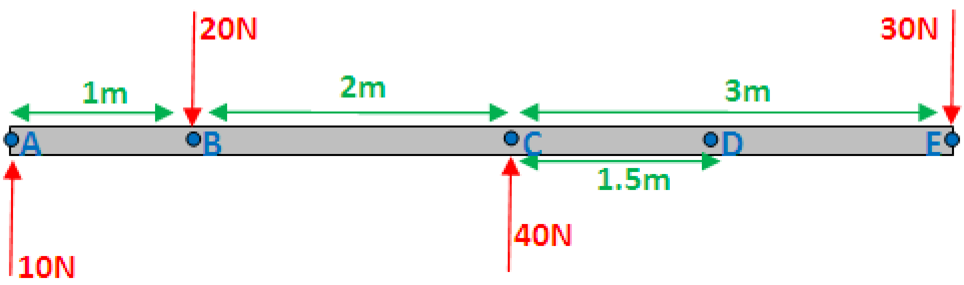




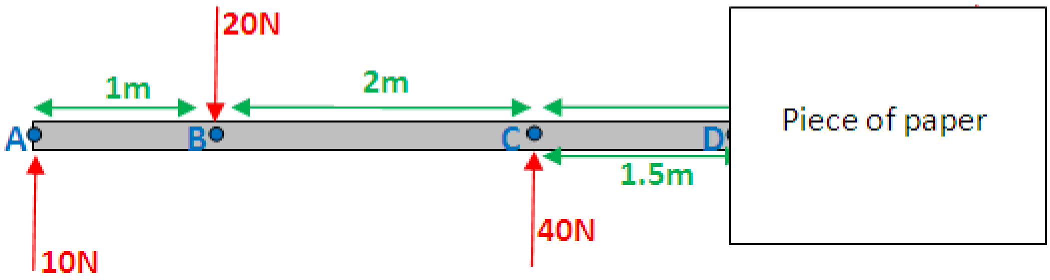
The idea of shear force might seem odd, maybe this example will help clarify. Imagine pushing an object along a kitchen table, with a 10N force. Even though you're applying the force only at one point on the object, it's not just that point of the object that moves forward. The whole object moves forward, which tells you that the force must have transferred all along the object, such that every atom of the object is experiencing this 10N force.

Please note that this is not the full explanation of what shear force actually is.

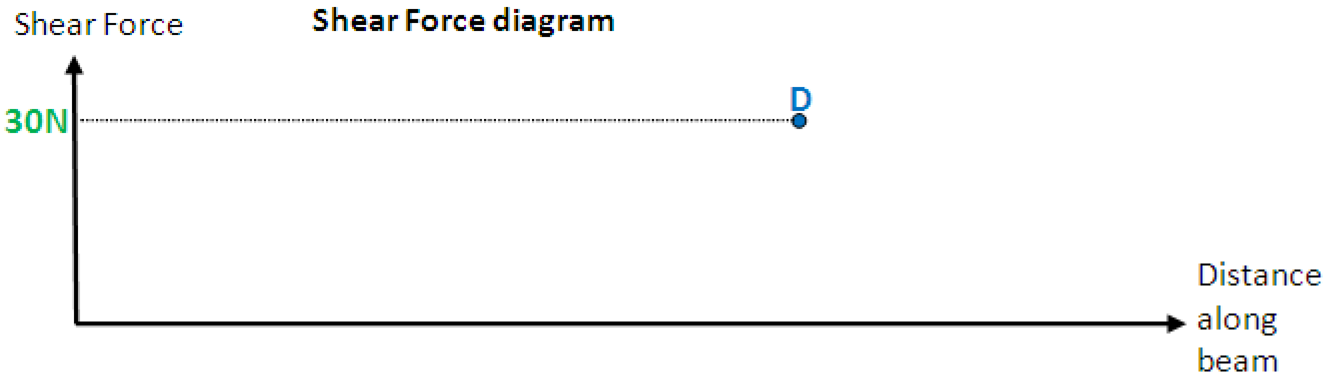
What if there is more than one force, as shown in the diagram below, what would the shear force diagram look like then?



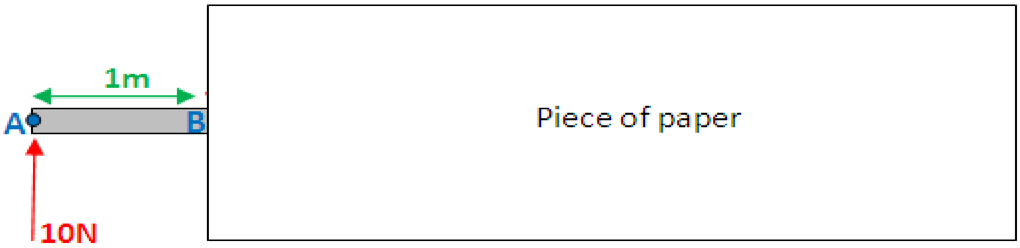
The way you go about this is by figuring out the shear force at points A,B,C,E (as there is an external force acting at these points). The way you work out the shear force at any point, is by covering (either with your hand or a piece of paper), everything to right of that point, and simply adding up the external forces. Then plot the point on the shear force diagram. For illustration purposes, this is done for point D:



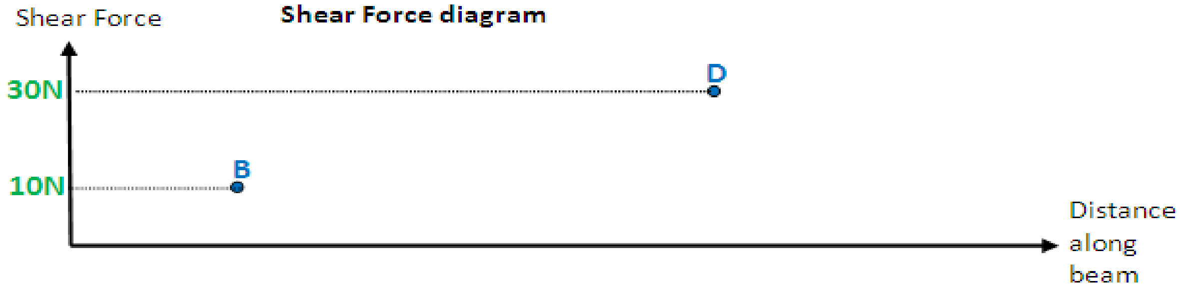
Shear force at D = 10N - 20N + 40N = 30Newtons



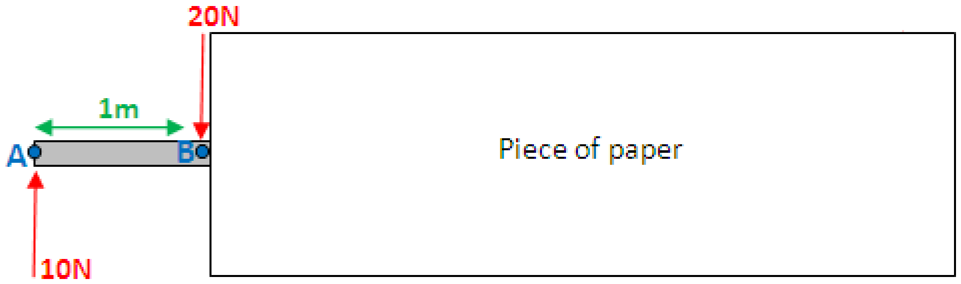
Now, let's do this for point B. BUT - slight complication - there's a force acting at point B, are you going to include it? The answer is both yes and no. You need to take 2 measurements. Firstly put your piece of paper, so it's JUST before point B:



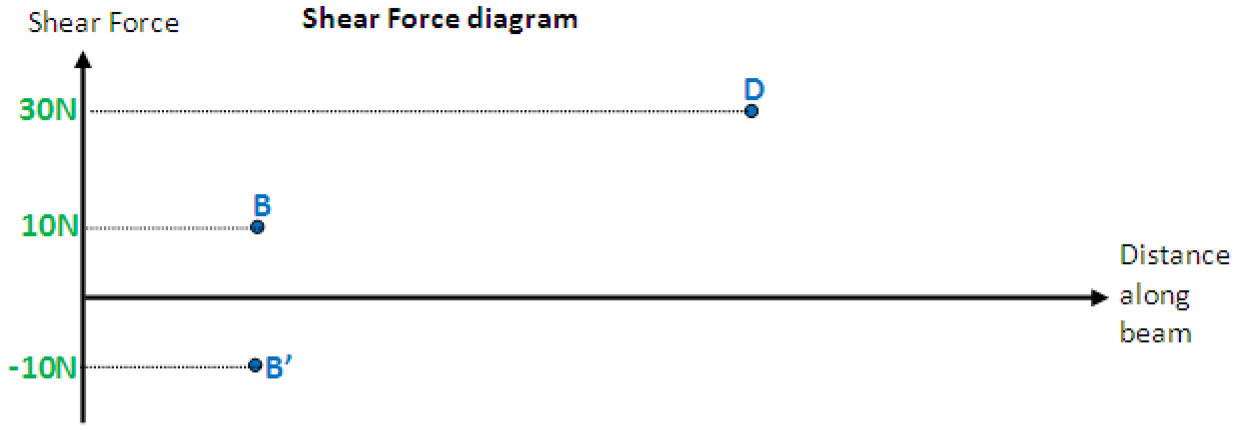
Shear force at B = 10N



Now place your paper JUST after point B:

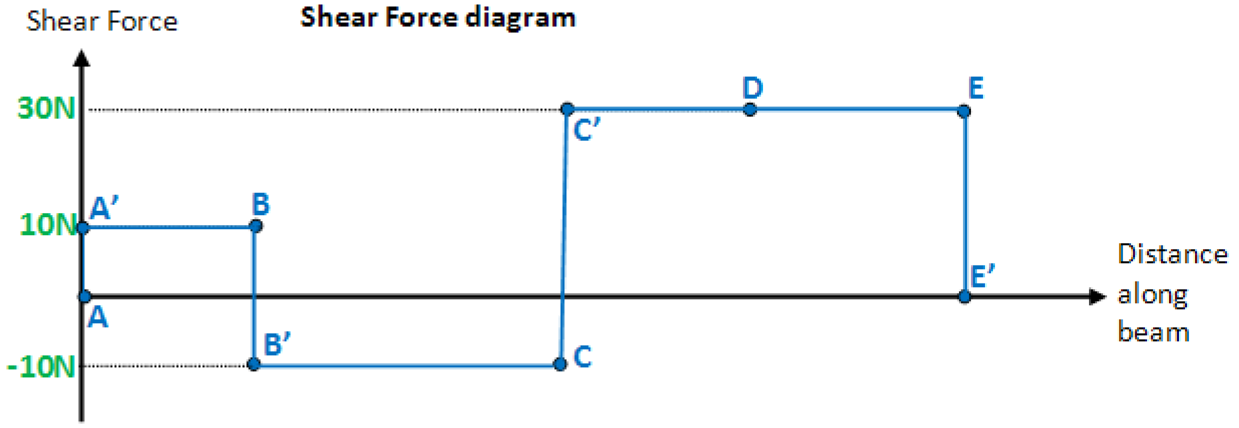


Shear force at B = 10N - 20N = -10N



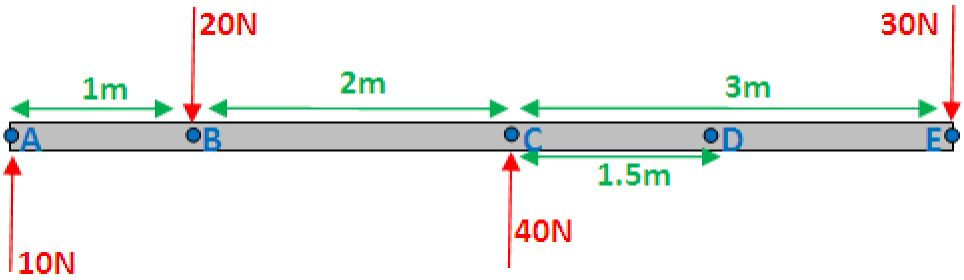
(B' is vertically below B)

Now, do point A, D and E, and finally join the points. your diagram should look like the one below. If you don't understand why, leave a message on the discussion section of this page (its at the top), I will elaborate on the explanation:

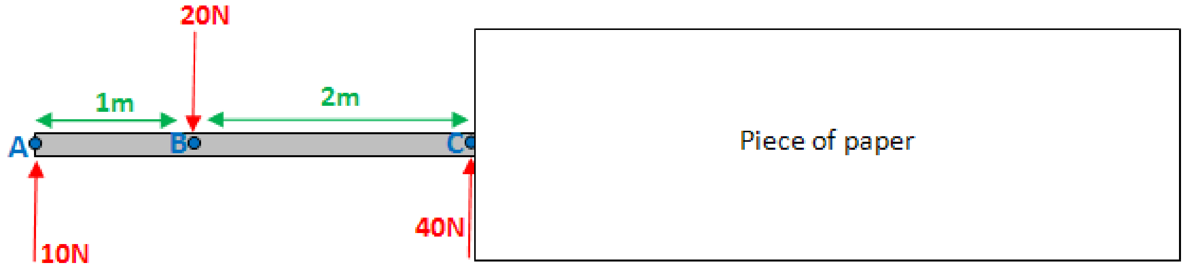


Notice how nothing exciting happens at point D, which is why you wouldn't normally analyse the shear force at that point. For clarity, when doing these diagrams it is recommended you move you paper from left to right, and hence analyse points A,B, C, and E, in that order. You can also do this procedure covering the left side instead of the right, your diagram will be "upside down" though. Both diagrams are correct.

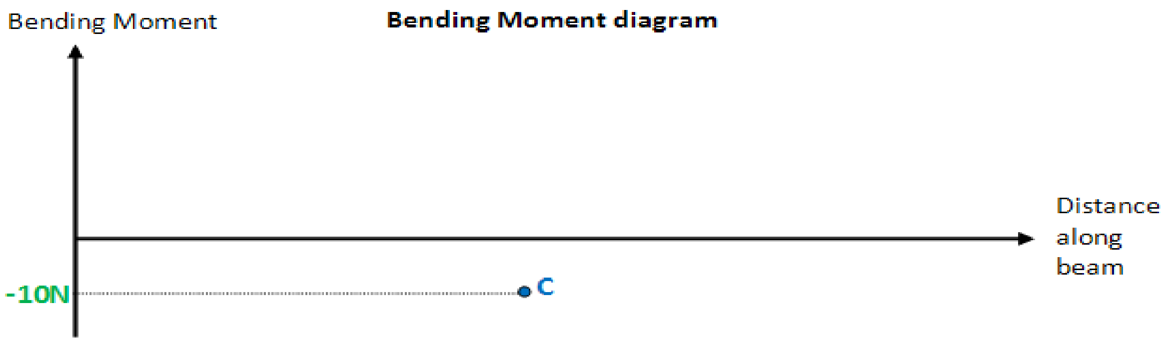
Bending moment refers to the internal moment that causes something to bend. When you bend a ruler, even though apply the forces/moments at the ends of the ruler, bending occurs all along the ruler, which indicates that there is a bending moment acting all along the ruler. Hence bending moment is shown on a bending moment diagram. The same case from before will be used here:



To work out the bending moment at any point, cover (with a piece of paper) everything to the right of that point, and take moments about that point. (I will take clockwise moments to be positive). To illustrate, I shall work out the bending moment at point C:

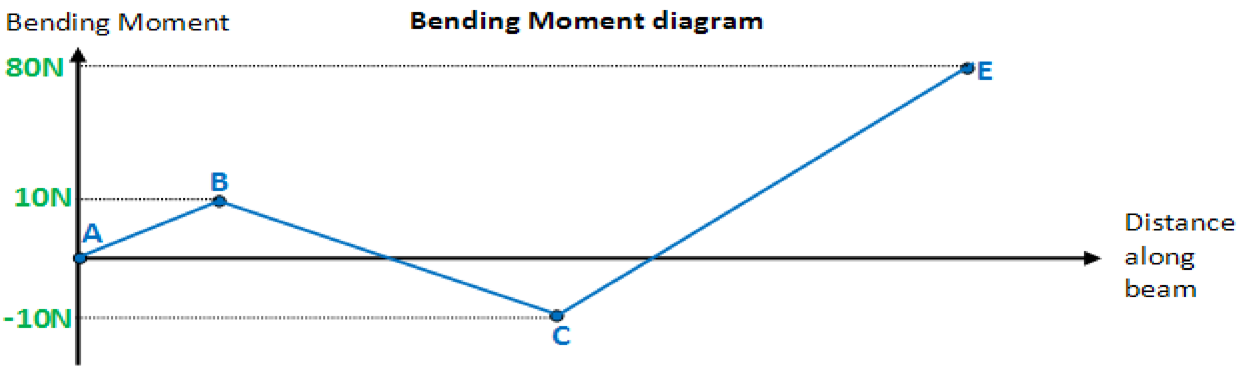


Bending moment at C = 10Nx3m - 20Nx2m = -10Nm



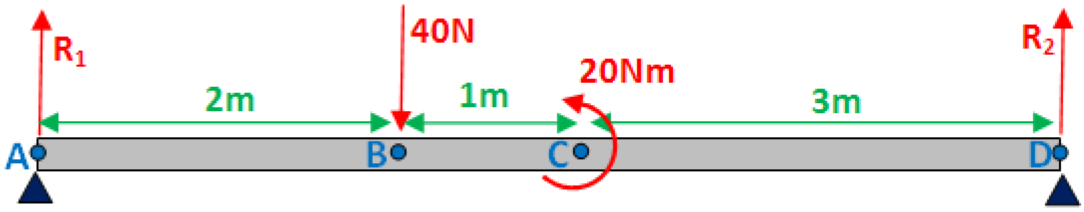
Notice that there's no need to work out the bending moment "just before and just after" point C, (as in the case for the shear force diagram). This is because the 40N force at point C exerts no moment about point C, either way.

Repeating the procedure for points A,B and E, and joining all the points:



Normally you would expect the diagram to start and end at zero, in this case it doesn't. This is my fault, and it happened because I accidentally chose my forces such that there is moment disequilibrium. i.e. take moments about any point (without covering the right of the point), and you'll notice that the moments aren't balanced, as they should be. It also means that if you're covering the left side as opposed to the right, you will get a completely different diagram. Sorry about this... Upon inspection, the forces are unbalanced, so it is immediately expected that the diagram will most likely not be balanced.

Point moments are something that you may not have come across before. Below, a point moment of 20Nm is exerted at point C. Work out the reaction of A and D:



Force equilibrium: $R_1 + R_2 = 40$

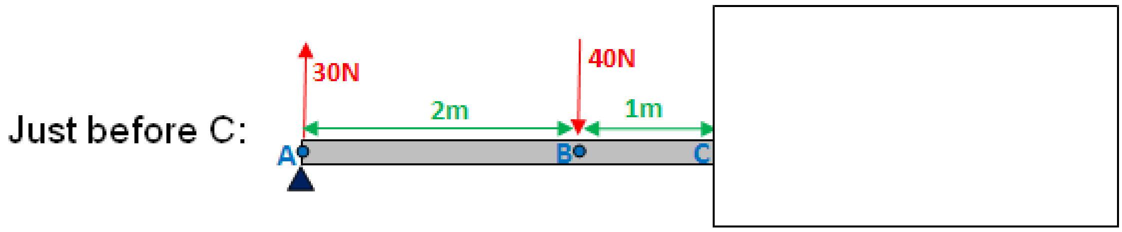
Taking moments about A (clockwise is positive): $40 \cdot 2 - 20 - 6 \cdot R_2 = 0$

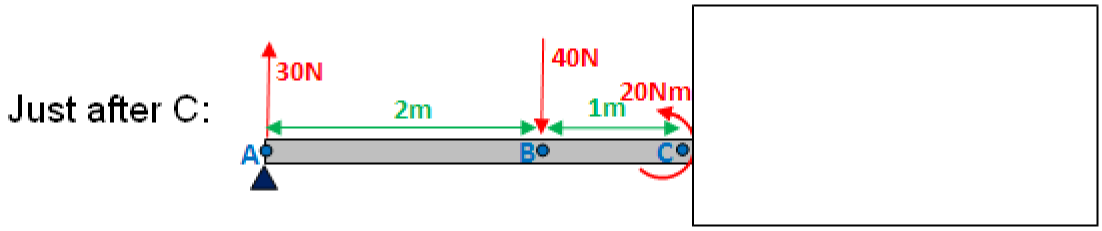
$R_1 = 30N$, $R_2 = 10N$

If instead you were to take moments about D you would get: $- 20 - 40 \cdot 4 + 6 \cdot R_1 = 0$

I think it's important for you to see that wherever you take moments about, the point moment is always taken as a negative (because it's a counter clockwise moment).

So how does a point moment affect the shear force and bending moment diagrams? Well. It has absolutely no effect on the shear force diagram. You can just ignore point C when drawing the shear force diagram. When drawing the bending moment diagram you will need to work out the bending moment just before and just after point C:





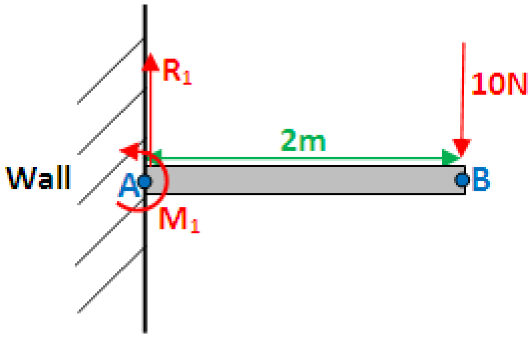
Just before: bending moment at C = $3 \cdot 30 - 1 \cdot 40 = 50Nm$

Just after: bending moment at C = $3 \cdot 30 - 1 \cdot 40 - 20 = 30Nm$

Then work out the bending moment at points A, B and D (no need to do before and after for these points). And plot.

Cantilever beam

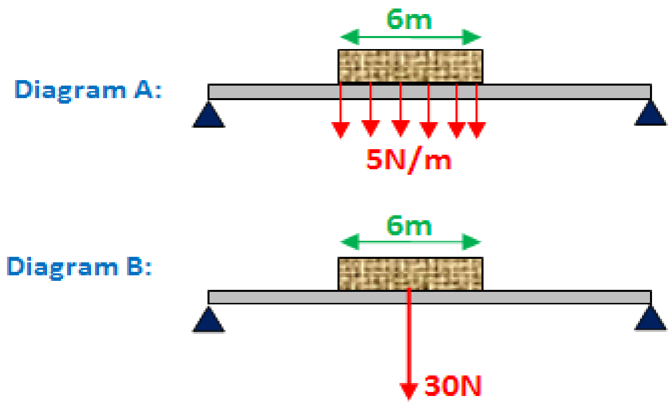
Until now, you may have only dealt with "simply supported beams" (like in the diagram above), where a beam is supported by 2 pivots at either end. Below is a cantilever beam, which means - a beam that rigidly attached to a wall. Just like a pivot, the wall is capable of exerting an upwards reaction force R_1 on the beam. However it is also capable of exerting a point moment M_1 on the beam.



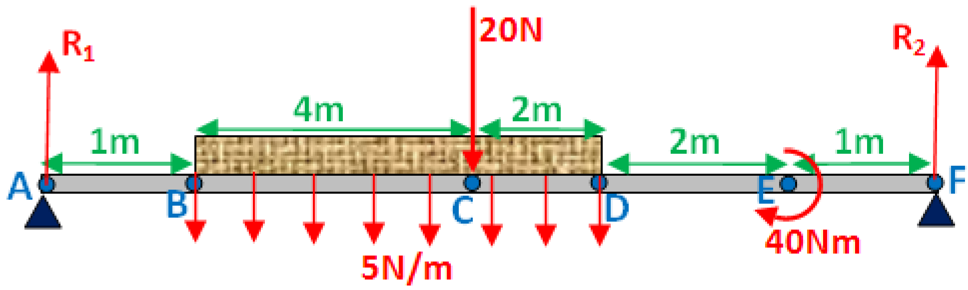
Force equilibrium: $R_1 = 10N$

Taking moments about A: $-M_1 + 10 \cdot 2 = 0 \rightarrow M_1 = 20Nm$

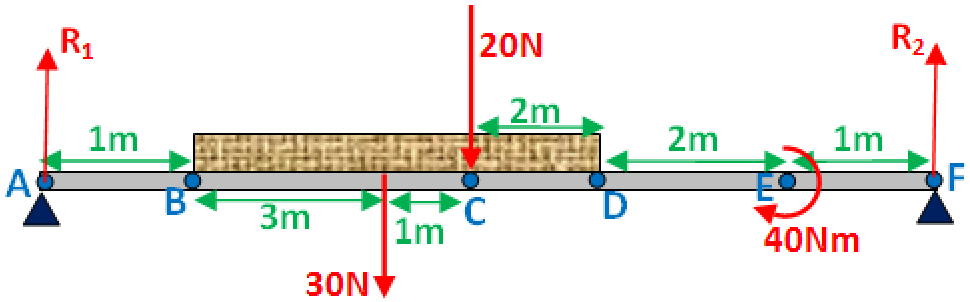
Below is a brick lying on a beam? The weight of the brick is uniformly distributed on the beam (shown in diagram A). [Actually, whoever wrote this, this is not true; the weight of the brick is not uniformly distributed along the beam, it is concentrated in the middle, according to the diagram.] The brick has a weight of 5N per meter of brick (5N/m). Since the brick is 6 meters long the total weight of the brick is 30N. This is shown in diagram B. Diagram B is a simplification of diagram A. As you will see, you will need to be able to convert a type A diagram to a type B.



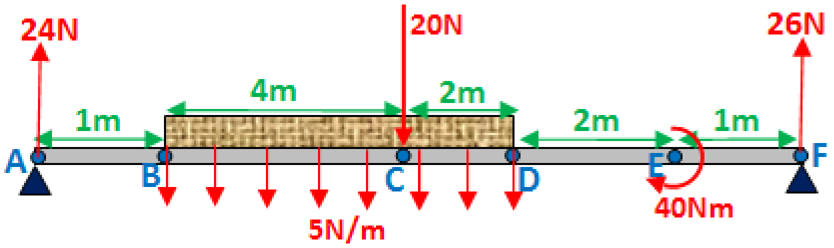
To make your life more difficult I have added an external force at point C, and a point moment to the diagram below. This is the most difficult type of question I can think of, and I will do the shear force and bending moment diagram for it, step by step.



Firstly identify the key points at which you will work out the shear force and bending moment at. These will be points: A,B,C,D,E and F.
As you would have noticed when working out the bending moment and shear force at any given point, sometimes you just work it out at the point, and sometimes you work it out just before and after. Here is a summary: When drawing a shear force diagram, if you are dealing with a point force (points A,C and F in the above diagram), work out the shear force before and after the point. Otherwise (for points B and D), just work it out right at that point. When drawing a bending moment diagram, if you are dealing with a point moment (point E), work out the bending moment before and after the point. Otherwise (for points A,B,C,D, and F), work out the bending moment at the point.
After identifying the key points, you want to work out the values of R_1 and R_2. You now need to convert to a type B diagram, as shown below. Notice the 30N force acts right in the middle between points B and D.

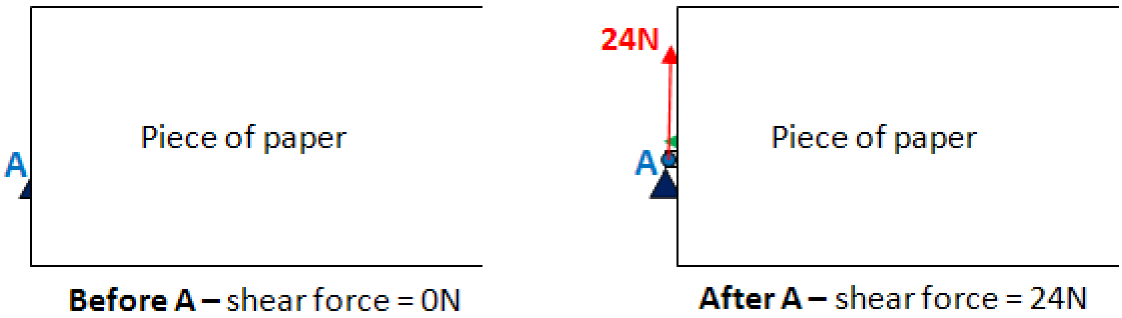


Force equilibrium: $R_1 + R_2 = 50$
Take moments about A: $4 \cdot 30 + 5 \cdot 20 + 40 - 10 \cdot R_2 = 0$
$R_1 = 24N$, $R_2 = 26N$
Update original diagram:

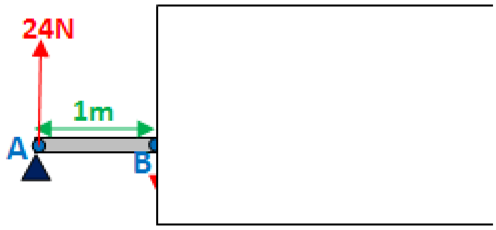


Shear force diagram [edit]

Point A:



point B:

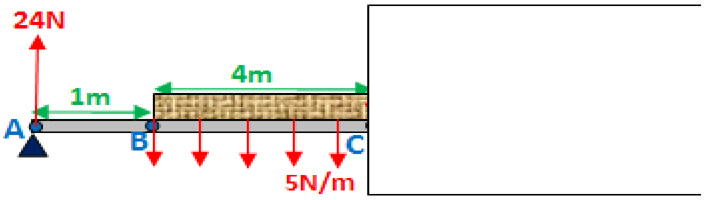


Shear force at B = 24N

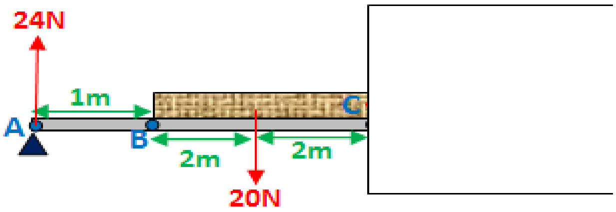
Notice that the uniformly distributed load has no effect on point B.

point C:

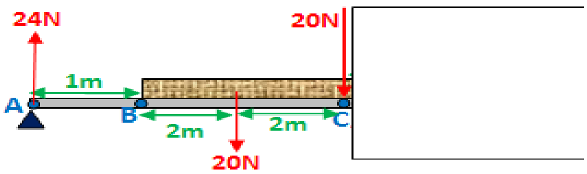
Just before C:



Now convert to a type B diagram. Total weight of brick from point B to C = 5x4 = 20N

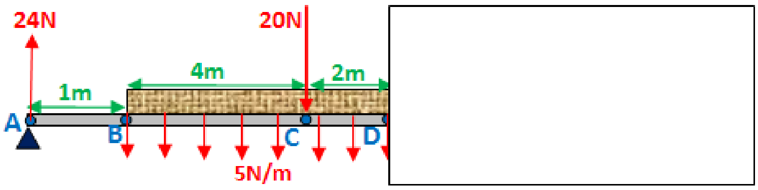


Shear force before C: 24 - 20 = 4N

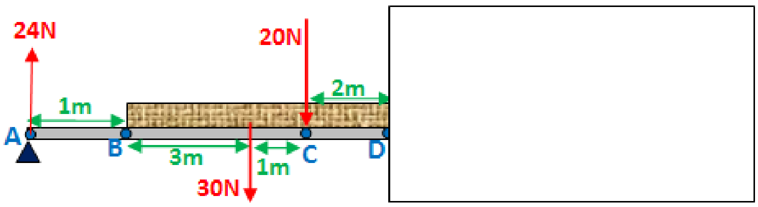


Shear force after C: 24 - 20 - 20 = -16N

Point D:



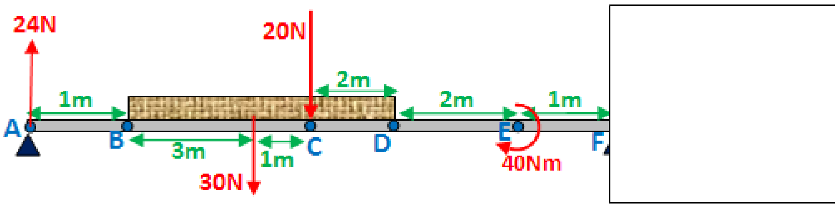
Convert to type B diagram:



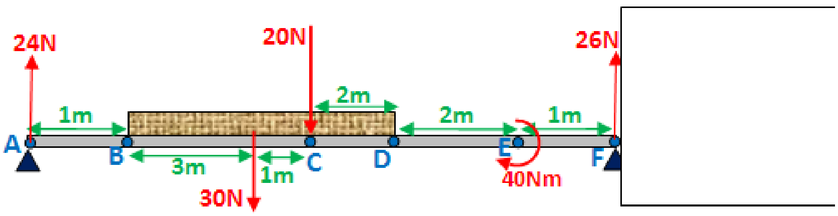
Shear force at D: 24 - 30 - 20 = -26N

point F:

(I have already converted to a type B diagram, below)

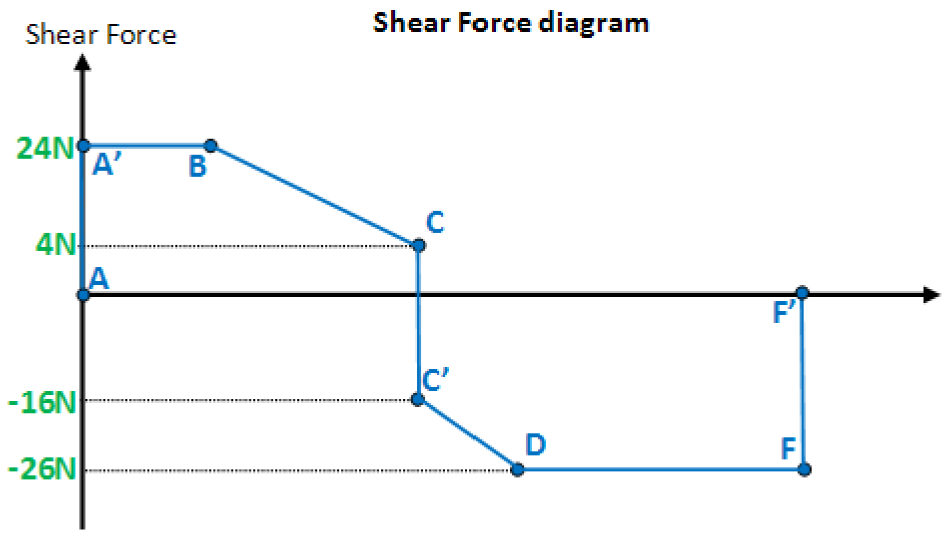


Shear force before F: 24 -30 -20 = -26N



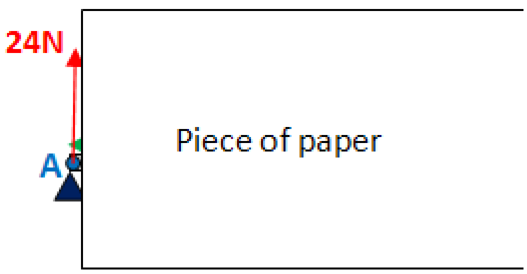
Shear force after F: 24 -30 -20 +26 = 0N

Finally plot all the points on the shear force diagram and join them up:



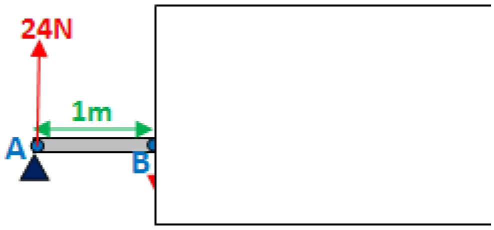
Bending moment diagram[edit]

Point A



Bending moment at A: 0Nm

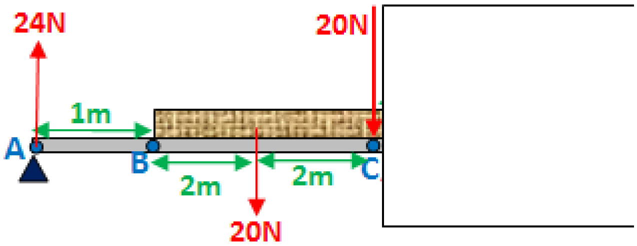
Point B



Bending moment at B: 24·1 = 24Nm

point C:

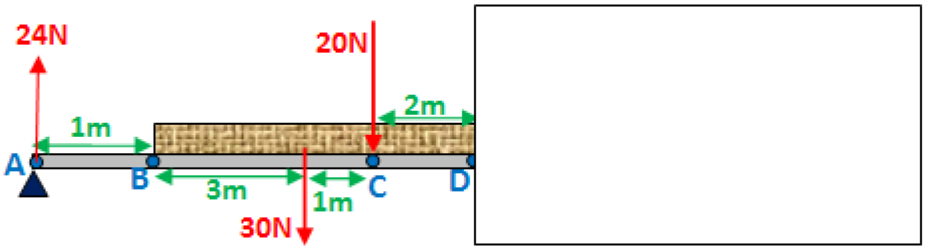
(I have already converted to a type B diagram, below)



Bending moment at C: 24·5 - 20·2 = 80Nm

point D:

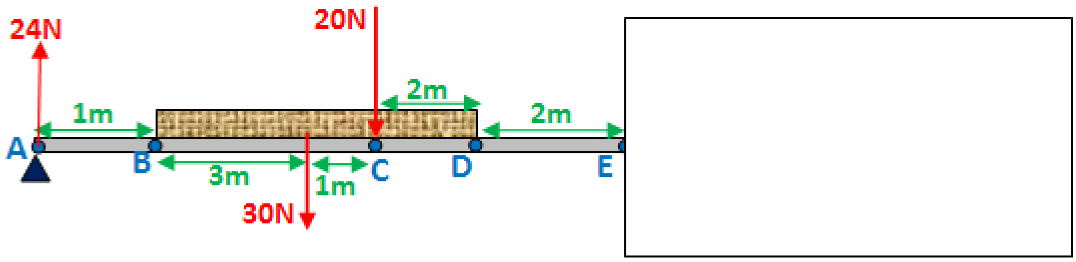
(I have already converted to a type B diagram, below)



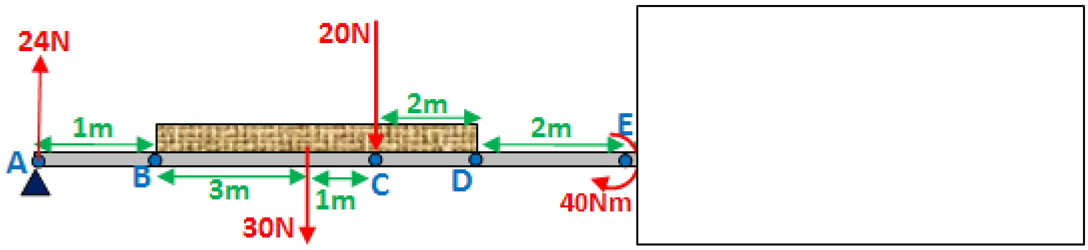
Bending moment at D: 24·7 - 30·3 - 20·2 = 38Nm

Point E:

(I have already converted to a type B diagram, below)

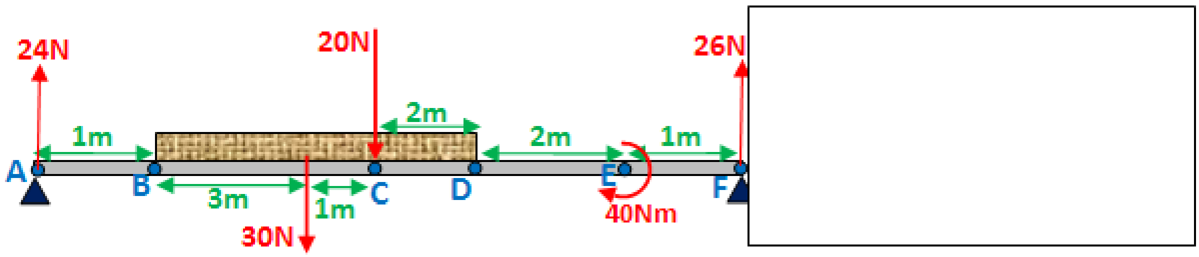


Bending moment just before E: 24·9 - 30·5 -20·4 = -14Nm

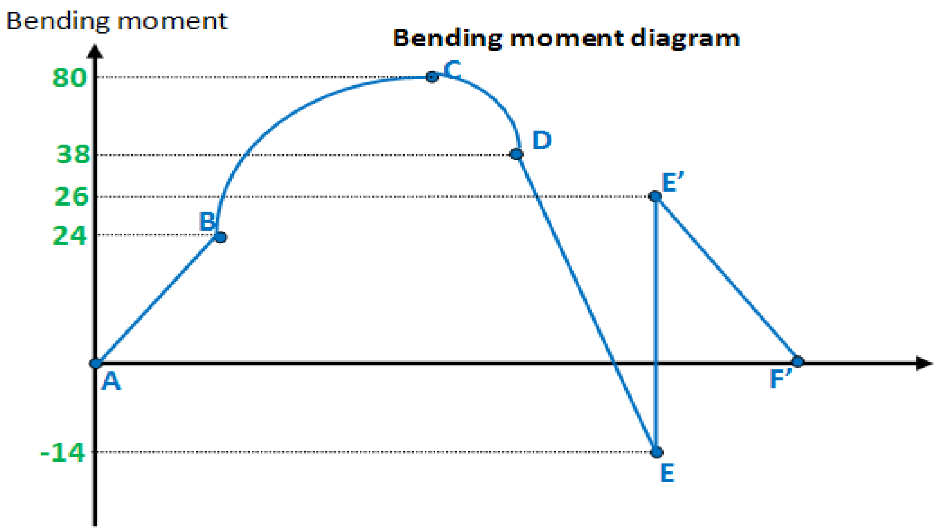


Bending moment just after E: 24·9 - 30·5 -20·4 + 40 = 26Nm

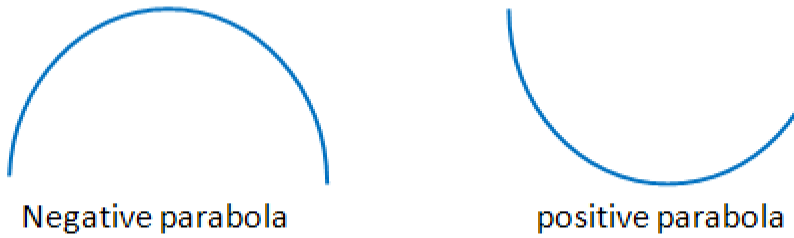
point F:
(I have already converted to a type B diagram, below)



Bending moment at F: 24·10 - 30·6 - 20·5 + 40 = 0Nm
Finally, plot the points on the bending moment diagram. Join all the points up, EXCEPT those that are under the uniformly distributed load (UDL), which are points B,C and D. As seen below, you need to draw a curve between these points. Unless requested, I will not explain why this happens.



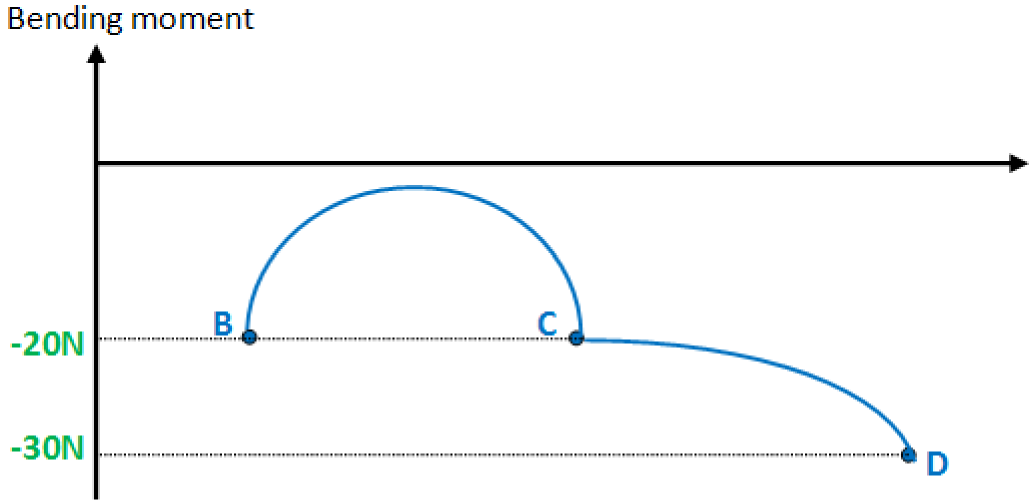
Note: The diagram is not at all drawn to scale.
I have drawn 2 curves. One from B to C, one from C to D. Notice that each of these curves resembles some part of a negative parabola.



Rule: When drawing a bending moment diagram, under a UDL, you must connect the points with a curve. This curve must resemble some part of a negative parabola.
Note: The convention used throughout this page is "clockwise moments are taken as positive". If the convention was "counter-clockwise moments are taken as positive", you would need to draw a positive parabola.

Hypothetical scenario

For a hypothetical question, what if points B, C and D, were plotted as shown below. Notice how I have drawn the curves for this case.



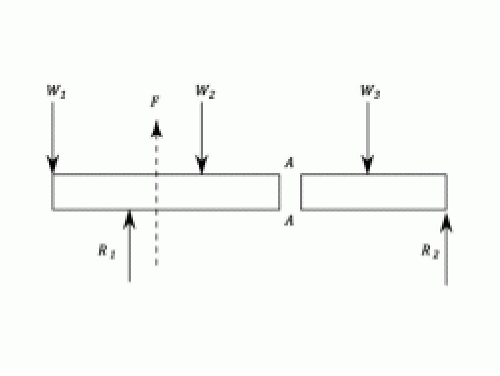
If you wanted to find the peak of the curve, how would you do it? Simple. On the original diagram (used at the start of the question) add an additional point (point G), centrally between point B and C. Then work out the bending moment at point G.

That's it! If you have found this article useful, please comment in the discussion section (at the top of the page), as this will help me decide whether to write more articles like this. Also please comment if there are other topics you want covered, or you would like something in this article to be written more clearly.

Shear Forces occurs when two parallel forces act out of alignment with each other. For example, in a large boiler made from sections of sheet metal plate riveted together, there is an equal and opposite force exerted on the rivets, owing to the expansion and contraction of the plates.

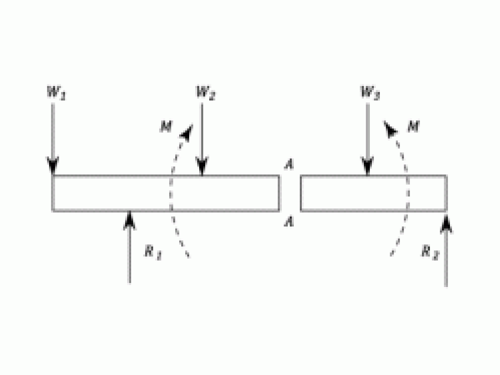
Bending Moments are rotational forces within the beam that cause bending. At any point within a beam, the Bending Moment is the sum of: each external force multiplied by the distance that is perpendicular to the direction of the force.

Shear Force



The shearing force (SF) at any section of a beam represents the tendency for the portion of the beam on one side of the section to slide or shear laterally relative to the other portion.
The diagram shows a beam carrying loads W_1 , W_2 and W_3. It is simply supported at two points where the reactions are R_1 and R_2. Assume that the beam is divided into two parts by a section XX. The resultant of the loads and reaction acting on the left of AA is F vertically upwards, and since the whole beam is in equilibrium, the resultant force to the right of AA must be F downwards. F is called the **Shearing Force** at the section AA. It may be defined as follows:-
The shearing force at any section of a beam is the algebraic sum of the lateral components of the forces acting on either side of the section.
Where forces are neither in the lateral or axial direction they must be resolved in the usual way and only the lateral components are used to calculate the shear force.

Bending Moments



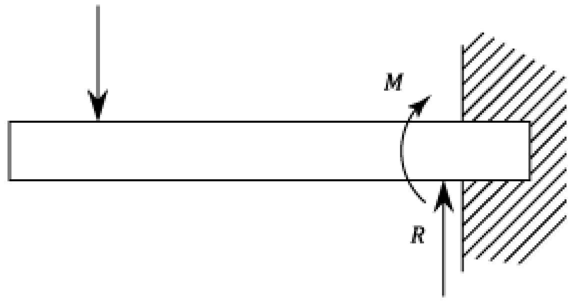
In a similar manner it can be seen that if the Bending moments (BM) of the forces to the left of AA are clockwise, then the bending moment of the forces to the right of AA must be anticlockwise.
Bending Moment at AA is defined as the algebraic sum of the moments about the section of all forces acting on either side of the section.
Bending moments are considered positive when the moment on the left portion is clockwise and on the right anticlockwise. This is referred to as a **sagging** bending moment as it tends to make the beam concave upwards at AA. A negative bending moment is termed **hogging**.

Types of Load

A beam is normally horizontal and the loads vertical. Other cases which occur are considered to be exceptions.
A **Concentrated load** is one which can be considered to act at a point, although in practice it must be distributed over a small area.
A **Distributed load** is one which is spread in some manner over the length, or a significant length, of the beam. It is usually quoted at a weight per unit length of beam. It may either be uniform or vary from point to point.

Types of Support

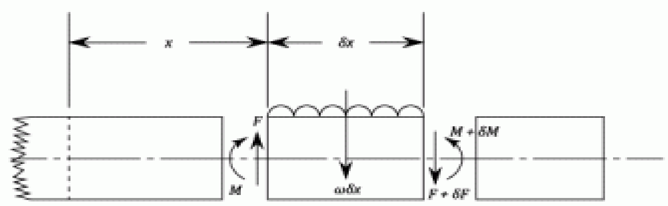
A **Simple** or **free** support is one on which the beam is rested and which exerts a reaction on the beam. It is normal to assume that the reaction acts at a point, although it may in fact act act over a short length of beam.



A **Built-in** or **encastre'** support is frequently met . The effect is to fix the direction of the beam at the support. In order to do this the support must exert a "fixing" moment M and a reaction R on the beam. A beam which is fixed at one end in this way is called a **Cantilever**. If both ends are fixed in this way the reactions are not statically determinate.
In practice, it is not usually possible to obtain perfect fixing and the fixing moment applied will be related to the angular movement of the support. When in doubt about the rigidity, it is safer to assume that the beam is freely supported.

The Relationship between W, F, M.

In the following diagram δx is the length of a small slice of a loaded beam at a distance x from the origin O



Let the shearing force at the section x be F and at $x = \delta x \quad be \quad F + \delta F$. Similarly, the bending moment is M at x, and$M + \delta M \quad at \quad x + \delta x$. If w is the mean rate of loading of the lengthδx, then the total load is $w\delta x$, acting approximately (exactly if uniformly distributed) through the centre C. The element must be in equilibrium under the action of these forces and couples and the following equations can be obtained:-

Taking Moments about C:

$$M + F.\frac{\delta x}{2} + (F + \delta F)\,\frac{\delta x}{2} = M + \delta M$$

Neglecting the product$\delta F.\delta x$ in the limit:

$$F = \frac{dM}{dx}$$

Resolving vertically:

$$w\delta x + F + \delta F = F$$

$$Or \qquad w = -\frac{dF}{dx}$$

$$= -\frac{d^2 M}{dx^2} \qquad from\ equation\ (2)$$

From equation (2) it can be seen that if M is varying continuously, zero shearing force corresponds to either maximum or minimum bending moment. It can be seen from the examples that "peaks" in the bending moment diagram frequently occur at concentrated loads or reactions, and these are not given by $F = \frac{dM}{dx} = 0$; although they may in fact represent the greatest bending moment on the beam. Consequently, it is not always sufficient to investigate the points of zero shearing force when determining the maximum bending moment.

At a point on the beam where the type of bending is changing from sagging to hogging, the bending moment must be zero, and this is called a point of *inflection* or *contraflexure*.

By integrating equation (2) between the x = a and x = b then:

$$M_b - M_a = \int_a^b F\,dx$$

Which shows that the increase in bending moment between two sections is the area under the shearing force diagram.

Similarly integrating equation (4)

$$F_a - F_b = \int_a^b w\,dx$$

equals the area under the load distribution diagram.

Integrating equation (5) gives:

$$M_a - M_b = \int\int_a^b w\,dx.dx$$

These relations can be very valuable when the rate of loading cannot be expressed in an algebraic form as they provide a means of graphical solution.

Concentrated Loads

Example

Problem

A Cantilever of length *l* carries a concentrated load *W* at its free end. Draw the Shear Force (SF) and Bending Moment (BM) diagrams.

Workings

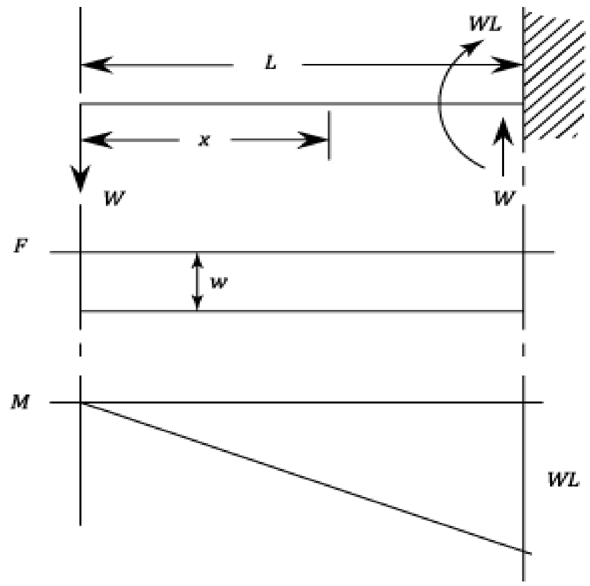
Consider the forces to the left of a section at a distance *x* from the free end.

Then F = - W and is constant along the whole cantilever i.e. for all values of *x*

Taking Moments about the section gives M = - W *x* so that the maximum Bending Moment occurs when *x* = *l* i.e. at the fixed end.

$\hat{M} = W\,l \quad (Hogging)$

From equilibrium considerations it can be seen that the fixing moment applied at the built in end is WL and the reaction is W. Hence the **SF** and **BM** diagrams are as follows:



The following general conclusions can be drawn when only concentrated loads and reactions are involved.

- **The shearing force suffers sudden changes when passing through a load point. The change is equal to the load.**
- **The bending Moment diagram is a series of straight lines between loads. The slope of the lines is equal to the shearing force between the loading points**.

Uniformly Distributed Loads

Example

Problem

Draw the SF and BM diagrams for a Simply supported beam of length *l* carrying a uniformly distributed load *w* per unit length which occurs across the whole Beam.

Workings

The Total Load carried is *wl* and by symmetry the reactions at both end supports are each wl/2

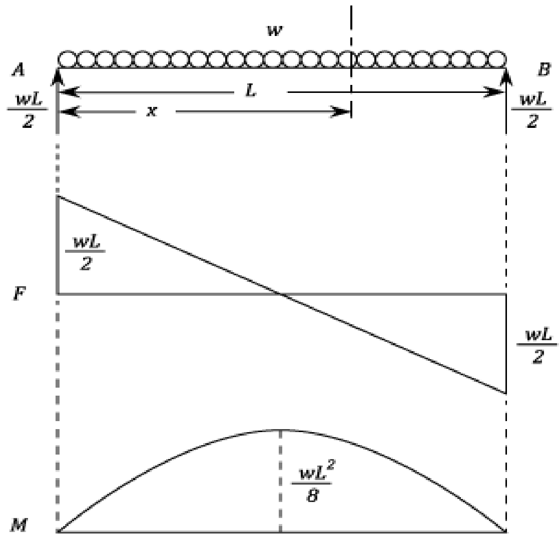
If *x* is the distance of the section measured from the left-hand support then:

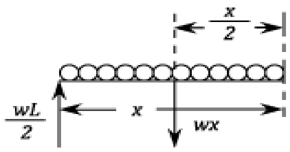
$$F = \frac{wl}{2} - wx = w\left(\frac{l}{2} - x\right)$$

This give a straight line graph equal to the rate of loading. The end values of Shearing Force are $\pm\frac{w\,l}{2}$

The Bending Moment at the section is found by assuming that the distributed load acts through its center of gravity which is *x/2* from the section.

$$Hence \qquad M = \left(\frac{w\,l}{2}\right) x - (wx)\,\frac{x}{2}$$





$$= \left(\frac{w\,l}{2}\right)(l - x)$$

This is a parabolic curve having a value of zero at each end. The maximum is at the center and corresponds to zero shear force.

From Equation (2)

$$\hat{M} = \left(\frac{w\,l}{4}\right)\left(l - \frac{l}{2}\right)$$

Putting *x* = *l/2*

$$\hat{M} = \frac{w\,l^2}{8}$$

Combined Loads.

Example

Problem

A Beam 25 ft. long is supported at A and B and is loaded as shown. Sketch the SF and BM diagrams and find (a) the position and magnitude of the maximum Bending Moment and (b) the position of the point of contra flexure.

Workings

Taking Moments about B

$$20\,R_a = 10 \times 15 + 2 \times 5 - 3 \times 5$$

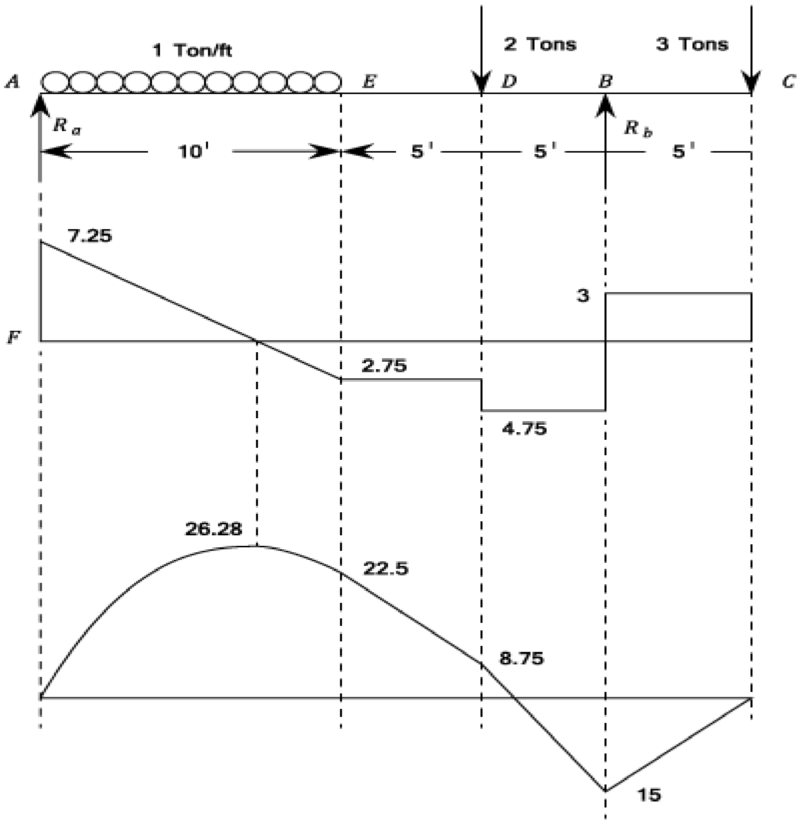
(The distributed load is taken as acting at its centre of gravity.)

$$\therefore \quad R_a = 7.25\ tons$$

$$\therefore \quad R_b = Total\ Load - R_a = 10 + 2 + 3 - 7.25 = 7.75\ tons$$

The Shearing Force

Starting at A F = 7.25. As the section moves away from A F decreases at a uniform rate of w per unit length (i.e. f =7.25 - wx) and reaches a value of - 2.75 at E. Between E and D, F is constant (There is no load on Ed) and at D it suffers a sudden decrease of 2 tons (the load at D) . Similarly there is an increase at B of 7.75 tons (the reaction at B). This results in a value of F = 3 tons at B which remains constant between B and C. Note this value agrees with the load at C.



Bending Moment from A to E:

$$M = R_a\,x - \frac{w\,x^2}{2} = 7.25\,x - \frac{x^2}{2} \quad since \quad (x = 1)$$

This is a parabola which can be sketched by taking several values of x. Beyond E the value of x for the distributed load remains constant at 5 ft. from A between E and D

$M = 7.25\,x - 10(X - 5) \;= -2.75x + 50$

This produces a straight line between E and D. Similar equations apply for sections DB and BC. However it is only necessary to evaluate M at the points D and B since M is zero at C. The diagram consists in straight lines between these values.

At D

$M = -2.75 \times 15 + 50 = 8.75\ \ tons - ft.$

At B

$M = -\,3 \times 5 = -\,15 \quad tons - ft.$

This last value was calculated for the portion BC

We were required to find the position and magnitude of the maximum BM. This occurs where the shearing force is zero. i.e.at 7.25 ft. from A

$$\therefore \qquad \hat{M} = 7.25 \times 7.25 - \frac{7.25^2}{2} = 26.28\ \ tons\ -\ ft.$$

The point of contra flexure occurs when the bending moment is zero and this is between D and B at:

$$\left(\frac{15}{15+8.75}\right) \times 5 = 3.16\ \ ft.\ from\ B$$

Varying Distributed Loads.

Example

Problem

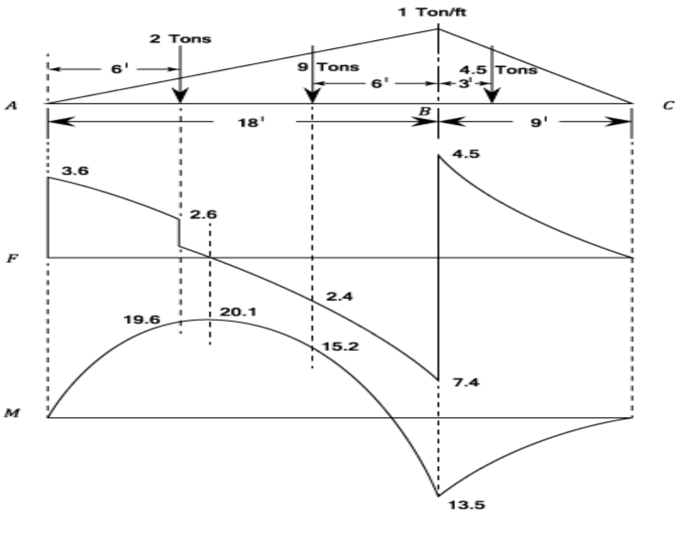
A Beam ABC, 27 ft. long, is simply supported at A and B 18 ft. across and carries a load of 2 tons at 6 ft. from A together with a distributed load whose intensity varies in linear fashion from zero at A and C to 1ton/ft. at B.

Workings

Draw the Shear Force and Bending Moment diagrams and calculate the position and magnitude of the maximum B.M. (U.L.)

The Total Load on the beam (i.e. the load plus the mean rate of loading of 1/2 tons/ft) is given by:

$Load = 2 + \frac{1}{2} \times 27 = 15.5\,tons$



The Total distribute load on $AB = \frac{1}{2} \times 18 = 9 \ \ tons$ and on $BC = \frac{1}{2} \times 9 = 4.5 \ \ tons$ each of which act through their centres of gravity. These are $\frac{2}{3} \times 18 = 12 \ \ ft.$ from A and $\frac{2}{3} \times 9 = 6 \ \ ft.$ from C in the other case.

(**Note**. These are the centroids of the triangles which represent the load distribution)

Taking Moments about B

$$R_1 = \left(\frac{2 \times 12 + 9 \times 6 \ - 4.5 \times 3}{18} \right) = 3.6 \ \ tons$$

$$\therefore \quad R_2 = 2 + 9 + 4.5 - 3.6 = 11.9 \ \ tons$$

At a distance *x* (<18)from A the loading is x/18 tons/ft.. The Total distributed load on this length is:

$$(Mean \ rate \ of \ loading) \times x = \frac{1}{2} \left(\frac{x}{18} \right) \ x = \frac{x^2}{36} \ \ tons$$

The centre of gravity of this load is $\frac{2}{3} \ x$ from A. **For 0<x<6**

$$F = 3.6 - \frac{x^2}{36}$$

At x = 6 ft.

$$F = 2.6 \ \ tons$$

$$M = 3.6\,x - \left(\frac{x^2}{36} \right) \times \frac{x}{3} = 3.6\,x - \frac{x^3}{108}$$

At x = 6 ft.

$$M = 19.6 \ \ tons - ft.$$

6< x <18

$$f = 3.6 - 2 - \frac{x^2}{36}$$

$$At \ x = 12 \ \ ft. \qquad F \ = -2.4 \ \ tons$$

$$At \ x = 18 \ \ ft. \qquad F \ = -7.4 \ \ tons$$

$$F = 0 \quad when \quad x = 6\,\sqrt{1.6} = 7.58 \ ft.$$

$$M = 3.6x - 2(x - 6) - \frac{x^3}{108} = 1.6x + 12 - \frac{x^3}{108}$$

$$At \ x = 12 \ \ ft \qquad M = 15.2 \ \ tons - ft.$$

$$At \ x = 12 \ \ ft \qquad M \ = -13.5 \ \ tons - ft.$$

The maximum Bending Moment occurs at zero shearing force 1.e.x = 7.58 ft.

$$\therefore \quad \hat{M} = 20.1 \ \ tons - ft.$$

The section BC can be more easily calculated by using a variable X measured from C. Then by a similar argument:-

$$F = \frac{1}{2} \left(\frac{X}{9} \right) X = \frac{X^2}{18} \ \ tons$$

$$At \quad X = 9 \ \ ft. \qquad F = 4.5 \ \ tons.$$

$$M \ = -\frac{1}{2} \left(\frac{X}{9} \right) X \left(\frac{X}{3} \right) \ = [-\frac{X^3}{54} \ \ ton - ft.$$

$At \quad X = 9 \ \ ft. \qquad\qquad = -13.5 \ \ tons - ft.$

The complete diagrams are shown. It can e seen that for a uniformly varying distributed load, the Shearing Force diagram consists of a series of parabolic curves and the Bending Moment diagram is made up of "cubic" discontinuities occurring at concentrated loads or reactions. It has been shown that Shearing Forces can be obtained by integrating the loading function and Bending Moment by integrating the Shearing Force, from which it follows that the curves produced will be of a successively "higher order" in x

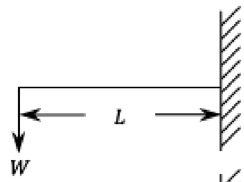
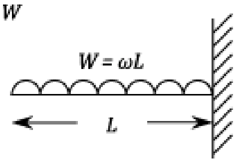
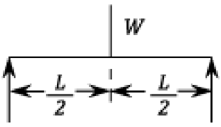
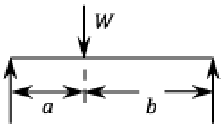
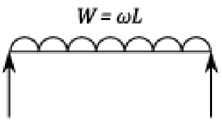
Shearing Force F

$$F = \frac{dM}{dx}$$

Mending Moment M

Rate of loading w

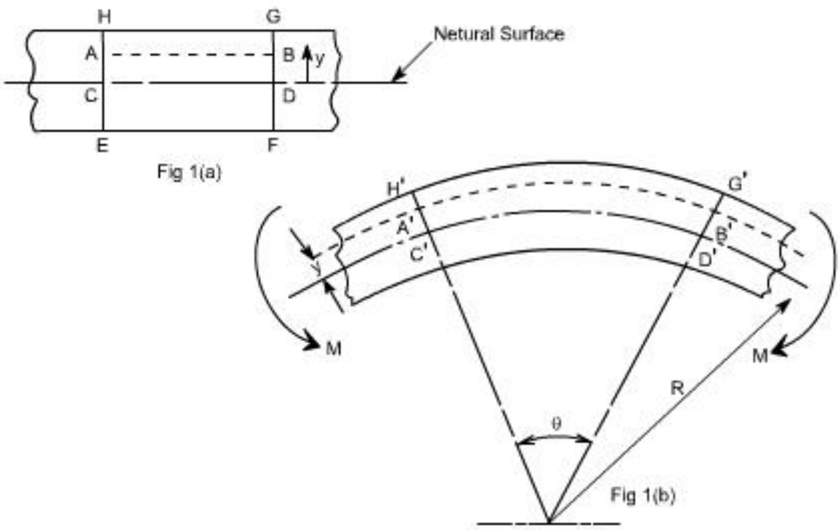
$$w = -\frac{dF}{dx} = -\frac{d^2M}{dx^2}$$

LOADING	$\hat{F}$	$\hat{M}$
	W	WL (Fixed End)
	W (Fixed End)	$\frac{WL}{2}$ (Fixed End)
	$\frac{W}{2}$	$\frac{WL}{4}$ (Centre)
	$\frac{Wb}{L}$	$\frac{Wab}{L}$ (Load)
	$\frac{W}{2}$ (Support)	$\frac{WL}{8}$ (Centre)

2. STRESSES IN BEAMS

Bending stresses

Simple Bending Theory OR Theory of Flexure for Initially Straight Beams
(The normal stress due to bending are called flexure stresses)
Preamble:
When a beam having an arbitrary cross section is subjected to a transverse loads the beam will bend. In addition to bending the other effects such as twisting and buckling may occur, and to investigate a problem that includes all the combined effects of bending, twisting and buckling could become a complicated one. Thus we are interested to investigate the bending effects alone, in order to do so; we have to put certain constraints on the geometry of the beam and the manner of loading.
Assumptions:
The constraints put on the geometry would form the **assumptions:**
1. Beam is initially **straight,** and has a **constant cross-section.**
2. Beam is made of **homogeneous material** and the beam has a **longitudinal plane of symmetry.**
3. Resultant of the applied loads lies in the plane of symmetry.
4. The geometry of the overall member is such that bending not buckling is the primary cause of failure.
5. Elastic limit is nowhere exceeded and **'E'** is same in tension and compression.
6. Plane cross - sections remains plane before and after bending.



Let us consider a beam initially unstressed as shown in fig 1(a). Now the beam is subjected to a constant bending moment (i.e. 'Zero Shearing Force') along its length as would be obtained by applying equal couples at each end. The beam will bend to the radius R as shown in Fig 1(b)

As a result of this bending, the top fibers of the beam will be subjected to tension and the bottom to compression it is reasonable to suppose, therefore, **that somewhere between the two there are points at which the stress is zero. The locus of all such points is known as neutral axis.** The radius of curvature R is then measured to this axis. For symmetrical sections the N. A. is the axis of symmetry but whatever the section N. A. will always pass through the centre of the area or centroid.

The above restrictions have been taken so as to eliminate the possibility of 'twisting' of the beam.

Concept of pure bending:

Loading restrictions:

As we are aware of the fact internal reactions developed on any cross-section of a beam may consists of a resultant normal force, a resultant shear force and a resultant couple. In order to ensure that the bending effects alone are investigated, we shall put a constraint on the loading such that the resultant normal and the resultant shear forces are zero on any cross-section perpendicular to the longitudinal axis of the member,

That means F = 0

Since $\frac{dM}{dX} = F = 0$ or M = constant.

Thus, the zero shear force means that the bending moment is constant or the bending is same at every cross-section of the beam. Such a situation may be visualized or envisaged when the beam or some portion of the beam, as been loaded only by pure couples at its ends. It must be recalled that the couples are assumed to be loaded in the plane of symmetry.

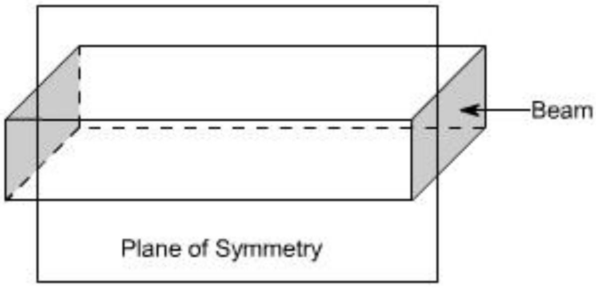


Fig (1)

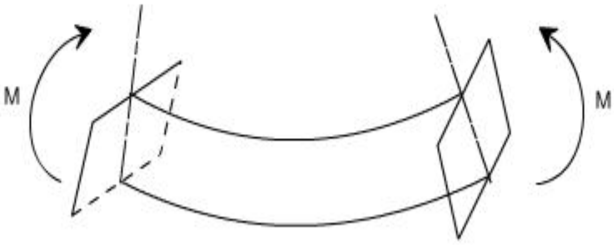
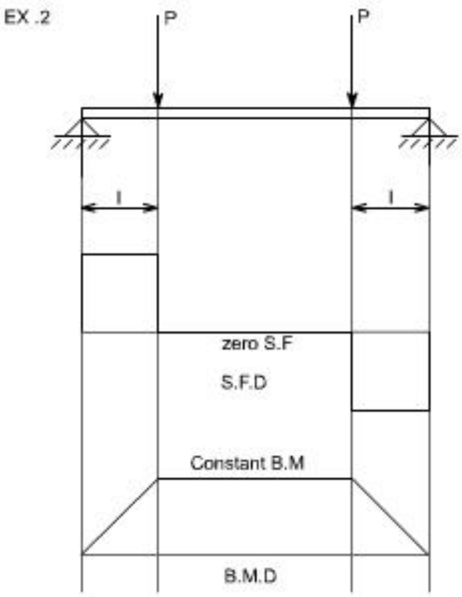
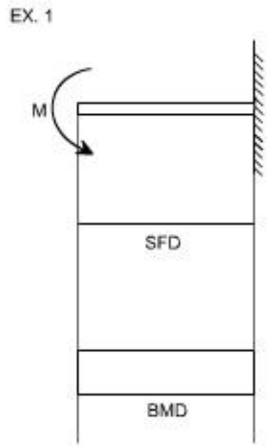


Fig (2)

When a member is loaded in such a fashion it is said to be in **pure bending.** The examples of pure bending have been indicated in EX 1and EX 2 as shown below:

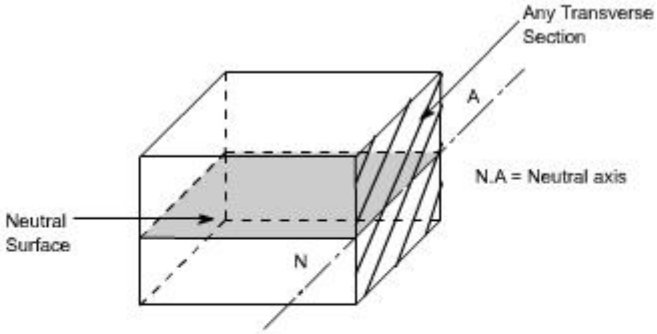




When a beam is subjected to pure bending are loaded by the couples at the ends, certain cross-section gets deformed and we shall have to make out the conclusion that,

1. Plane sections originally perpendicular to longitudinal axis of the beam remain plane and perpendicular to the longitudinal axis even after bending , i.e. the cross-section A'E', B'F' (refer Fig 1(a)) do not get warped or curved.

2. In the deformed section, the planes of this cross-section have a common intersection i.e. any time originally parallel to the longitudinal axis of the beam becomes an arc of circle.



We know that when a beam is under bending the fibres at the top will be lengthened while at the bottom will be shortened provided the bending moment M acts at the ends. In between these there are some fibres which remain unchanged in length that is they are not strained, that is they do not carry any stress. The plane containing such fibres is called neutral surface.

The line of intersection between the neutral surface and the transverse exploratory section is called the neutral axis Neutral axis **(N A)** .

Bending Stresses in Beams or Derivation of Elastic Flexural formula :

In order to compute the value of bending stresses developed in a loaded beam, let us consider the two cross-sections of a beam **HE** and **GF,** originally parallel as shown in fig 1(a).when the beam is to bend it is assumed that these sections remain parallel i.e. **H'E'** and **G’F’,** the final position of the sections, are still straight lines, they then subtend some angle □ .

Consider now fiber AB in the material, at a distance y from the N.A, when the beam bends this will stretch to A'B'

Therefore,

$$\text{strain in fibre AB} = \frac{\text{change in length}}{\text{orginal length}}$$

$$= \frac{A'B' - AB}{AB} \qquad \text{But } AB = CD \text{ and } CD = C'D'$$

refer to fig1(a) and fig1(b)

$$\therefore \text{strain} = \frac{A'B' - C'D'}{C'D'}$$

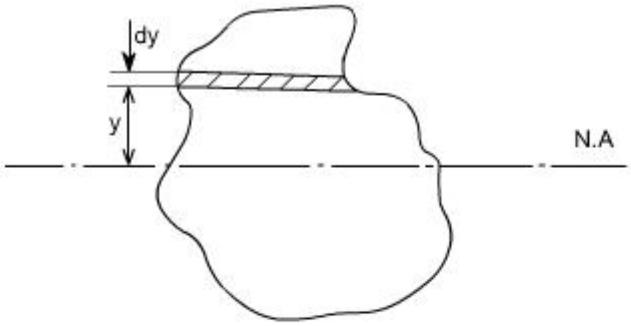
Since CD and C'D' are on the neutral axis and it is assumed that the Stress on the neutral axis zero. Therefore, there won't be any strain on the neutral axis

$$= \frac{(R+y)\theta - R\theta}{R\theta} = \frac{R\theta + y\theta - R\theta}{R\theta} = \frac{y}{R}$$

$$\text{However } \frac{\text{stress}}{\text{strain}} = E \qquad \text{where } E = \text{Young's Modulus of elasticity}$$

Therefore, equating the two strains as obtained from the two relations i.e,

$$\frac{\sigma}{E} = \frac{y}{R} \text{ or } \frac{\sigma}{y} = \frac{E}{R} \qquad \text{.............(1)}$$



Consider any arbitrary a cross-section of beam, as shown above now the strain on a fibre at a distance 'y' from the N.A, is given by the expression

$$\sigma = \frac{E}{R} y$$

if the shaded strip is of area 'dA'

then the force on the strip is

$$F = \sigma \, \delta A = \frac{E}{R} y \, \delta A$$

Moment about the neutral axis would be $= F.y = \frac{E}{R} y^2 \delta A$

The toatl moment for the whole cross-section is therefore equal to

$$M = \sum \frac{E}{R} y^2 \, \delta A = \frac{E}{R} \sum y^2 \delta A$$

Now the term $\sum y^2 \delta A$ is the property of the material and is called as a second moment of area of the cross-section and is denoted by a symbol I.

Therefore

$$M = \frac{E}{R} I \quad \quad \ldots\ldots\ldots(2)$$

combining equation 1 and 2 we get

$$\boxed{\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}}$$

This equation is known as the Bending Theory Equation. The above proof has involved the assumption of pure bending without any shear force being present. Therefore this termed as the pure bending equation. This equation gives distribution of stresses which are normal to cross-section i.e. in x-direction.

Section Modulus:

From simple bending theory equation, the maximum stress obtained in any cross-section is given as

$$\sigma_{max}{}^{m} = \frac{M}{I} y_{max}{}^{m}$$

For any given allowable stress the maximum moment which can be accepted by a particular shape of cross-section is therefore

$$M = \frac{I}{y_{max}{}^{m}} \sigma_{max}{}^{m}$$

For ready comparison of the strength of various beam cross-section this relationship is sometimes written in the form

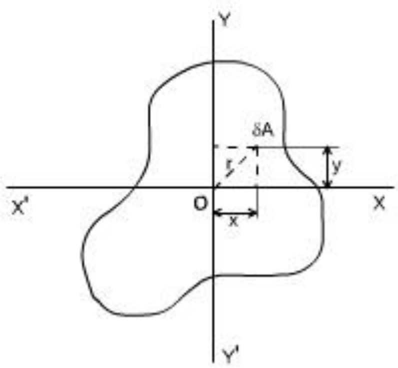
$$M = Z\,\sigma_{max}^{m} \quad \text{where } Z = \frac{I}{y_{max}^{m}}$$ **Is termed as section modulus**

The higher value of Z for a particular cross-section, the higher the bending moment which it can withstand for a given maximum stress.

Theorems to determine second moment of area: There are two theorems which are helpful to determine the value of second moment of area, which is required to be used while solving the simple bending theory equation.

Second Moment of Area:

Taking an analogy from the mass moment of inertia, the second moment of area is defined as the summation of areas times the distance squared from a fixed axis. (This property arised while we were driving bending theory equation). This is also known as the moment of inertia. An alternative name given to this is second moment of area, because the first moment being the sum of areas times their distance from a given axis and the second moment being the square of the distance or $\int y^2\, dA$.



Consider any cross-section having small element of area d A then by the definition

I_x(Mass Moment of Inertia about x-axis) = $\int y^2\, dA$ and I_y(Mass Moment of Inertia about y-axis) $= \int x^2\, dA$

Now the moment of inertia about an axis through 'O' and perpendicular to the plane of figure is called the polar moment of inertia. (The polar moment of inertia is also the area moment of inertia).

i.e,

J = polar moment of inertia

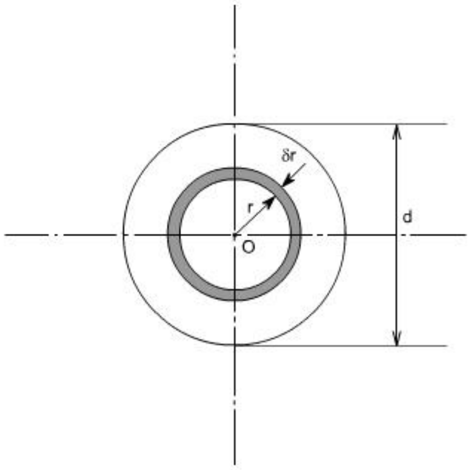
$$= \int r^2 dA$$
$$= \int (x^2 + y^2) dA$$
$$= \int x^2 dA + \int y^2 dA$$
$$= I_X + I_Y$$
$$\text{or } J = I_X + I_Y \qquad \ldots\ldots\ldots\ (1)$$

The relation (1) is known as the **perpendicular axis theorem** and may be stated as follows:

The sum of the Moment of Inertia about any two axes in the plane is equal to the moment of inertia about an axis perpendicular to the plane, the three axes being concurrent, i.e, the three axes exist together.

CIRCULAR SECTION:

For a circular x-section, the polar moment of inertia may be computed in the following manner



Consider any circular strip of thickness δr located at a radius 'r'.
Than the area of the circular strip would be dA = 2δr. δr

$$J = \int r^2 \, dA$$

Taking the limits of intergration from 0 to d/2

$$J = \int_0^{\frac{d}{2}} r^2 \, 2\pi r \, \delta r$$

$$= 2\pi \int_0^{\frac{d}{2}} r^3 \, \delta r$$

$$J = 2\pi \left[\frac{r^4}{4} \right]_0^{\frac{d}{2}} = \frac{\pi d^4}{32}$$

however, by perpendicular axis theorem

$$J = I_x + I_y$$

But for the circular cross-section, the I_x and I_y are both equal being moment of inertia about a diameter

$$I_{dia} = \frac{1}{2} J$$

$$I_{dia} = \frac{\pi d^4}{64}$$

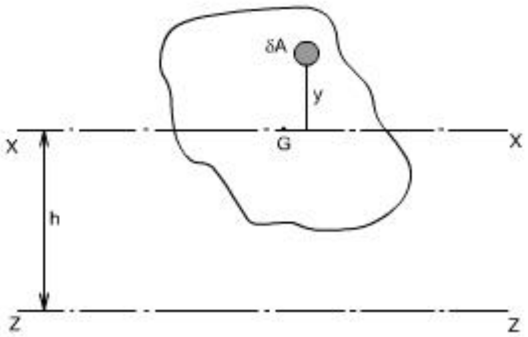
for a hollow circular section of diameter D and d, the values of J and I are defined as

$$J = \frac{\pi (D^4 - d^4)}{32}$$

Thus $$I = \frac{\pi (D^4 - d^4)}{64}$$

Parallel Axis Theorem:

The moment of inertia about any axis is equal to the moment of inertia about a parallel axis through the centroid plus the area times the square of the distance between the axes.



If ‘ZZ' is any axis in the plane of cross-section and ‘XX' is a parallel axis through the centroid G, of the cross-section, then

$$I_z = \int (y + h)^2 \; dA \text{ by definition (moment of inertia about an axis ZZ)}$$

$$= \int (\; + 2yh + h^2) dA$$

$$= \int y^2 dA + h^2 \int dA + 2h \int y dA$$

$$\text{Since } \int y dA = 0$$

$$= \int y^2 dA + h^2 \int dA$$

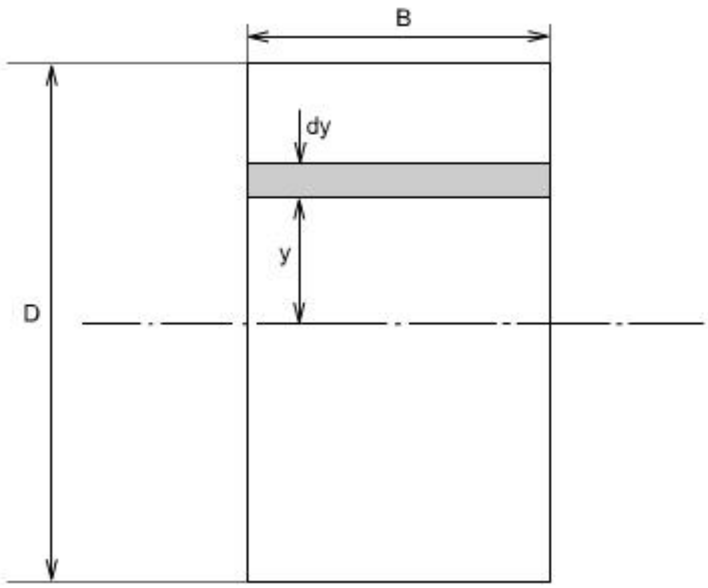
$$= \int y^2 dA + h^2 A$$

$$I_z = I_x + Ah^2 \qquad I_x = I_G \text{ (since cross-section axes also pass through G)}$$

$$\text{Where } A = \text{Total area of the section}$$

Rectangular Section:

For a rectangular x-section of the beam, the second moment of area may be computed as below:



Consider the rectangular beam cross-section as shown above and an element of area **dA**, thickness **dy**, breadth **B** located at a distance **y** from the neutral axis, which by symmetry passes through the centre of section. The second moment of area **I** as defined earlier would be

$$I_{N.A} = \int y^2 dA$$

Thus, for the rectangular section the second moment of area about the neutral axis i.e., an axis through the centre is given by

$$I_{N.A} = \int_{\frac{-D}{2}}^{\frac{D}{2}} y^2 (B\,dy)$$

$$= B \int_{\frac{-D}{2}}^{\frac{D}{2}} y^2\,dy$$

$$= B\left[\frac{y^3}{3}\right]_{\frac{D}{2}}^{\frac{-D}{2}}$$

$$= \frac{B}{3}\left[\frac{D^3}{8} - \left(\frac{-D^3}{8}\right)\right]$$

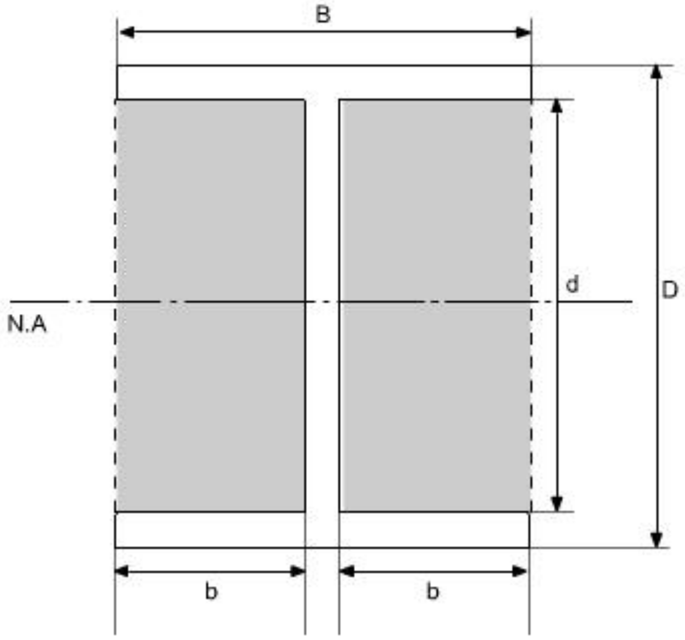
$$= \frac{B}{3}\left[\frac{D^3}{8} + \frac{D^3}{8}\right]$$

$$I_{N.A} = \frac{BD^3}{12}$$

Similarly, the second moment of area of the rectangular section about an axis through the lower edge of the section would be found using the same procedure but with integral limits of **0** to **D** .

Therefore $$I = B\left[\frac{y^3}{3}\right]_0^D = \frac{BD^3}{3}$$

These standards formulas prove very convenient in the determination of I_{NA} for build up sections which can be conveniently divided into rectangles. For instance if we just want to find out the Moment of Inertia of an I - section, then we can use the above relation.



$$I_{N.A} = I_{\text{of dotted rectangle}} - I_{\text{of shaded portion}}$$

$$\therefore\ I_{N.A} = \frac{BD^3}{12} - 2\left(\frac{bd^3}{12}\right)$$

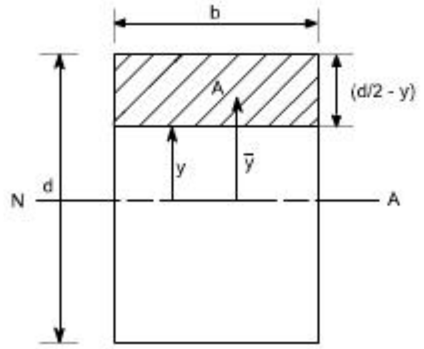
$$I_{N.A} = \frac{BD^3}{12} - \frac{bd^3}{6}$$

Shearing stress distribution in typical cross-sections:

Let us consider few examples to determine the sheer stress distribution in a given X- sections

Rectangular x-section:

Consider a rectangular x-section of dimension b and d



A is the area of the x-section cut off by a line parallel to the neutral axis. $\bar{y}$ is the distance of the centroid of A from the neutral axis

$$\tau = \frac{F.A.\bar{y}}{I.z}$$

$$\text{for this case, } A = b\left(\frac{d}{2} - y\right)$$

$$\text{While} \quad \bar{y} = \left[\frac{1}{2}\left(\frac{d}{2} - y\right) + y\right]$$

$$\text{i.e} \quad \bar{y} = \frac{1}{2}\left(\frac{d}{2} + y\right) \text{ and } z = b\,;\, I = \frac{b.d^3}{12}$$

substituting all these values, in the formula

$$\tau = \frac{F.A.\bar{y}}{I.z}$$

$$= \frac{F.b.\left(\frac{d}{2} - y\right).\frac{1}{2}.\left(\frac{d}{2} + y\right)}{b.\frac{b.d^3}{12}}$$

$$= \frac{\frac{F}{2}.\left\{\left(\frac{d}{2}\right)^2 - y^2\right\}}{\frac{b.d^3}{12}}$$

$$= \frac{6.F.\left\{\left(\frac{d}{2}\right)^2 - y^2\right\}}{b.d^3}$$

This shows that there is a parabolic distribution of shear stress with y.

The maximum value of shear stress would obviously beat the location y = 0.

$$\text{Such that } \tau_{max} = \frac{6.F}{b.d^3} \cdot \frac{d^2}{4}$$

$$= \frac{3.F}{2.b.d}$$

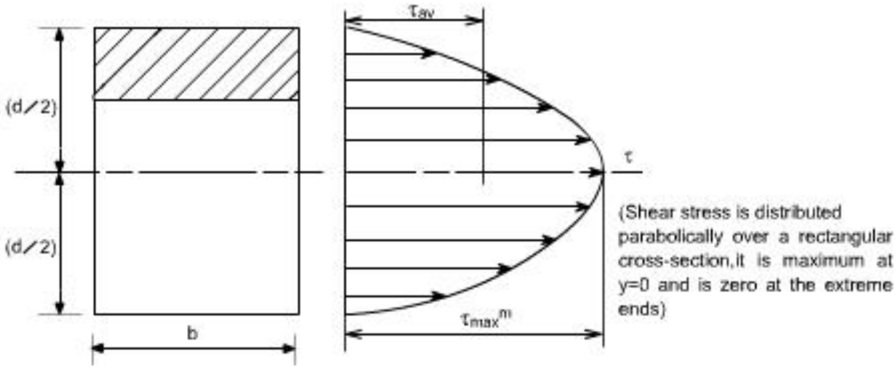
So $\boxed{\tau_{max} = \frac{3.F}{2.b.d}}$ The value of τ_{max} occurs at the neutral axis

The mean shear stress in the beam is defined as

$$\tau_{mean} \text{ or } \tau_{avg} = F/A = F/b.d$$

So $\tau_{max} = 1.5\ \tau_{mean} = 1.5\ \tau_{avg}$

Therefore the shear stress distribution is shown as below.



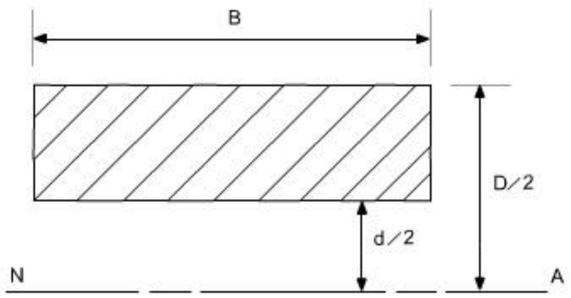
It may be noted that the shear stress is distributed parabolically over a rectangular cross-section, it is maximum at y = 0 and is zero at the extreme ends.

I - section :

Consider an I - section of the dimension shown below.

The shear stress distribution for any arbitrary shape is given as $\tau = \frac{F A \bar{y}}{Z I}$

Let us evaluate the quantity $A\bar{y}$, the $A\bar{y}$ quantity for this case comprise the contribution due to flange area and web area



Flange area

$$\text{Area of the flange} = B\left(\frac{D-d}{2}\right)$$

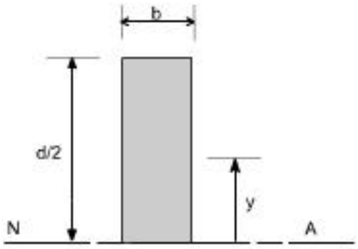
Distance of the centroid of the flange from the N.A

$$\bar{y} = \frac{1}{2}\left(\frac{D-d}{2}\right) + \frac{d}{2}$$

$$\bar{y} = \left(\frac{D+d}{4}\right)$$

Hence,

$$A\bar{y}\Big|_{Flange} = B\left(\frac{D-d}{2}\right)\left(\frac{D-d}{4}\right)$$



Web Area

Area of the web

$$A = b\left(\frac{d}{2} - y\right)$$

Distance of the centroid from N.A

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} - y\right) + y$$

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} + y\right)$$

Therefore,

$$A\bar{y}\big|_{web} = b\left(\frac{d}{2} - y\right)\frac{1}{2}\left(\frac{d}{2} + y\right)$$

Hence,

$$A\bar{y}\big|_{Total} = B\left(\frac{D - d}{2}\right)\left(\frac{D + d}{4}\right) + b\left(\frac{d}{2} - y\right)\left(\frac{d}{2} + y\right)\frac{1}{2}$$

Thus,

$$A\bar{y}\big|_{Total} = B\left(\frac{D^2 - d^2}{8}\right) + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

Therefore shear stress,

$$\tau = \frac{F}{bI}\left[\frac{B\left(D^2 - d^2\right)}{8} + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)\right]$$

To get the maximum and minimum values of ▯ substitute in the above relation.

y = 0 at N. A. And y = d/2 at the tip.

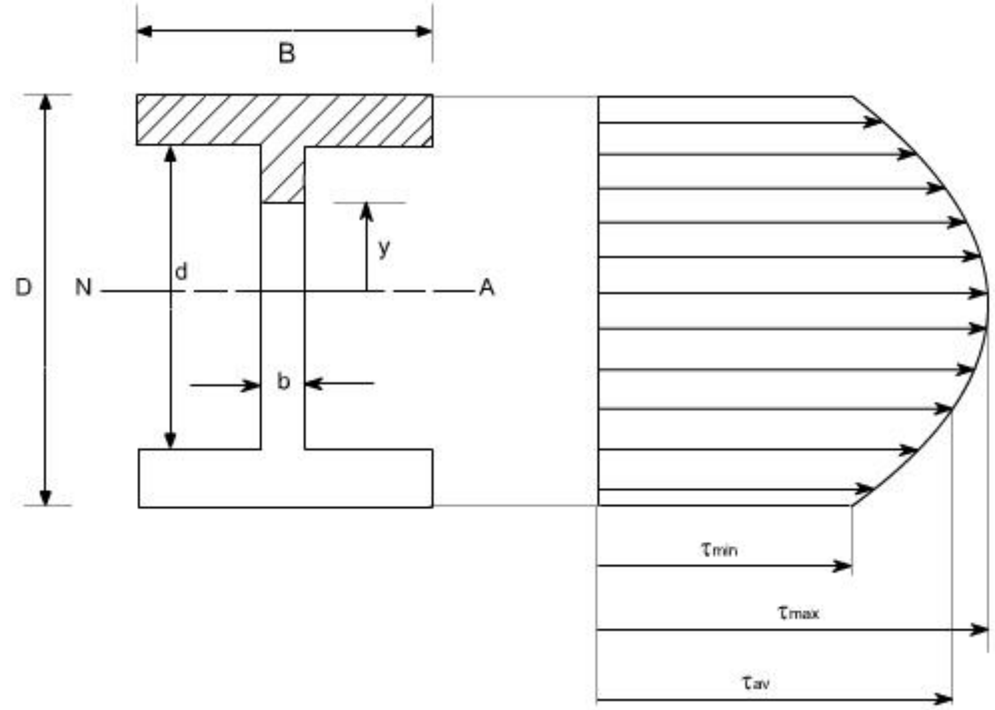
The maximum shear stress is at the neutral axis. I.e. for the condition y = 0 at N. A.

Hence, τ_{max} at $y = 0 = \frac{F}{8bI}\left[B\left(D^2 - d^2\right) + bd^2\right]$ .....(2)

The minimum stress occur at the top of the web, the term bd 2 goes off and shear stress is given by the following expression

τ_{min} at $y = d/2 = \frac{F}{8bI}\left[B\left(D^2 - d^2\right)\right]$ .... (3)

The distribution of shear stress may be drawn as below, which clearly indicates a parabolic

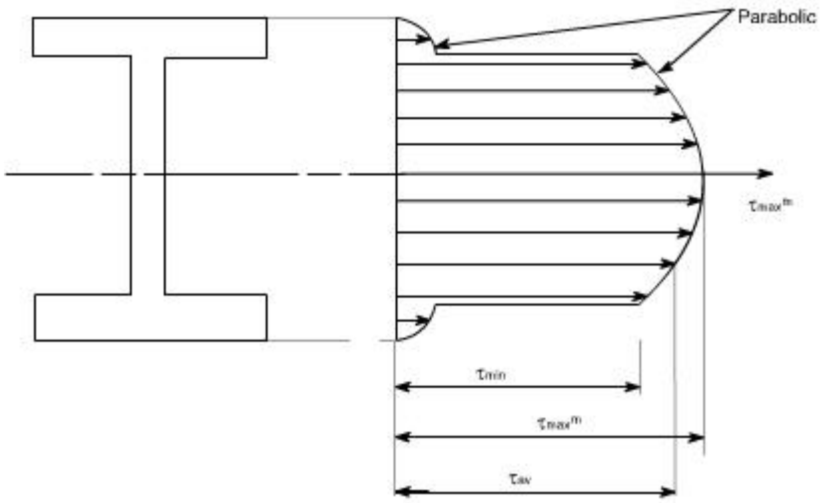


distribution

$$\tau_{max^m} = \frac{F}{8bI}\left[B\left(D^2 - d^2\right) + bd^2\right]$$

Note: from the above distribution we can see that the shear stress at the flanges is not zero, but it has some value, this can be analyzed from equation (1). At the flange tip or flange or web interface y = d/2.Obviously than this will have some constant value and than onwards this will have parabolic distribution.

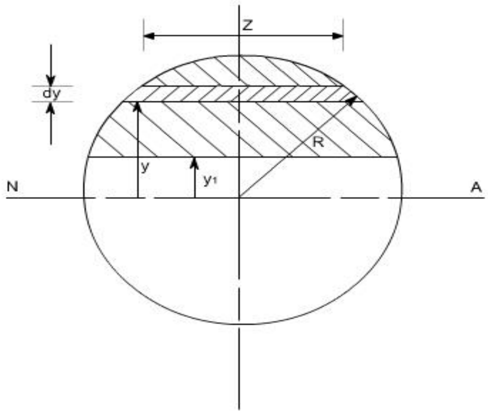
In practice it is usually found that most of shearing stress usually about 95% is carried by the web, and hence the shear stress in the flange is neglible however if we have the concrete analysis i.e. if we analyze the shearing stress in the flange i.e. writing down the expression for shear stress for flange and web separately, we will have this type of variation.



This distribution is known as the "top – hat" distribution. Clearly the web bears the most of the shear stress and bending theory we can say that the flange will bear most of the bending stress.

Shear stress distribution in beams of circular cross-section:

Let us find the shear stress distribution in beams of circular cross-section. In a beam of circular cross-section, the value of Z width depends on y.



Using the expression for the determination of shear stresses for any arbitrary shape or a arbitrary section.

$$\tau = \frac{FA\,\bar{y}}{Z\,I} = \frac{FA\int y\,dA}{Z\,I}$$

Where dA is the area moment of the shaded portion or the first moment of area.

Here in this case 'dA' is to be found out using the Pythagoras theorem

$$\left(\frac{Z}{2}\right)^2 + y^2 = R^2$$

$$\left(\frac{Z}{2}\right)^2 = R^2 - y^2 \text{ or } \frac{Z}{2} = \sqrt{R^2 - y^2}$$

$$Z = 2\sqrt{R^2 - y^2}$$

$$dA = Z\,dy = 2.\sqrt{R^2 - y^2}\,.dy$$

$I_{N.A}$ for a circular cross-section $= \dfrac{\pi R^4}{4}$

Hence,

$$\tau = \frac{FA\,\bar{y}}{Z\,I} = \frac{F}{\dfrac{\pi R^4}{4}\,2\sqrt{R^2 - y^2}}\int_{y_1}^{R} 2\,y\sqrt{R^2 - y^2}\,dy$$

Where R = radius of the circle.

[The limits have been taken from y_1 to R because we have to find moment of area the shaded portion]

$$= \frac{4\,F}{\pi R^4\sqrt{R^2 - y^2}}\int_{y_1}^{R} y\sqrt{R^2 - y^2}\,dy$$

The integration yields the final result to be

$$\tau = \frac{4\,F\left(R^2 - y_1^2\right)}{3\,\pi R^4}$$

Again this is a parabolic distribution of shear stress, having a maximum value when $y_1 = 0$

$$\tau_{max^m}\big|_{y_1 = 0} = \frac{4\,F}{3\pi R^2}$$

Obviously at the ends of the diameter the value of $y_1 = \pm R$ thus $\tau = 0$

so this again a parabolic distribution; maximum at the neutral axis

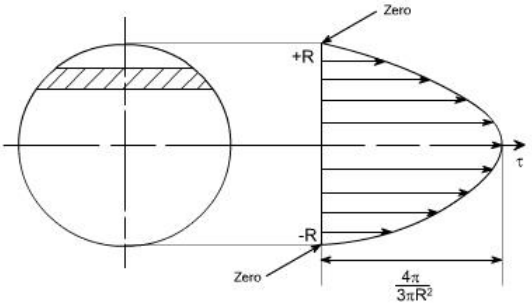
Also

$$\tau_{avg} \text{ or } \tau_{mean} = \frac{F}{A} = \frac{F}{\pi R^2}$$

Hence,

$$\boxed{\tau_{max^m} = \frac{4}{3}\,\tau_{avg}}$$

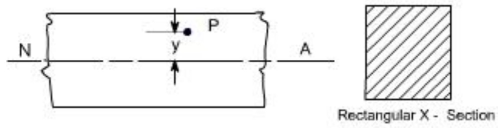
The distribution of shear stresses is shown below, which indicates a parabolic distribution

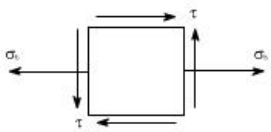


Principal Stresses in Beams

It becomes clear that the bending stress in beam is not a principal stress, since at any distance y from the neutral axis; there is a shear stress or we are assuming a plane stress situation

In general the state of stress at a distance y from the neutral axis will be as follows.





At some point 'P' in the beam, the value of bending stresses is given as

$$\sigma_b = \frac{My}{I} \text{ for a beam of rectangular cross - section of dimensions b and d; } I = \frac{bd^3}{12}$$

$$\sigma_b = \frac{12\,My}{bd^3}$$

whereas the value shear stress in the rectangular cross - section is given as

$$\tau = \frac{6F}{bd^3}\left[\frac{d^2}{4} - y^2\right]$$

Hence the values of principle stress can be determined from the relations,

$$\sigma_1, \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau_{xy}^2}$$

Letting $\sigma_y = 0$; $\sigma_x = \sigma_b$, the values of σ_1 and σ_2 can be computed as

$$\text{Hence } \sigma_1 / \sigma_2 = \frac{1}{2}\left(\frac{12My}{bd^3}\right) \pm \frac{1}{2}\sqrt{\left(\frac{12\,My}{bd^3}\right)^2 + 4\left(\frac{6\,F}{bd^3}\left(\frac{d^2}{4} - y^2\right)\right)^2}$$

$$\sigma_1, \sigma_2 = \frac{6}{bd^3}\left[My \pm \sqrt{\left\{M^2y^2 + F^2\left(\frac{d^2}{4} - y^2\right)^2\right\}}\right]$$

Also,

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \text{putting } \sigma_y = 0$$

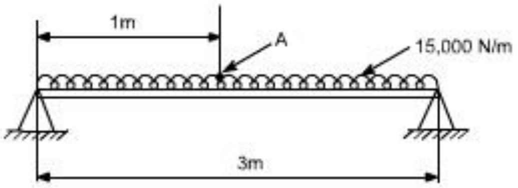
we get,

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x}$$

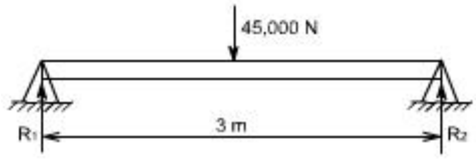
After substituting the appropriate values in the above expression we may get the inclination of the principal planes.

Illustrative examples: Let us study some illustrative examples, pertaining to determination of principal stresses in a beam

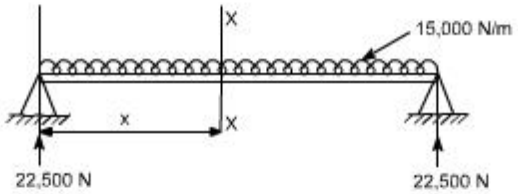
1. Find the principal stress at a point A in a uniform rectangular beam 200 mm deep and 100 mm wide, simply supported at each end over a span of 3 m and carrying a uniformly distributed load of 15,000 N/m.



Solution: The reaction can be determined by symmetry



$R_1 = R_2$ = 22,500 N



Consider any cross-section X-X located at a distance x from the left end.

Hence,

S. F $_{\text{at XX}}$ =22,500 – 15,000 x

B.M $_{\text{at XX}}$ = 22,500 x – 15,000 x (x/2) = 22,500 x – 15,000. x^2 / 2

Therefore,

S. F $_{\text{at X}}$ = 1 m = 7,500 N

B. M $_{\text{at X}}$ = 1 m = 15,000 N

$$S.F|_{x=1m} = 7{,}500\,N$$

$$B.M|_{x=1m} = 15{,}000\,N.m$$

$$\sigma_x = \frac{My}{I}$$

$$= \frac{15{,}000 \times 5 \times 10^{-2} \times 12}{10 \times 10^{-12} \times \left(20 \times 10^{-2}\right)^3}$$

$$\sigma_x = 11.25\,MN/m^2$$

For the compution of shear stresses

$$\tau = \frac{6F}{bd^3}\left[\frac{d^2}{4} - y^2\right] \qquad \text{putting } y = 50\,mm,\ d = 200\,mm$$

$$F = 7500\,N$$

$$\tau = 0.422\,MN/m^2$$

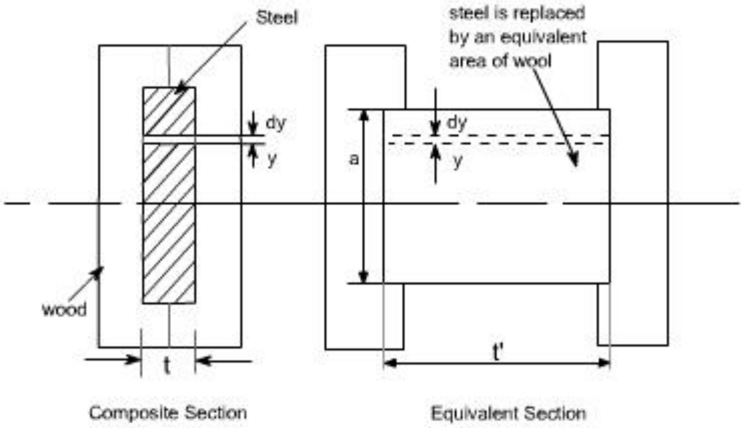
Now substituting these values in the principal stress equation,

We get 11.27 MN/m^{2,} - 0.025 MN/m^2

Bending Of Composite or Fletched Beams

A composite beam is defined as the one which is constructed from a combination of materials. If such a beam is formed by rigidly bolting together two timber joists and a reinforcing steel plate, then it is termed as a fletched beam.

The bending theory is valid when a constant value of Young's modulus applies across a section it cannot be used directly to solve the composite-beam problems where two different materials, and therefore different values of E, exist. The method of solution in such a case is to replace one of the materials by an equivalent section of the other.



Consider, a beam as shown in figure in which a steel plate is held centrally in an appropriate recess/pocket between two blocks of wood .Here it is convenient to replace the steel by an equivalent area of wood, retaining the same bending strength. i.e. the moment at any section must be the same in the equivalent section as in the original section so that the force at any given dy in the equivalent beam must be equal to that at the strip it replaces.

$$\sigma.t = \sigma'.t' \text{ or } \boxed{\frac{\sigma}{\sigma'} = \frac{t'}{t}}$$

recalling $\sigma = E.\varepsilon$

Thus

$$\varepsilon E t = \varepsilon' E' t'$$

Again, for true similarity the strains must be equal,

$$\varepsilon = \varepsilon' \text{ or } E t = E' t' \text{ or } \boxed{\frac{E}{E'} = \frac{t'}{t}}$$

Thus, $\boxed{t' = \frac{E}{E'}.t}$

Hence to replace a steel strip by an equivalent wooden strip the thickness must be multiplied by the modular ratio E/E'.

The equivalent section is then one of the same materials throughout and the simple bending theory applies. The stress in the wooden part of the original beam is found directly and that in the steel found from the value at the same point in the equivalent material as follows by utilizing the given relations.

$$\frac{\sigma}{\sigma'} = \frac{t'}{t}$$

$$\frac{\sigma}{\sigma'} = \frac{E}{E'}$$

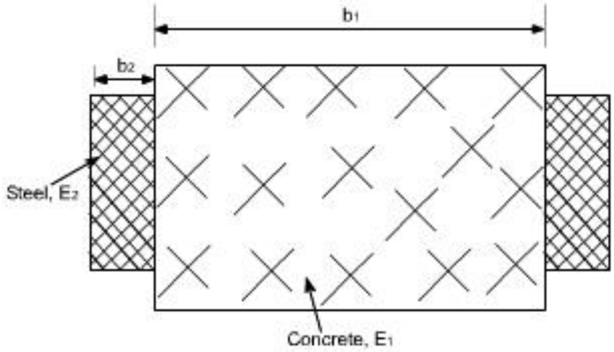
Stress in steel = modular ratio x stress in equivalent wood

The above procedure of course is not limited to the two materials treated above but applies well for any material combination. The wood and steel flitched beam was nearly chosen as a just for the sake of convenience.

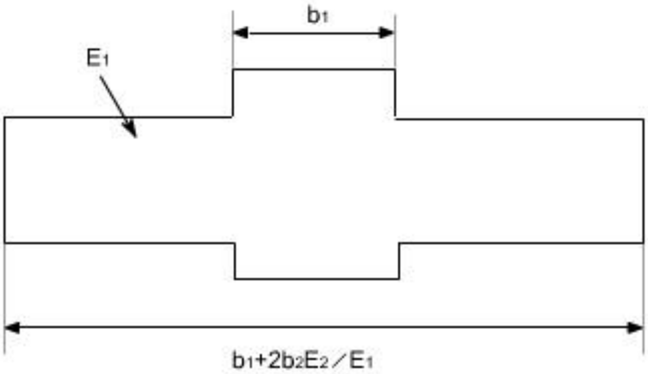
Assumption

In order to analyze the behavior of composite beams, we first make the assumption that the materials are bonded rigidly together so that there can be no relative axial movement between them. This means that all the assumptions, which were valid for homogenous beams are valid except the one assumption that is no longer valid is that the Young's Modulus is the same throughout the beam.

The composite beams need not be made up of horizontal layers of materials as in the earlier example. For instance, a beam might have stiffening plates as shown in the figure below.



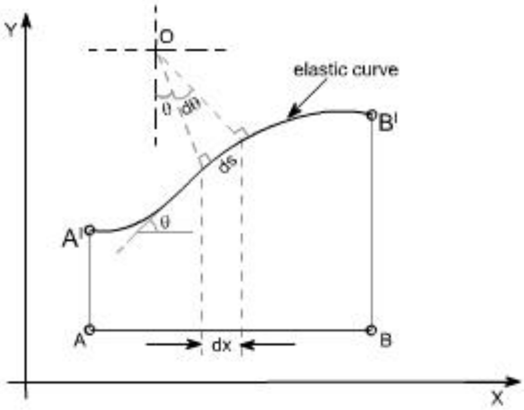
Again, the equivalent beam of the main beam material can be formed by scaling the breadth of the plate material in proportion to modular ratio. Bearing in mind that the strain at any level is same in both materials, the bending stresses in them are in proportion to the Young's modulus.



THE AREA-MOMENT / MOMENT-AREA METHODS:

The area moment method is a semi graphical method of dealing with problems of deflection of beams subjected to bending. The method is based on a geometrical interpretation of definite integrals. This is applied to cases where the equation for bending moment to be written is cumbersome and the loading is relatively simple.

Let us recall the figure, which we referred while deriving the differential equation governing the beams.



It may be noted that d is an angle subtended by an arc element ds and M is the bending moment to which this element is subjected.
We can assume,
ds = dx [since the curvature is small]
Hence, R d = ds

$$\frac{d\theta}{ds} = \frac{1}{R} = \frac{M}{EI}$$

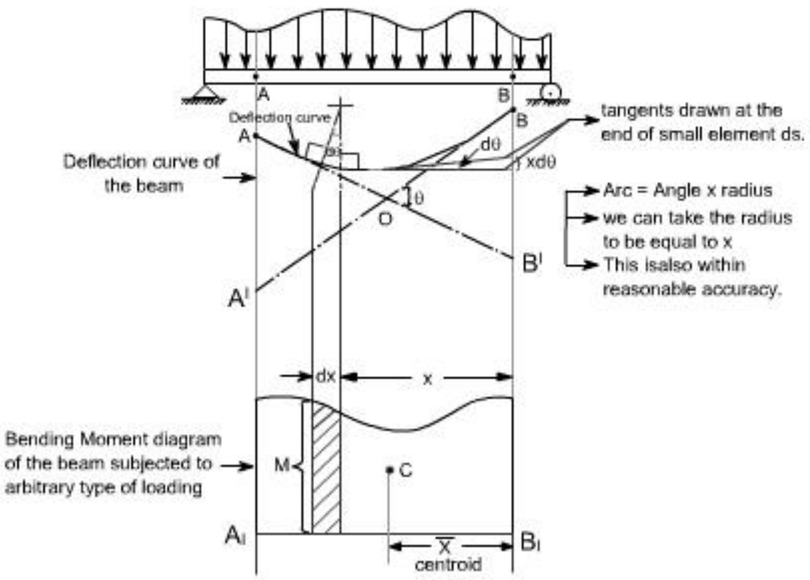
$$\frac{d\theta}{ds} = \frac{M}{EI}$$

But for small curvature[but θ is the angle, slope is $\tan\theta = \frac{dy}{dx}$ for small angles $\tan\theta \approx \theta$, hence $\theta \cong \frac{dy}{dx}$ so we get $\frac{d^2y}{dx^2} = \frac{M}{EI}$ by putting ds ≈ dx]

Hence,

$$\frac{d\theta}{dx} = \frac{M}{EI} \text{ or } \boxed{d\theta = \frac{M.dx}{EI}} \quad ------(1)$$

The relationship as described in equation (1) can be given a very simple graphical interpretation with reference to the elastic plane of the beam and its bending moment diagram



Refer to the figure shown above consider AB to be any portion of the elastic line of the loaded beam and $A_1B_1$is its corresponding bending moment diagram.

Let AO = Tangent drawn at A

BO = Tangent drawn at B

Tangents at A and B intersects at the point O.

Futher, AA ' is the deflection of A away from the tangent at B while the vertical distance B'B is the deflection of point B away from the tangent at A. All these quantities are further understood to be very small.

Let ds ≈ dx be any element of the elastic line at a distance x from B and an angle between at its tangents be d . Then, as derived erlier

$$d\theta = \frac{M.dx}{EI}$$

This relationship may be interpreted as that this angle is nothing but the area M.dx of the shaded bending moment diagram divided by EI.

From the above relationship the total angle between the tangents A and B may be determined as

$$\boxed{\theta = \int_A^B \frac{Mdx}{EI} = \frac{1}{EI}\int_A^B Mdx}$$

Since this integral represents the total area of the bending moment diagram, hence we may conclude this result in the following theorem

Theorem I:

$$\left\{\begin{matrix}\text{slope or } \theta \\ \text{between any two points}\end{matrix}\right\} = \left\{\begin{matrix}\frac{1}{EI} \times \text{area of B.M diagram between} \\ \text{corresponding portion of B.M diagram}\end{matrix}\right\}$$

Now let us consider the deflection of point B relative to tangent at A, this is nothing but the vertical distance BB'. It may be note from the bending diagram that bending of the element ds contributes to this deflection by an amount equal to x deach of this intercept may be considered as the arc of a circle of radius x subtended by the angle]

Hence the total distance B'B becomes $\delta = \int_A^B x\,d\theta$

The limits from A to B have been taken because A and B are the two points on the elastic curve, under consideration]. Let us substitute the value of d = M dx / EI as derived earlier

$$\delta = \int_A^B x\frac{Mdx}{EI} = \int_A^B \frac{Mdx}{EI}.x$$

[This is infact the moment of area of the bending moment diagram]

Since M dx is the area of the shaded strip of the bending moment diagram and x is its distance from B, we therefore conclude that right hand side of the above equation represents first moment area with respect to B of the total bending moment area between A and B divided by EI.

Therefore, we are in a position to state the above conclusion in the form of theorem as follows:

Theorem II:

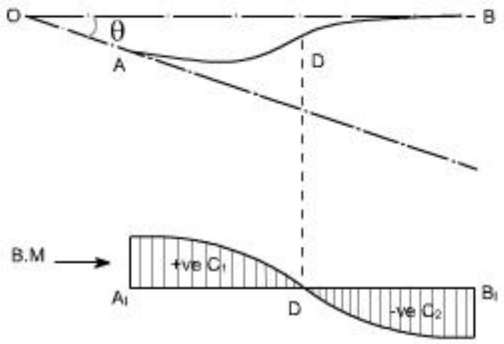
Deflection of point 'B' relative to point A $= \frac{1}{EI} \times \left\{\begin{matrix}\text{first moment of area with respect} \\ \text{to point B, of the total B.M diagram}\end{matrix}\right\}$

Further, the first moment of area, according to the definition of centroid may be written as $A\bar{x}$, where $\bar{x}$ is equal to distance of centroid and a is the total area of bending moment

Thus, $\boxed{\delta_A = \frac{1}{EI}A\bar{X}}$

Therefore, the first moment of area may be obtained simply as a product of the total area of the B.M diagram between the points A and B multiplied by the distance $\bar{x}$ to its centroid C.

If there exists an inflection point or point of contreflexure for the elastic line of the loaded beam between the points A and B, as shown below,



Then, adequate precaution must be exercised in using the above theorem. In such a case B. M diagram gets divide into two portions +ve and –ve portions with centroids $C_1$and C_2. Then to find an angle ▯ between the tangents at the points A and B

$$\theta = \int_A^D \frac{Mdx}{EI} - \int_D^B \frac{Mdx}{EI}$$

And similarly for the deflection of B away from the tangent at A becomes

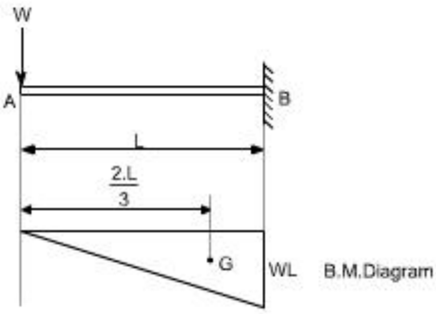
$$\delta = \int_A^D \frac{M.dx}{EI}.x - \int_B^D \frac{M.dx}{EI}.x$$

Illustrative Examples: Let us study few illustrative examples, pertaining to the use of these theorems

Example 1:

1. A cantilever is subjected to a concentrated load at the free end. It is required to find out the deflection at the free end.

Fpr a cantilever beam, the bending moment diagram may be drawn as shown below



Let us workout this problem from the zero slope condition and apply the first area - moment theorem

$$\text{slope at A} = \frac{1}{EI}\left[\text{Area of B.M diagram between the points A and B}\right]$$

$$= \frac{1}{EI}\left[\frac{1}{2}L.WL\right]$$

$$= \frac{WL^2}{2EI}$$

The deflection at A (relative to B) may be obtained by applying the second area - moment theorem

NOTE: In this case the point B is at zero slopes.

Thus,

$$\delta = \frac{1}{EI}[\text{first moment of area of B.M diagram between A and B about A}]$$

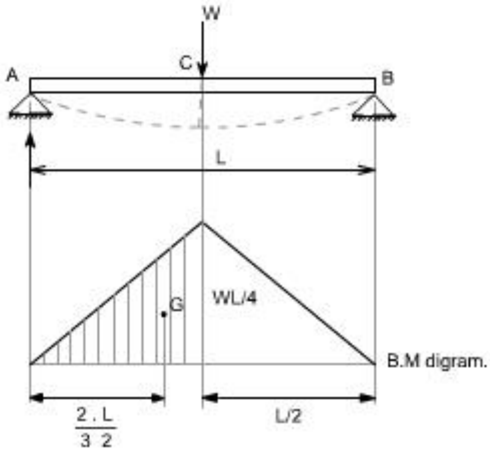
$$= \frac{1}{EI}[A\bar{y}]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}L.WL\right)\frac{2}{3}L\right]$$

$$= \frac{WL^3}{3EI}$$

Example 2: Simply supported beam is subjected to a concentrated load at the mid span determine the value of deflection.

A simply supported beam is subjected to a concentrated load W at point C. The bending moment diagram is drawn below the loaded beam.



Again working relative to the zero slopes at the centre C.

$$\text{slope at A} = \frac{1}{EI}[\text{Area of B.M diagram between A and C}]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{WL}{4}\right)\right]$$ we are taking half area of the B.M because we have to work out this relative to a zero slope

$$= \frac{WL^2}{16EI}$$

Deflection of A relative to C = central deflection of C

or

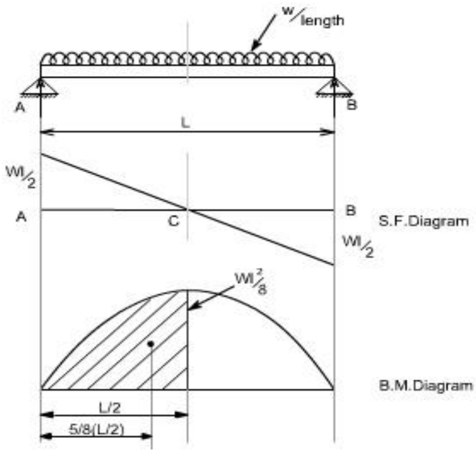
$$\delta_C = \frac{1}{EI}[\text{Moment of B.M diagram between points A and C about A}]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{WL}{4}\right)\frac{2}{3}L\right]$$

$$= \frac{WL^3}{48EI}$$

Example 3: A simply supported beam is subjected to a uniformly distributed load, with an intensity of loading W / length. It is required to determine the deflection.

The bending moment diagram is drawn, below the loaded beam; the value of maximum B.M is equal to $Wl^2 / 8$



So by area moment method,

$$\text{Slope at point C w.r.t point A} = \frac{1}{EI}[\text{Area of B.M diagram between point A and C}]$$

$$= \frac{1}{EI}\left[\left(\frac{2}{3}\right)\left(\frac{WL^2}{8}\right)\left(\frac{L}{2}\right)\right]$$

$$= \frac{WL^3}{24EI}$$

$$\text{Deflection at point C relative to A} = \frac{1}{EI}[A\,\bar{y}]$$

$$= \frac{1}{EI}\left[\left(\frac{WL^3}{24}\right)\left(\frac{5}{8}\right)\left(\frac{L}{2}\right)\right]$$

$$= \frac{5}{384EI}.WL^4$$

Unit 3.BEAMS AND COLUMNS

Deflection of Beams

Introduction:

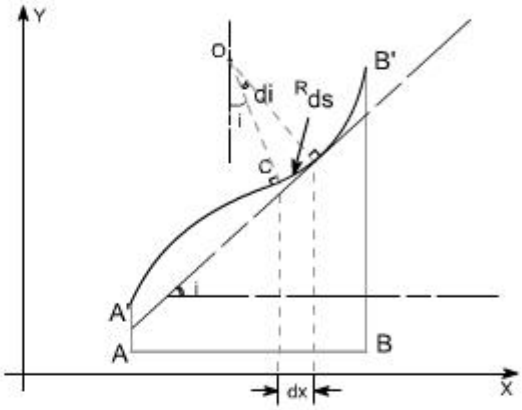
In all practical engineering applications, when we use the different components, normally we have to operate them within the certain limits i.e. the constraints are placed on the performance and behavior of the components. For instance we say that the particular component is supposed to operate within this value of stress and the deflection of the component should not exceed beyond a particular value.

In some problems the maximum stress however, may not be a strict or severe condition but there may be the deflection which is the more rigid condition under operation. It is obvious therefore to study the methods by which we can predict the deflection of members under lateral loads or transverse loads, since it is this form of loading which will generally produce the greatest deflection of beams.

Assumption: The following assumptions are undertaken in order to derive a differential equation of elastic curve for the loaded beam

1. Stress is proportional to strain i.e. hooks law applies. Thus, the equation is valid only for beams that are not stressed beyond the elastic limit.
2. The curvature is always small.
3. Any deflection resulting from the shear deformation of the material or shear stresses is neglected.

It can be shown that the deflections due to shear deformations are usually small and hence can be ignored.



Consider a beam AB which is initially straight and horizontal when unloaded. If under the action of loads the beam deflects to a position A'B' under load or infact we say that the axis of the beam bends to a shape A'B'. It is customary to call A'B' the curved axis of the beam as the elastic line or deflection curve.

In the case of a beam bent by transverse loads acting in a plane of symmetry, the bending moment M varies along the length of the beam and we represent the variation of bending moment in B.M diagram. Further, it is assumed that the simple bending theory equation holds good.

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$$

If we look at the elastic line or the deflection curve, this is obvious that the curvature at every point is different; hence the slope is different at different points.

To express the deflected shape of the beam in rectangular co-ordinates let us take two axes x and y, x-axis coincide with the original straight axis of the beam and the y – axis shows the deflection.
Further, let us consider element ds of the deflected beam. At the ends of this element let us construct the normal which intersect at point O denoting the angle between these two normal be di But for the deflected shape of the beam the slope i at any point C is defined,

$$\tan i = \frac{dy}{dx} \quad ......(1) \quad \text{or} \quad i = \frac{dy}{dx} \quad \text{Assuming } \tan i = i$$

Futher

$$ds = R\,di$$

however,

$$ds = dx \text{ [usually for small curvature]}$$

Hence

$$ds = dx = R\,di$$

$$\text{or} \quad \boxed{\frac{di}{dx} = \frac{1}{R}}$$

substituting the value of i, one get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{R} \text{ or } \frac{d^2y}{dx^2} = \frac{1}{R}$$

From the simple bending theory

$$\frac{M}{I} = \frac{E}{R} \text{ or } M = \frac{EI}{R}$$

so the basic differential equation governing the deflection of beams is

$$\boxed{M = EI\frac{d^2y}{dx^2}}$$

This is the differential equation of the elastic line for a beam subjected to bending in the plane of symmetry. Its solution y = f(x) defines the shape of the elastic line or the deflection curve as it is frequently called.

Relationship between shear force, bending moment and deflection: The relationship among shear force, bending moment and deflection of the beam may be obtained as
Differentiating the equation as derived

$$\frac{dM}{dx} = EI\frac{d^3y}{dx^3} \quad \text{Recalling } \frac{dM}{dx} = F$$

Thus,

$$F = EI\frac{d^3y}{dx^3}$$

Therefore, the above expression represents the shear force whereas rate of intensity of loading can also be found out by differentiating the expression for shear force

$$\text{i.e } w = -\frac{dF}{dx}$$

$$w = -EI\frac{d^4y}{dx^4}$$

Therefore if 'y' is the deflection of the loaded beam, then the following important relations can be arrived at

$$\boxed{slope = \frac{dy}{dx}}$$

$$\boxed{B.M = EI\frac{d^2y}{dx^2}}$$

$$\boxed{Shear\ force = EI\frac{d^3y}{dx^3}}$$

$$\boxed{load\ distribution = EI\frac{d^4y}{dx^4}}$$

Methods for finding the deflection: The deflection of the loaded beam can be obtained various methods. The one of the methods for finding the deflection of the beam is the direct integration method, i.e. the method using the differential equation which we have derived.

Direct integration method: The governing differential equation is defined as

$$M = EI\frac{d^2y}{dx^2} \quad \text{or} \quad \frac{M}{EI} = \frac{d^2y}{dx^2}$$

on integrating one get,

$$\frac{dy}{dx} = \int \frac{M}{EI}dx + A \text{ ---- this equation gives the slope of the loaded beam.}$$

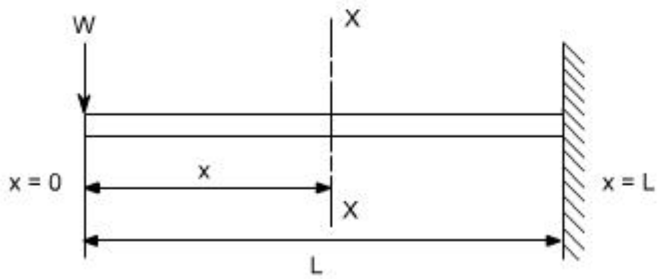
Integrate once again to get the deflection.

$$y = \int\int \frac{M}{EI}dx + Ax + B$$

Where A and B are constants of integration to be evaluated from the known conditions of slope and deflections for the particular value of x.

Illustrative examples : let us consider few illustrative examples to have a familiarity with the direct integration method

Case 1: Cantilever Beam with Concentrated Load at the end:- A cantilever beam is subjected to a concentrated load W at the free end, it is required to determine the deflection of the beam



In order to solve this problem, consider any X-section X-X located at a distance x from the left end or the reference, and write down the expressions for the shear force and the bending moment

$S.F|_{x-x} = -W$

$B.M|_{x-x} = -W.x$

Therefore $M|_{x-x} = -W.x$

the governing equation $\frac{M}{EI} = \frac{d^2y}{dx^2}$

substituting the value of M interms of x then integrating the equation one get

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{Wx}{EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{Wx}{EI}dx$$

$$\frac{dy}{dx} = -\frac{Wx^2}{2EI} + A$$

Integrating once more,

$$\int \frac{dy}{dx} = \int -\frac{Wx^2}{2EI}dx + \int A\,dx$$

$$y = -\frac{Wx^3}{6EI} + Ax + B$$

The constants A and B are required to be found out by utilizing the boundary conditions as defined below

i.e at x= L ; y= 0 -------------------- (1)

at x = L ; dy/dx = 0 -------------------- (2)

Utilizing the second condition, the value of constant A is obtained as

$$A = \frac{Wl^2}{2EI}$$

While employing the first condition yields

$$y = -\frac{WL^3}{6EI} + AL + B$$

$$B = \frac{WL^3}{6EI} - AL$$

$$= \frac{WL^3}{6EI} - \frac{WL^3}{2EI}$$

$$= \frac{WL^3 - 3WL^3}{6EI} = -\frac{2WL^3}{6EI}$$

$$B = -\frac{WL^3}{3EI}$$

Substituting the values of A and B we get

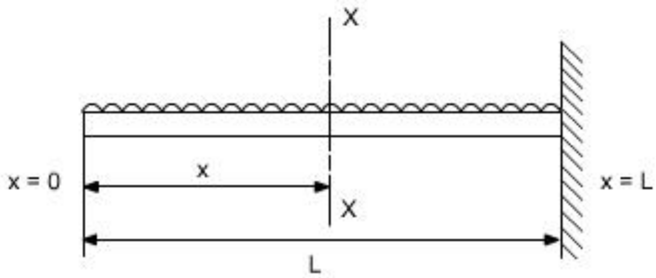
$$y = \frac{1}{EI}\left[-\frac{Wx^3}{6EI} + \frac{WL^2 x}{2EI} - \frac{WL^3}{3EI}\right]$$

The slope as well as the deflection would be maximum at the free end hence putting x=0 we get,

$$\boxed{y_{max} = -\frac{WL^3}{3EI}}$$

$$\boxed{(\text{Slope})_{max} m = +\frac{WL^2}{2EI}}$$

Case 2: A Cantilever with Uniformly distributed Loads: - In this case the cantilever beam is subjected to U.d.l with rate of intensity varying w / length. The same procedure can also be adopted in this case



$$S.F|_{x-x} = -w$$

$$B.M|_{x-x} = -w.x.\frac{x}{2} = w\left(\frac{x^2}{2}\right)$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{wx^2}{2EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{wx^2}{2EI}dx$$

$$\frac{dy}{dx} = -\frac{wx^3}{6EI} + A$$

$$\int \frac{dy}{dx} = \int -\frac{wx^3}{6EI}dx + \int A\,dx$$

$$y = -\frac{wx^4}{24EI} + Ax + B$$

Boundary conditions relevant to the problem are as follows:

1. At x = L; y = 0
2. At x= L; dy/dx = 0

The second boundary conditions yields

$$A = +\frac{wx^3}{6EI}$$

whereas the first boundary conditions yields

$$B = \frac{wL^4}{24EI} - \frac{wL^4}{6EI}$$

$$B = -\frac{wL^4}{8EI}$$

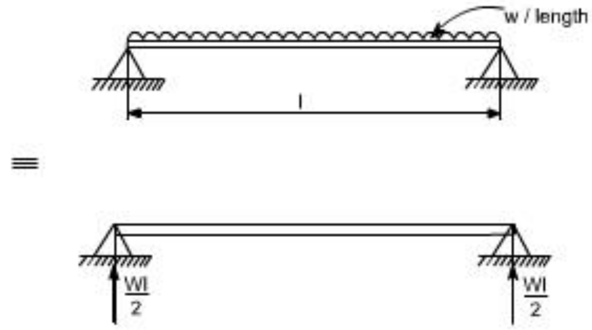
$$\text{Thus, } y = \frac{1}{EI}\left[-\frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{wL^4}{8}\right]$$

So y_{maxm} will be at $x = 0$

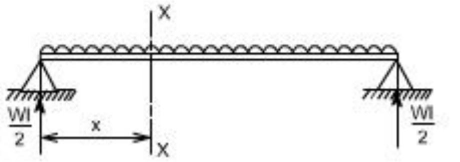
$$\boxed{y_{maxm} = -\frac{wL^4}{8EI}}$$

$$\boxed{\left(\frac{dy}{dx}\right)_{maxm} = \frac{wL^3}{6EI}}$$

Case 3: Simply Supported beam with uniformly distributed Loads: - In this case a simply supported beam is subjected to a uniformly distributed load whose rate of intensity varies as w / length.



In order to write down the expression for bending moment consider any cross-section at distance of x meter from left end support.



$$S.F|_{x-x} = w\left(\frac{l}{2}\right) - w.x$$

$$B.M|_{x-x} = w.\left(\frac{l}{2}\right).x - w.x.\left(\frac{x}{2}\right)$$

$$= \frac{wl.x}{2} - \frac{wx^2}{2}$$

The differential equation which gives the elastic curve for the deflected beam is

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{1}{EI}\left[\frac{wl.x}{2} - \frac{wx^2}{2}\right]$$

$$\frac{dy}{dx} = \int \frac{wlx}{2EI} dx - \int \frac{wx^2}{2EI} dx + A$$

$$= \frac{wlx^2}{4EI} - \frac{wx^3}{6EI} + A$$

Integrating, once more one gets

$$y = \frac{wlx^3}{12EI} - \frac{wx^4}{24EI} + A.x + B \qquad \text{-----}(1)$$

Boundary conditions which are relevant in this case are that the deflection at each support must be zero.

i.e. at x = 0; y = 0 : at x = l; y = 0

Let us apply these two boundary conditions on equation (1) because the boundary conditions are on y, This yields B = 0.

$$0 = \frac{wl^4}{12EI} - \frac{wl^4}{24EI} + A.l$$

$$A = -\frac{wl^3}{24EI}$$

So the equation which gives the deflection curve is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

Futher

In this case the maximum deflection will occur at the centre of the beam where x = L/2 [i.e. at the position where the load is being applied].So if we substitute the value of x = L/2

$$\text{Then}\quad y_{max^m} = \frac{1}{EI}\left[\frac{wL}{12}\left(\frac{L^3}{8}\right) - \frac{w}{24}\left(\frac{L^4}{16}\right) - \frac{wL^3}{24}\left(\frac{L}{2}\right)\right]$$

$$\boxed{y_{max^m} = -\frac{5wL^4}{384EI}}$$

Conclusions

(i) The value of the slope at the position where the deflection is maximum would be zero.

(ii) The value of maximum deflection would be at the centre i.e. at x = L/2.

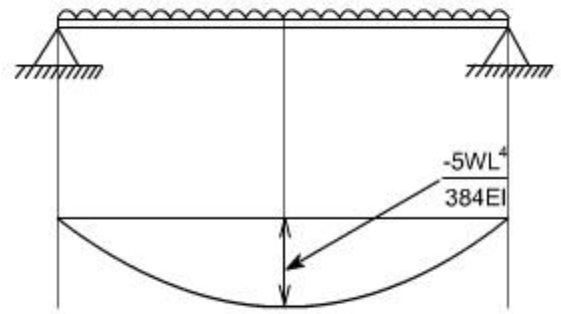
The final equation which is governs the deflection of the loaded beam in this case is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

By successive differentiation one can find the relations for slope, bending moment, shear force and rate of loading.

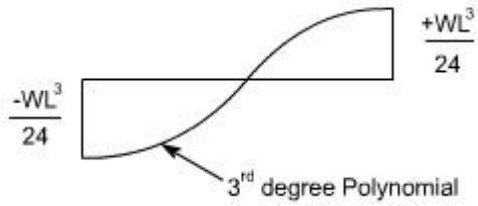
Deflection (y)

$$yEI = \left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$



Slope (dy/dx)

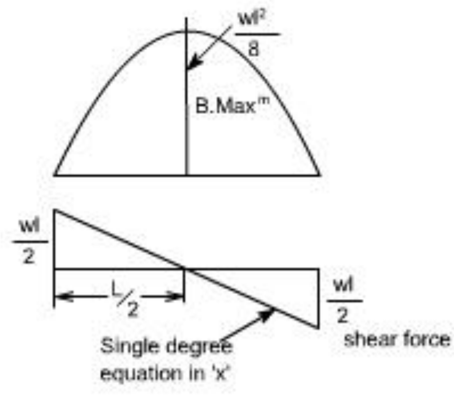
$$EI.\frac{dy}{dx} = \left[\frac{3wLx^2}{12} - \frac{4wx^3}{24} - \frac{wL^3}{24}\right]$$



So the bending moment diagram would be

Bending Moment

$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left[\frac{wLx}{2} - \frac{wx^2}{2}\right]$$



Shear Force

Shear force is obtained by taking

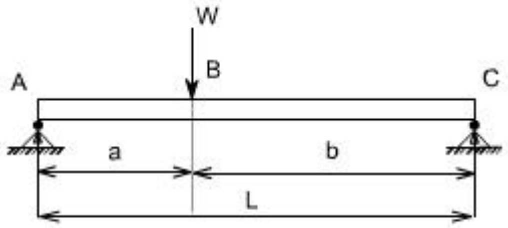
Third derivative.

$$EI\frac{d^3y}{dx^3} = \frac{wL}{2} - w.x$$

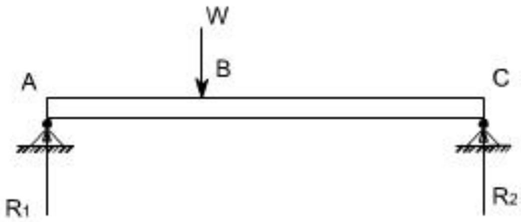
Rate of intensity of loading

$$EI\frac{d^4y}{dx^4} = -w$$

Case 4: The direct integration method may become more involved if the expression for entire beam is not valid for the entire beam. Let us consider a deflection of a simply supported beam which is subjected to a concentrated load W acting at a distance 'a' from the left end.



Let R_1 & R_2 be the reactions then,



B.M for the portion AB

$$M|_{AB} = R_1.x \quad 0 \leq x \leq a$$

B.M for the portion BC

$$M|_{BC} = R_1.x - W(x - a) \quad a \leq x \leq l$$

so the differential equation for the two cases would be,

$$EI\frac{d^2y}{dx^2} = R_1 x$$

$$EI\frac{d^2y}{dx^2} = R_1 x - W(x - a)$$

These two equations can be integrated in the usual way to find 'y' but this will result in four constants of integration two for each equation. To evaluate the four constants of integration, four independent boundary conditions will be needed since the deflection of each support must be zero, hence the boundary conditions (a) and (b) can be realized.

Further, since the deflection curve is smooth, the deflection equations for the same slope and deflection at the point of application of load i.e. at x = a. Therefore four conditions required to evaluate these constants may be defined as follows:

(a) at $x = 0$; $y = 0$ in the portion AB i.e. $0 \leq x \leq a$

(b) at $x = l$; $y = 0$ in the portion BC i.e. $a \leq x \leq l$

(c) at $x = a$; dy/dx, the slope is same for both portion

(d) at $x = a$; y, the deflection is same for both portion

By symmetry, the reaction R_1 is obtained as

$$R_1 = \frac{Wb}{a+b}$$

Hence,

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x \qquad 0 \le x \le a \text{ --------(1)}$$

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x - W(x-a) \qquad a \le x \le l \text{ ---------(2)}$$

integrating (1) and (2) we get,

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k_1 \qquad 0 \le x \le a \text{ --------(3)}$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k_2 \qquad a \le x \le l \text{ --------(4)}$$

Using condition (c) in equation (3) and (4) shows that these constants should be equal, hence letting $K_1 = K_2 = K$

Hence

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k \qquad 0 \le x \le a \text{ ------(3)}$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k \qquad a \le x \le l \text{ ------- (4)}$$

Integrating agian equation (3) and (4) we get

$$EIy = \frac{Wb}{6(a+b)}x^3 + kx + k_3 \qquad 0 \le x \le a \text{ ------(5)}$$

$$EIy = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4 \qquad a \le x \le l \text{ ------(6)}$$

Utilizing condition (a) in equation (5) yields

$$k_3 = 0$$

Utilizing condition (b) in equation (6) yields

$$0 = \frac{Wb}{6(a+b)}l^3 - \frac{W(l-a)^3}{6} + kl + k_4$$

$$k_4 = -\frac{Wb}{6(a+b)}l^3 + \frac{W(l-a)^3}{6} - kl$$

But $a + b = l$,

Thus,

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b)$$

Now lastly k_3 is found out using condition (d) in equation (5) and equation (6), the condition (d) is that,

At x = a; y; the deflection is the same for both portion

Therefore $y|_{\text{from equation 5}} = y|_{\text{from equation 6}}$

or

$$\frac{Wb}{6(a+b)}x^3 + kx + k_3 = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

$$\frac{Wb}{6(a+b)}a^3 + ka + k_3 = \frac{Wb}{6(a+b)}a^3 - \frac{W(a-a)^3}{6} + ka + k_4$$

Thus, $k_4 = 0$;

OR

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b) = o$$

$$k(a+b) = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6}$$

$$k = -\frac{Wb(a+b)}{6} + \frac{Wb^3}{6(a+b)}$$

so the deflection equations for each portion of the beam are

$$EIy = \frac{Wb}{6(a+b)}x^3 + kx + k_3$$

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \qquad \text{- - - - } \mathbf{for\ 0 \le x \le a} \text{ - - - - - (7)}$$

and for other portion

$$EIy = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

Substituting the value of 'k' in the above equation

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{W(x-a)^3}{6} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \qquad \text{For } \mathbf{for\ a \le x \le l} \text{ - - - - - (8)}$$

so either of the equation (7) or (8) may be used to find the deflection at x = a

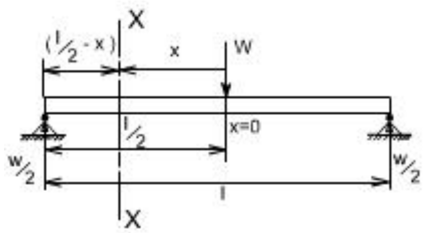
hence substituting x = a in either of the equation we get

$$Y|_{x=a} = -\frac{Wa^2b^2}{3EI(a+b)}$$

OR if a = b = l/2

$$\boxed{Y_{max^m} = -\frac{WL^3}{48EI}}$$

ALTERNATE METHOD: There is also an alternative way to attempt this problem in a simpler way. Let us considering the origin at the point of application of the load,



$$S.F|_{xx} = \frac{W}{2}$$

$$B.M|_{xx} = \frac{W}{2}\left(\frac{l}{2} - x\right)$$

substituting the value of M in the governing equation for the deflection

$$\frac{d^2y}{dx^2} = \frac{\frac{W}{2}\left(\frac{l}{2} - x\right)}{EI}$$

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{WLx}{4} - \frac{Wx^2}{4}\right] + A$$

$$y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^2}{12}\right] + Ax + B$$

Boundary conditions relevant for this case are as follows

(i) at x = 0; dy/dx= 0

hence, A = 0

(ii) at x = l/2; y = 0 (because now l / 2 is on the left end or right end support since we have taken the origin at the centre)

Thus,

$$0 = \left[\frac{WL^3}{32} - \frac{WL^3}{96} + B\right]$$

$$B = -\frac{WL^3}{48}$$

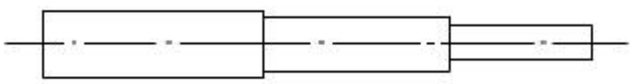
Hence he equation which governs the deflection would be

$$y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^3}{12} - \frac{WL^3}{48}\right]$$

Hence

$Y_{max^m}\Big\|_{at\,x=0}$	$= -\frac{WL^3}{48EI}$	At the centre
$\left(\frac{dy}{dx}\right)_{max^m}\Big\|_{at\,x=\pm\frac{L}{2}}$	$= \pm\frac{WL^2}{16EI}$	At the ends

Hence the integration method may be bit cumbersome in some of the case. Another limitation of the method would be that if the beam is of non uniform cross section,



I.e. it is having different cross-section then this method also fails.

So there are other methods by which we find the deflection like

1. Macaulay's method in which we can write the different equation for bending moment for different sections.
2. Area moment methods
3. Energy principle methods

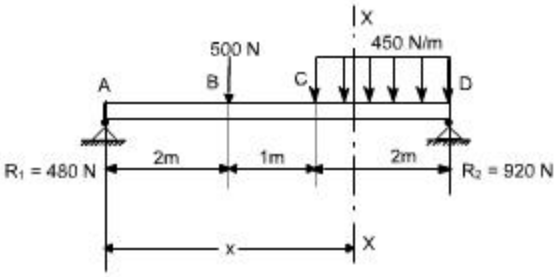
Macaulay's Methods

If the loading conditions change along the span of beam, there is corresponding change in moment equation. This requires that a separate moment equation be written between each change of load point and that too integration be made for each such moment equation. Evaluation of the constants introduced by each integration can become very involved. Fortunately, these complications can be avoided by writing single moment equation in such a way that it becomes continuous for entire length of the beam in spite of the discontinuity of loading.

Note: In Macaulay's method some author's take the help of unit function approximation (i.e. Laplace transform) in order to illustrate this method, however both are essentially the same.

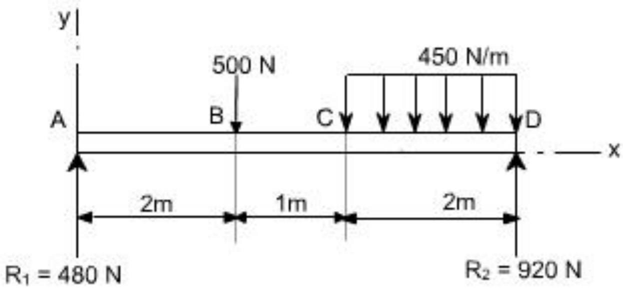
For example consider the beam shown in fig below:

Let us write the general moment equation using the definition M = (∑ M)$_L$, Which means that we consider the effects of loads lying on the left of an exploratory section. The moment equations for the portions AB,BC and CD are written as follows



$$M_{AB} = 480\,x \text{ N.m}$$
$$M_{BC} = \left[480\,x - 500(x-2)\right] \text{N.m}$$
$$M_{CD} = \left[480\,x - 500(x-2) - \frac{450}{2}(x-3)^2\right] \text{N.m}$$

It may be observed that the equation for M_{CD} will also be valid for both M_{AB} and M_{BC} provided that the terms (x - 2) and $(x - 3)^2$are neglected for values of x less than 2 m and 3 m, respectively. In other words, the terms (x - 2) and $(x - 3)^2$ are nonexistent for values of x for which the terms in parentheses are negative.

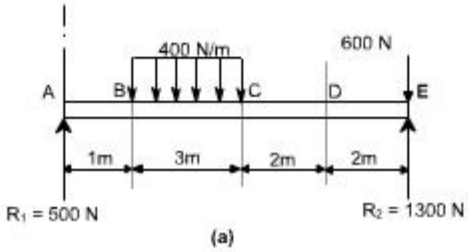


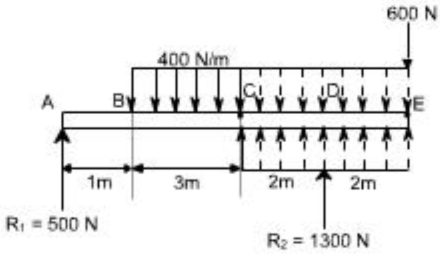
As an clear indication of these restrictions, one may use a nomenclature in which the usual form of parentheses is replaced by pointed brackets, namely, ‹ ›. With this change in nomenclature, we obtain a single moment equation

$$M = \left(480x - 500(x-2) - \frac{450}{2}(x-3)^2\right) \text{N.m}$$

Which is valid for the entire beam if we postulate that the terms between the pointed brackets do not exists for negative values; otherwise the term is to be treated like any ordinary expression.

As an another example, consider the beam as shown in the fig below. Here the distributed load extends only over the segment BC. We can create continuity, however, by assuming that the distributed load extends beyond C and adding an equal upward-distributed load to cancel its effect beyond C, as shown in the adjacent fig below. The general moment equation, written for the last segment DE in the new nomenclature may be written as:





$$M = \left(500x - \frac{400}{2}(x-1)^2 + \frac{400}{2}(x-4)^2 + 1300(x-6)\right) N.m$$

It may be noted that in this equation effect of load 600 N won't appear since it is just at the last end of the beam so if we assume the exploratory just at section at just the point of application of 600 N than x = 0 or else we will here take the X - section beyond 600 N which is invalid.

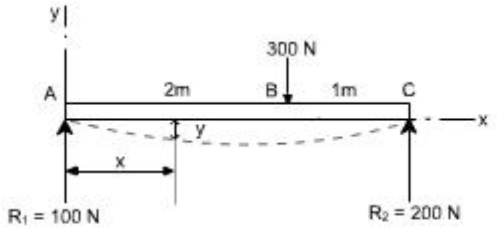
Procedure to solve the problems

(i). After writing down the moment equation which is valid for all values of 'x' i.e. containing pointed brackets, integrate the moment equation like an ordinary equation.

(ii). While applying the B.C's keep in mind the necessary changes to be made regarding the pointed brackets.

Illustrative Examples:

1. A concentrated load of 300 N is applied to the simply supported beam as shown in Fig. Determine the equations of the elastic curve between each change of load point and the maximum deflection in the beam.



Solution: writing the general moment equation for the last portion BC of the loaded beam,

$$EI\frac{d^2y}{dx^2} = M = \left(100x - 300\langle x-2\rangle\right)N.m \qquad ....(1)$$

Integrating twice the above equation to obtain slope and the deflection

$$EI\frac{dy}{dx} = \left(50x^2 - 150\langle x-2\rangle^2 + C_1\right)N.m^2 \qquad ....(2)$$

$$EIy = \left(\frac{50}{3}x^3 - 50\langle x-2\rangle^3 + C_1x + C_2\right)N.m^3 \qquad ....(3)$$

To evaluate the two constants of integration. Let us apply the following boundary conditions:

1. At point A where x = 0, the value of deflection y = 0. Substituting these values in Eq. (3) we find C_2 = 0.keep in mind that $< x - 2 >^3$ is to be neglected for negative values.

2. At the other support where x = 3m, the value of deflection y is also zero.

Substituting these values in the deflection Eq. (3), we obtain

$$0 = \left(\frac{50}{3}3^3 - 50(3-2)^3 + 3.C_1\right) \text{or } C_1 = -133N.m^2$$

Having determined the constants of integration, let us make use of Esq. (2) and (3) to rewrite the slope and deflection equations in the conventional form for the two portions.

segment AB ($0 \le x \le 2m$)

$$EI\frac{dy}{dx} = \left(50x^2 - 133\right)N.m^2 \qquad ....(4)$$

$$EIy = \left(\frac{50}{3}x^3 - 133x\right)N.m^3 \qquad ....(5)$$

segment BC ($2m \le x \le 3m$)

$$EI\frac{dy}{dx} = \left(50x^2 - 150(x-2)^2 - 133x\right)N.m^2 \qquad ....(6)$$

$$EIy = \left(\frac{50}{3}x^3 - 50(x-2)^3 - 133x\right)N.m^3 \qquad ....(7)$$

Continuing the solution, we assume that the maximum deflection will occur in the segment AB. Its location may be found by differentiating Eq. (5) with respect to x and setting the derivative to be equal to zero, or, what amounts to the same thing, setting the slope equation (4) equal to zero and solving for the point of zero slope.

We obtain

$50\,x^2 - 133 = 0$ or $x = 1.63$ m (It may be kept in mind that if the solution of the equation does not yield a value < 2 m then we have to try the other equations which are valid for segment BC)

Since this value of x is valid for segment AB, our assumption that the maximum deflection occurs in this region is correct. Hence, to determine the maximum deflection, we substitute x = 1.63 m in Eq (5), which yields

$$EIy\big|_{max}m = -145\,N.m^3 \qquad ....(8)$$

The negative value obtained indicates that the deflection y is downward from the x axis.quite usually only the magnitude of the deflection, without regard to sign, is desired; this is denoted by , the use of y may be reserved to indicate a directed value of deflection.

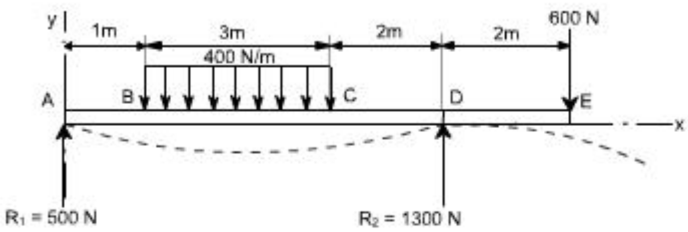
if E = 30 Gpa and I = 1.9×10^6 mm^4 = 1.9×10^{-6} m^4 , Eq. (h) becomes

$$y\big|_{max}m = (30\times10^9)(1.9\times10^{-6})$$

Then $$= -2.54\,mm$$

Example 2:

It is required to determine the value of EIy at the position midway between the supports and at the overhanging end for the beam shown in figure below.



Solution:

Writing down the moment equation which is valid for the entire span of the beam and applying the differential equation of the elastic curve, and integrating it twice, we obtain

$$EI\frac{d^2y}{dx^2} = M = \left(500x - \frac{400}{2}(x-1)^2 + \frac{400}{2}(x-4)^2 + 1300(x-6)\right)N.m$$

$$EI\frac{dy}{dx} = \left(250x^2 - \frac{200}{3}(x-1)^3 + \frac{200}{3}(x-4)^3 + 650(x-6)^2 + C_1\right)N.m$$

$$EIy = \left(\frac{250}{3}x^3 - \frac{50}{3}(x-1)^4 + \frac{50}{3}(x-4)^4 + \frac{650}{3}(x-6)^3 + C_1x + C_2\right)N.m^3$$

To determine the value of C_2, It may be noted that EIy = 0 at x = 0, which gives $C_2 = 0$.Note that the negative terms in the pointed brackets are to be ignored Next, let us use the condition that EIy = 0 at the right support where x = 6m.This gives

$$0 = \frac{250}{3}(6)^3 - \frac{50}{3}(5)^4 + \frac{50}{3}(2)^4 + 6C_1 \quad \text{or} \quad C_1 = -1308\text{N.m}^2$$

Finally, to obtain the midspan deflection, let us substitute the value of x = 3m in the deflection equation for the segment BC obtained by ignoring negative values of the bracketed terms $\langle x - 4 \rangle^4$ and $\langle x - 6 \rangle^3$. We obtain

$$EIy = \frac{250}{3}(3)^3 - \frac{50}{3}(2)^4 - 1308(3) = -1941 \text{ N.m}^3$$

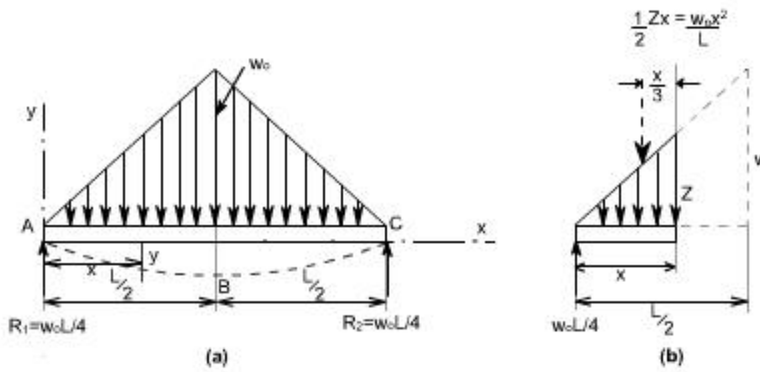
For the overhanging end where x=8 m,we have

$$EIy = \left(\frac{250}{3}(8)^3 - \frac{50}{3}(7)^4 + \frac{50}{3}(4)^4 + \frac{650}{3}(2)^3 - 1308(8)\right)$$

$$= -1814\text{N.m}^3$$

Example 3:

A simply supported beam carries the triangularly distributed load as shown in figure. Determine the deflection equation and the value of the maximum deflection.



Solution:

Due to symmetry, the reactions is one half the total load of $1/2w_0L$, or $R_1 = R_2 = 1/4w_0L$.Due to the advantage of symmetry to the deflection curve from A to B is the mirror image of that from C to B. The condition of zero deflection at A and of zero slopes at B does not require the use of a general moment equation. Only the moment equation for segment AB is needed and this may be easily written with the aid of figure (b).

Taking into account the differential equation of the elastic curve for the segment AB and integrating twice, one can obtain

$$EI\frac{d^2y}{dx^2} = M_{AB} = \frac{w_0L}{4}x - \frac{w_0x^2}{L}.\frac{x}{3} \qquad ......(1)$$

$$EI\frac{dy}{dx} = \frac{w_0Lx^2}{8} - \frac{w_0x^4}{12L} + C_1 \qquad ......(2)$$

$$EIy = \frac{w_0Lx^3}{24} - \frac{w_0x^5}{60L} + C_1x + C_2 ......(3)$$

In order to evaluate the constants of integration, let us apply the B.C's we note that at the support A, y = 0 at x = 0.Hence from equation (3), we get $C_2 = 0$. Also, because of symmetry, the slope dy/dx = 0 at midspan where x = L/2.Substituting these conditions in equation (2) we get

$$0 = \frac{w_0L}{8}\left(\frac{L}{2}\right)^2 - \frac{w_0}{12L}\left(\frac{L}{2}\right)^4 + C_1 C_1 = -\frac{5w_0L^3}{192}$$

Hence the deflection equation from A to B (and also from C to B because of symmetry) becomes

$$EIy = \frac{w_0Lx^3}{24} - \frac{w_0x^5}{60L} - \frac{5w_0L^3x}{192}$$

Whichreducesto

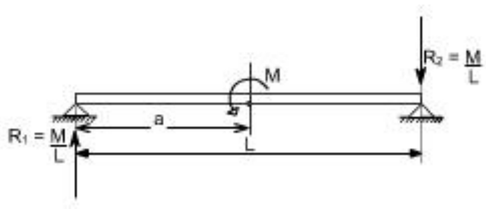
$$EIy = -\frac{w_0x}{960L}\left(25L^4 - 40L^2x^2 + 16x^4\right)$$

The maximum deflection at midspan,where x = L/2 is then found to be

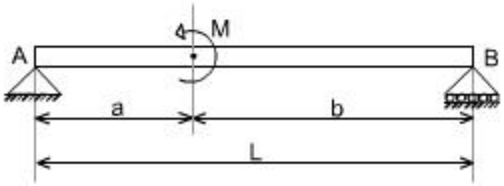
$$\boxed{EIy = -\frac{w_0L^4}{120}}$$

Example 4: couple acting

Consider a simply supported beam which is subjected to a couple M at a distance 'a' from the left end. It is required to determine using the Macaulay’s method.



To deal with couples, only thing to remember is that within the pointed brackets we have to take some quantity and this should be raised to the power zero.



Therefore, writing the general moment equation we get

$$M = R_1 x - M\langle x - a\rangle \text{ or } EI\frac{d^2y}{dx^2} = M$$

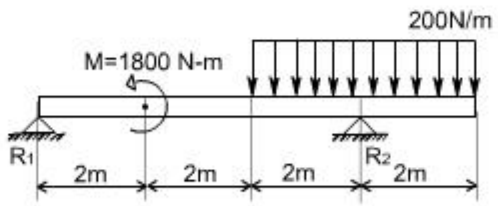
Integrating twice we get

$$EI\frac{dy}{dx} = R_1.\frac{x^2}{2} - M\langle x - a\rangle^1 + C_1$$

$$EI.y = R_1.\frac{x^3}{6} - \frac{M}{2}\langle x - a\rangle^2 + C_1 x + C_2$$

Example 5:

A simply supported beam is subjected to U.d.l in combination with couple M. It is required to determine the deflection.



This problem may be attempted in the same way. The general moment equation may be written as

$$M(x) = R_1 x - 1800\langle x - 2\rangle^0 - \frac{200\langle x - 4\rangle\langle x - 4\rangle}{2} + R_2\langle x - 6\rangle$$

$$= R_1 x - 1800\langle x - 2\rangle^0 - \frac{200\langle x - 4\rangle^2}{2} + R_2\langle x - 6\rangle$$

Thus,

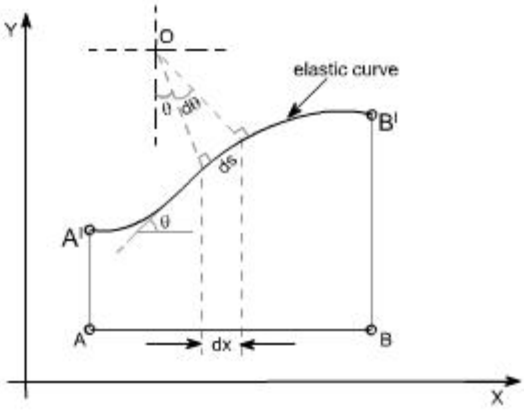
$$EI\frac{d^2y}{dx^2} = R_1 x - 1800\langle x - 2\rangle^0 - \frac{200\langle x - 4\rangle^2}{2} + R_2\langle x - 6\rangle$$

Integrate twice to get the deflection of the loaded beam.

THE AREA-MOMENT / MOMENT-AREA METHODS:

The area moment method is a semi graphical method of dealing with problems of deflection of beams subjected to bending. The method is based on a geometrical interpretation of definite integrals. This is applied to cases where the equation for bending moment to be written is cumbersome and the loading is relatively simple.

Let us recall the figure, which we referred while deriving the differential equation governing the beams.



It may be noted that d is an angle subtended by an arc element ds and M is the bending moment to which this element is subjected.
We can assume,
ds = dx [since the curvature is small]
Hence, R d = ds

$$\frac{d\theta}{ds} = \frac{1}{R} = \frac{M}{EI}$$

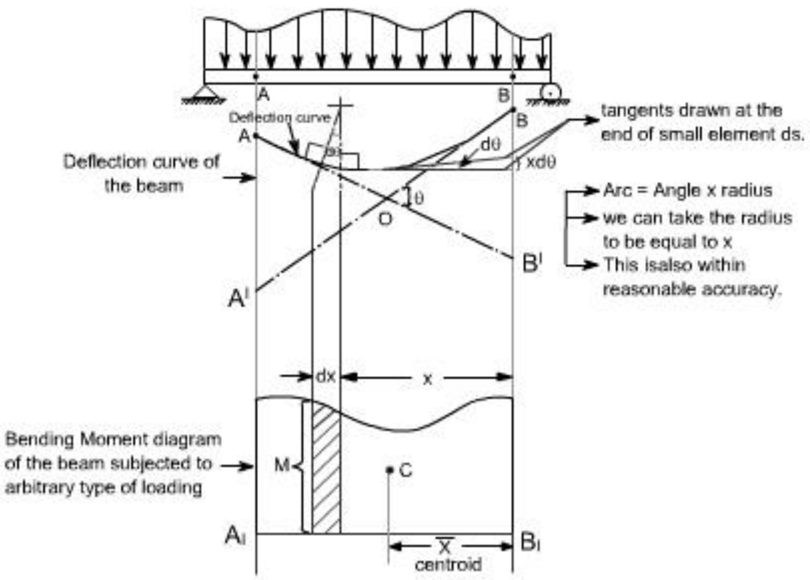
$$\frac{d\theta}{ds} = \frac{M}{EI}$$

But for small curvature[but θ is the angle, slope is $\tan\theta = \frac{dy}{dx}$ for small angles $\tan\theta \approx \theta$, hence $\theta \cong \frac{dy}{dx}$ so we get $\frac{d^2y}{dx^2} = \frac{M}{EI}$ by putting $ds \approx dx$]

Hence,

$$\frac{d\theta}{dx} = \frac{M}{EI} \text{ or } \boxed{d\theta = \frac{M.dx}{EI}} \text{ - - - - - (1)}$$

The relationship as described in equation (1) can be given a very simple graphical interpretation with reference to the elastic plane of the beam and its bending moment diagram



Refer to the figure shown above consider AB to be any portion of the elastic line of the loaded beam and $A_1B_1$is its corresponding bending moment diagram.

Let AO = Tangent drawn at A
BO = Tangent drawn at B
Tangents at A and B intersects at the point O.
Further, AA ' is the deflection of A away from the tangent at B while the vertical distance B'B is the deflection of point B away from the tangent at A. All these quantities are further understood to be very small.
Let ds ≈ dx be any element of the elastic line at a distance x from B and an angle between at its tangents be d . Then, as derived earlier

$$d\theta = \frac{M.dx}{EI}$$

This relationship may be interpreted as that this angle is nothing but the area M.dx of the shaded bending moment diagram divided by EI.
From the above relationship the total angle between the tangents A and B may be determined as

$$\boxed{\theta = \int_A^B \frac{Mdx}{EI} = \frac{1}{EI}\int_A^B Mdx}$$

Since this integral represents the total area of the bending moment diagram, hence we may conclude this result in the following theorem

Theorem I:

$$\left\{\begin{matrix}\text{slope or } \theta \\ \text{between any two points}\end{matrix}\right\} = \left\{\begin{matrix}\frac{1}{EI} \times \text{area of B.M diagram between} \\ \text{corresponding portion of B.M diagram}\end{matrix}\right\}$$

Now let us consider the deflection of point B relative to tangent at A, this is nothing but the vertical distance BB'. It may be note from the bending diagram that bending of the element ds contributes to this deflection by an amount equal to x each of this intercept may be considered as the arc of a circle of radius x subtended by the angle]

Hence the total distance B'B becomes $\delta = \int_A^B x d\theta$

The limits from A to B have been taken because A and B are the two points on the elastic curve, under consideration]. Let us substitute the value of d = M dx / EI as derived earlier

$$\delta = \int_A^B x \frac{Mdx}{EI} = \int_A^B \frac{Mdx}{EI}.x$$

[This is in fact the moment of area of the bending moment diagram]

Since M dx is the area of the shaded strip of the bending moment diagram and x is its distance from B, we therefore conclude that right hand side of the above equation represents first moment area with respect to B of the total bending moment area between A and B divided by EI.
Therefore, we are in a position to state the above conclusion in the form of theorem as follows:

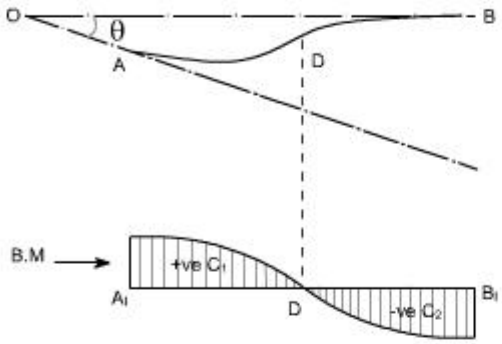
Theorem II:

Deflection of point 'B' relative to point A $= \frac{1}{EI} \times \left\{\begin{matrix}\text{first moment of area with respect} \\ \text{to point B, of the total B.M diagram}\end{matrix}\right\}$

Further, the first moment of area, according to the definition of centroid may be written as $A\bar{x}$, where $\bar{x}$ is equal to distance of centroid and a is the total area of bending moment

Thus, $\boxed{\delta_A = \frac{1}{EI} A\bar{X}}$

Therefore, the first moment of area may be obtained simply as a product of the total area of the B.M diagram between the points A and B multiplied by the distance $\bar{x}$ to its centroid C.
If there exists an inflection point or point of contra flexure for the elastic line of the loaded beam between the points A and B, as shown below,



Then, adequate precaution must be exercised in using the above theorem. In such a case B. M diagram gets divide into two portions +ve and –ve portions with centroids $C_1$and C_2. Then to find an angle ⍰ between the tangents at the points A and B

$$\theta = \int_A^D \frac{Mdx}{EI} - \int_D^B \frac{Mdx}{EI}$$

And similarly for the deflection of B away from the tangent at A becomes

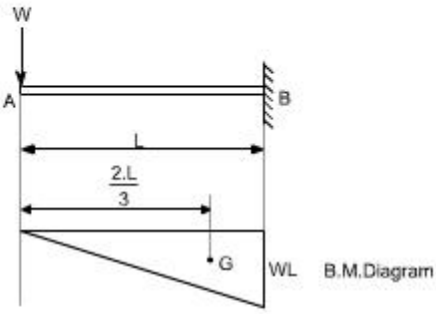
$$\delta = \int_A^D \frac{M.dx}{EI}.x - \int_B^D \frac{M.dx}{EI}.x$$

Illustrative Examples: Let us study few illustrative examples, pertaining to the use of these theorems

Example 1:

1. A cantilever is subjected to a concentrated load at the free end. It is required to find out the deflection at the free end.

For a cantilever beam, the bending moment diagram may be drawn as shown below



Let us workout this problem from the zero slope condition and apply the first area - moment theorem

$$\text{slope at A} = \frac{1}{EI}[\text{Area of B.M diagram between the points A and B}]$$

$$= \frac{1}{EI}\left[\frac{1}{2}L.WL\right]$$

$$= \frac{WL^2}{2EI}$$

The deflection at A (relative to B) may be obtained by applying the second area - moment theorem

NOTE: In this case the point B is at zero slopes.

Thus,

$$\delta = \frac{1}{EI}[\text{first moment of area of B.M diagram between A and B about A}]$$

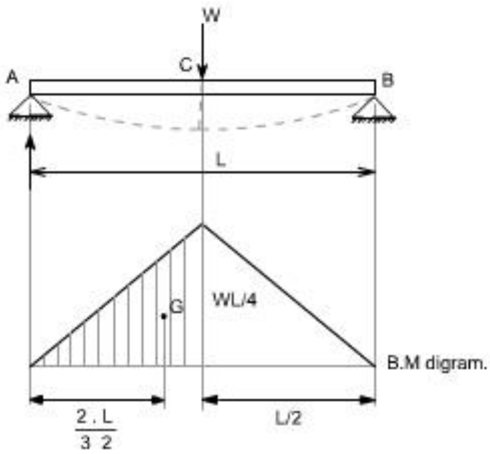
$$= \frac{1}{EI}[A\bar{y}]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}L.WL\right)\frac{2}{3}L\right]$$

$$= \frac{WL^3}{3EI}$$

Example 2: Simply supported beam is subjected to a concentrated load at the mid span determine the value of deflection.

A simply supported beam is subjected to a concentrated load W at point C. The bending moment diagram is drawn below the loaded beam.



Again working relative to the zero slope at the centre C.

$$\text{slope at A} = \frac{1}{EI}[\text{Area of B.M diagram between A and C}]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{WL}{4}\right)\right]$$ we are taking half area of the B.M because we have to work out this relative to a zero slope

$$= \frac{WL^2}{16EI}$$

Deflection of A relative to C = central deflection of C

or

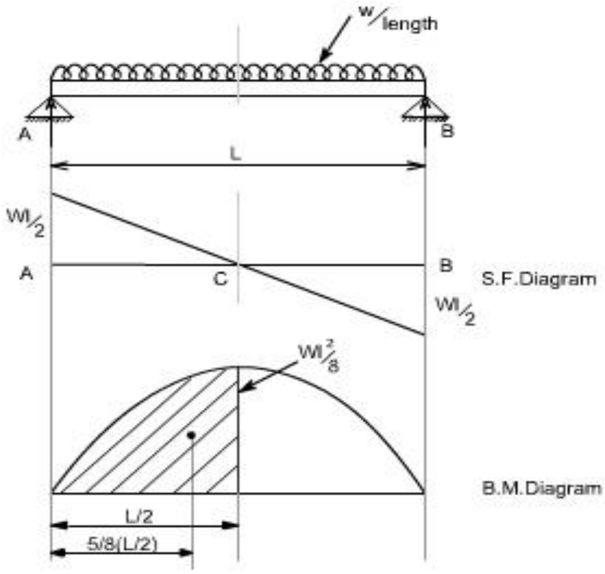
$$\delta_C = \frac{1}{EI}[\text{Moment of B.M diagram between points A and C about A}]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{WL}{4}\right)\frac{2}{3}L\right]$$

$$= \frac{WL^3}{48EI}$$

Example 3: A simply supported beam is subjected to a uniformly distributed load, with an intensity of loading W / length. It is required to determine the deflection.

The bending moment diagram is drawn, below the loaded beam; the value of maximum B.M is equal to Wl^2 / 8



So by area moment method,

$$\text{Slope at point C w.r.t point A} = \frac{1}{EI}\left[\text{Area of B.M diagram between point A and C}\right]$$

$$= \frac{1}{EI}\left[\left(\frac{2}{3}\right)\left(\frac{WL^2}{8}\right)\left(\frac{L}{2}\right)\right]$$

$$= \frac{WL^3}{24EI}$$

$$\text{Deflection at point C relative to A} = \frac{1}{EI}\left[A\,\bar{y}\right]$$

$$= \frac{1}{EI}\left[\left(\frac{WL^3}{24}\right)\left(\frac{5}{8}\right)\left(\frac{L}{2}\right)\right]$$

$$= \frac{5}{384EI}.WL^4$$

Principle of superposition

Members Subjected to Combined Loads

Combined Bending & Twisting: In some applications the shaft are simultaneously subjected to bending moment M and Torque T. The Bending moment comes on the shaft due to gravity or Inertia loads. So the stresses are set up due to bending moment and Torque.

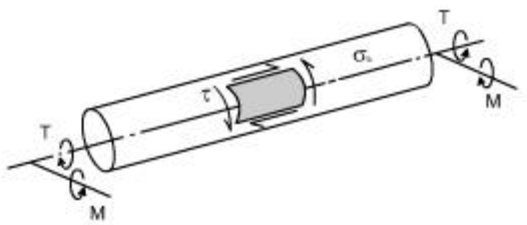
For design purposes it is necessary to find the principal stresses, maximum shear stress, whichever is used as a criterion of failure.

From the simple bending theory equation $\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$

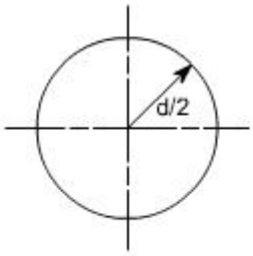
If σ_b is the maximum bending stresses due to bending.

$$\sigma_b = \frac{M.y}{I}$$

$$\sigma_b\big|_{max^m} = \frac{M}{I}.y\,max^m$$



For the case of circular shafts y_{max^m} – equal to d/2 since y is the distance from the neutral axis.



I is the moment of inertia for circular shafts

Hence then, the maximum bending stresses developed due to the application of bending moment M is

$$\sigma_b\Big|_{max^m} = \frac{M}{\pi d^4/64}\cdot\frac{d}{2}$$

$$\sigma_b\Big|_{max^m} = \frac{32M}{\pi d^3} \qquad (1)$$

From the torsion theory, the maximum shear stress on the surface of the shaft is given by the torsion equation

$$\frac{T}{J} = \frac{\tau'}{r} = \frac{G.\theta}{L}$$

$$\Rightarrow \frac{\tau'}{r} = \frac{T}{J}$$

Where ' is the shear stress at any radius r but when the maximum value is desired the value ofr should be maximum and the value of r is maximum at r = d/2

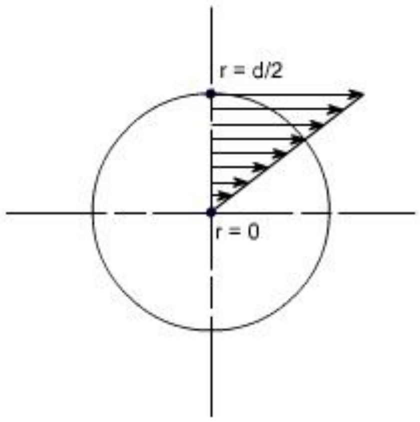
$$\text{Thus } \tau_{max^m} = \frac{T}{J}\cdot\frac{d}{2}$$

$$J = \frac{\pi d^4}{32}$$

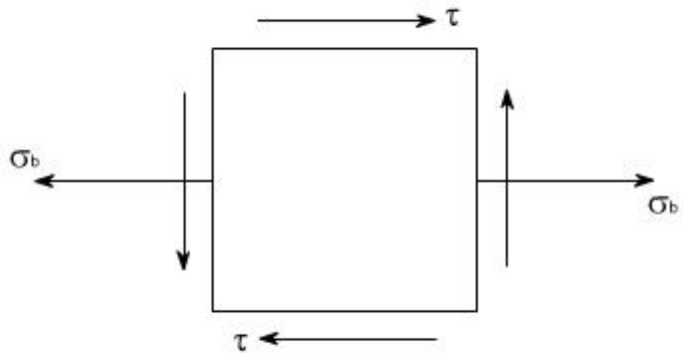
substituting the value of J,we get

$$\tau_{max^m} = \frac{16T}{\pi d^3} \qquad (2)$$

The nature of the shear stress distribution is shown below:



This can now be treated as the two – dimensional stress system in which the loading in a vertical plane in zero i.e. $\sigma_y = 0$ and $\sigma_x = \sigma_b$ and is shown below:



Thus, the principle stresses may be obtained as

$$\sigma_1, \sigma_2 = \left(\frac{\sigma_x + \sigma_y}{2}\right) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^{\;2}}$$

or

$$\sigma_1 = \frac{\sigma_b}{2} + \frac{1}{2}\sqrt{\sigma_b^{\;2} + 4\tau_{max}^{\;2}}$$

$$= \frac{32M}{\pi d^3 .2} + \frac{1}{2}\sqrt{\left(\frac{32M}{\pi d^3}\right)^2 + 4\left(\frac{16T}{\pi d^3}\right)^2}$$

$$= \frac{16M}{\pi d^3 2} + \frac{1}{2}\sqrt{\left(\frac{32M}{\pi d^3}\right)^2 + \left(\frac{2.16T}{\pi d^3}\right)^2}$$

$$= \frac{16}{\pi d^3}\left[M + \sqrt{M^2 + T^2}\right]$$

Equivalent Bending Moment:

Now let us define the term the equivalent bending moment which acting alone will produce the same maximum principal stress or bending stress. Let M_e be the equivalent bending moment, then due to bending

$$\sigma_1 = \frac{32M_e}{\pi d^3}$$

Futher

$$\sigma_1 = \frac{16}{\pi d^3}\left[M + \sqrt{M^2 + T^2}\right]$$

Thus, equating the two we get

$$\boxed{M_e = \frac{1}{2}\left[M + \sqrt{M^2 + T^2}\right]}$$

Equivalent Torque :

At we here already proved that σ_1 and σ_2 for the combined bending and twisting case are expressed by the relations:

$$\sigma_1, \sigma_2 = \frac{16}{\pi d^3}\left\{M \pm \sqrt{M^2 + T^2}\right\}$$

$$\text{or } \sigma_1 = \frac{16}{\pi d^3}\left[\left\{M + \sqrt{M^2 + T^2}\right\}\right]$$

$$\sigma_2 = \frac{16}{\pi d^3}\left[\left\{M - \sqrt{M^2 + T^2}\right\}\right]$$

$$\text{As } \tau_{max^m} = \frac{\sigma_1 - \sigma_2}{2}$$

$$\text{so } \tau_{max^m} = \frac{16}{\pi d^3}\left[\left\{M + \sqrt{M^2 + T^2}\right\}\right] - \frac{16}{\pi d^3}\left[\left\{M - \sqrt{M^2 + T^2}\right\}\right] \Big/ 2$$

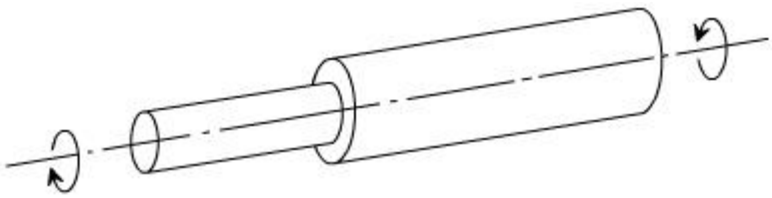
$$\tau_{max^m} = \frac{16}{\pi d^3}\sqrt{M^2 + T^2} = \frac{16}{\pi d^3}.T_e$$

where $\sqrt{M^2 + T^2}$ is defined as the equivalent torque, which acting alone would produce the same maximum shear stress as produced by the pure torsion

Thus, $\boxed{T_e = \sqrt{M^2 + T^2}}$

Composite shafts: (in series)

If two or more shaft of different material, diameter or basic forms are connected together in such a way that each carries the same torque, then the shafts are said to be connected in series & the composite shaft so produced is therefore termed as series – connected.



Here in this case the equilibrium of the shaft requires that the torque 'T' be the same throughout both the parts.

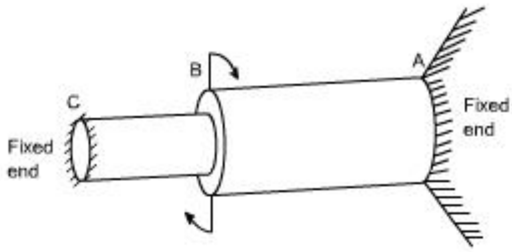
In such cases the composite shaft strength is treated by considering each component shaft separately, applying the torsion – theory to each in turn. The composite shaft will therefore be as weak as its weakest component. If relative dimensions of the various parts are required then a solution is usually effected by equating the torque in each shaft e.g. for two shafts in series

$$T = \frac{G_1 J_1 \theta_1}{L_1} = \frac{G_2 J_2 \theta_2}{L_2}$$

In some applications it is convenient to ensure that the angle of twist in each shaft are equal i.e. $\theta_1 = \theta_2$, so that for similar materials in each shaft $\frac{J_1}{L_1} = \frac{J_2}{L_2}$ or $\frac{L_1}{L_2} = \frac{J_1}{J_2}$

The total angle of twist at the free end must be the sum of angles $\theta_1 = \theta_2$ over each x - section

Composite shaft parallel connection: If two or more shafts are rigidly fixed together such that the applied torque is shared between them then the composite shaft so formed is said to be connected in parallel.



For parallel connection.

Total Torque $T = T_1 + T_2$

In this case the angle of twist for each portion are equal and $\frac{T_1 L_1}{G_1 J_1} = \frac{T_2 L_2}{G_2 J_2}$

for equal lengths(as is normally the case for parallel shafts) $\frac{T_1}{T_2} = \frac{G_1 J_1}{G_2 J_2}$

This type of configuration is statically indeterminate, because we do not know how the applied torque is apportioned to each segment, To deal such type of problem the procedure is exactly the same as we have discussed earlier,

Thus two equations are obtained in terms of the torques in each part of the composite shaft and the maximum shear stress in each part can then be found from the relations.

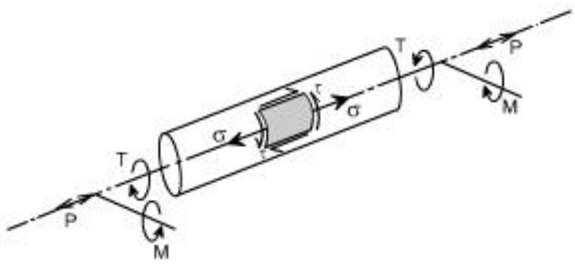
$$\tau_1 = \frac{T_1 R_1}{J_1}$$

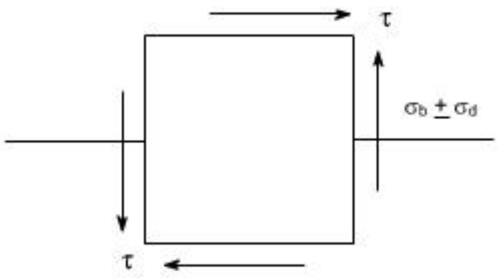
$$\tau_2 = \frac{T_2 R_2}{J_2}$$

Combined bending, Torsion and Axial thrust:

Sometimes, a shaft may be subjected to a combined bending, torsion and axial thrust. This type of situation arises in turbine propeller shaft

If P = Thrust load





Then $\sigma_d = P / A$ (stress due to thrust)

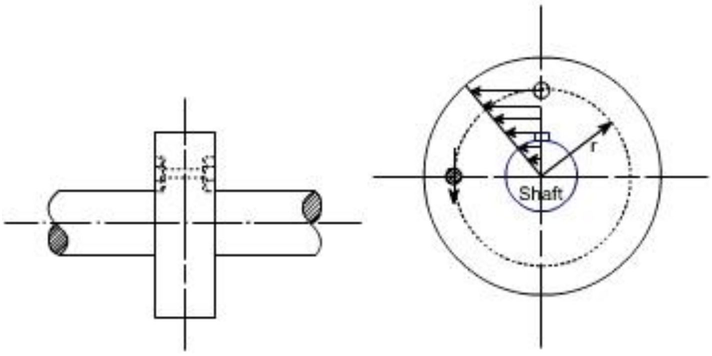
Where σ_d is the direct stress depending on the whether the steam is tensile on the whether the stress is tensile or compressive

This type of problem may be analyzed as discussed in earlier case.

Shaft couplings: In shaft couplings, the bolts fail in shear. In this case the torque capacity of the coupling may be determined in the following manner

Assumptions:

The shearing stress in any bolt is assumed to be uniform and is governed by the distance from its center to the centre of coupling.



Thus, the torque capacity of the coupling is given as $T = \left(\frac{\pi}{4} d_b^2\right) . \tau_b ' . r . n$

Where

d_b = diameter of bolt

τ'_b = maximum shear stress in bolt

n = no. of bolts

Euler‘s formula instability of columns

Limitations of Euler's Theory :

In practice the ideal conditions are never [i.e. the strut is initially straight and the end load being applied axially through centroid] reached. There is always some eccentricity and initial curvature present. These factors need to be accommodated in the required formulas.

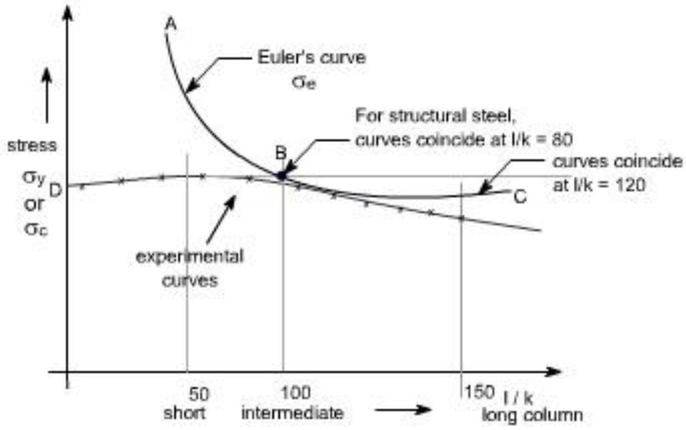
It is realized that, due to the above mentioned imperfections the strut will suffer a deflection which increases with load and consequently a bending moment is introduced which causes failure before the Euler's load is reached. In fact failure is by stress rather than by buckling and the deviation from the Euler value is more marked as the slenderness-ratio l/k is reduced. For values of $l/k < 120$ approx, the error in applying the Euler theory is too great to allow of its use. The stress to cause buckling from the Euler formula for the pin ended strut is

$$\text{Euler's stress, } \sigma_e = \frac{P_e}{A} = \frac{\pi^2 EI}{Al^2}$$

But, $I = Ak^2$

$$\sigma_e = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$$

A plot of σ_e versus l / k ratio is shown by the curve ABC.



Allowing for the imperfections of loading and strut, actual values at failure must lie within and below line CBD.

Other formulae have therefore been derived to attempt to obtain closer agreement between the actual failing load and the predicted value in this particular range of slenderness ratio i.e./k=40 to l/k=100.

(a) Straight – line formulae :

The permissible load is given by the formulae

$P = \sigma_y A\left[1 - n\left(\frac{l}{k}\right)\right]$ Where the value of index 'n' depends on the material used and the end conditions.

(b) Johnson parabolic formulae : The Johnson parabolic formulae is defined as

$P = \sigma_y A\left[1 - b\left(\frac{l}{k}\right)^2\right]$ Where the value of index 'b' depends on the end conditions.

(c) Rankin Gordon Formulae :

$$\frac{1}{P_R} = \frac{1}{P_e} + \frac{1}{P_c}$$

Where P_e = Euler crippling load

P_c = Crushing load or Yield point load in Compression

P_R = Actual load to cause failure or Rankin load

Since the Rankin formulae is a combination of the Euler and crushing load for a strut.

$$\frac{1}{P_R} = \frac{1}{P_e} + \frac{1}{P_c}$$

For a very short strut P_e is very large hence 1/ P $_w$ould be large so that 1/ P $_c$an be neglected.

Thus $P_R = P_c$, for very large struts, P_e is very small so 1/ P_e would be large and 1/ P_c can be neglected ,hence $P_R = P_e$

The Rankin formulae are therefore valid for extreme values of 1/k.It are also found to be fairly accurate for the intermediate values in the range under consideration. Thus rewriting the formula in terms of stresses, we have

$$\frac{1}{\sigma A} = \frac{1}{\sigma_e A} + \frac{1}{\sigma_y A}$$

$$\frac{1}{\sigma} = \frac{1}{\sigma_e} + \frac{1}{\sigma_y}$$

$$\frac{1}{\sigma} = \frac{\sigma_e + \sigma_y}{\sigma_e . \sigma_y}$$

$$\sigma = \frac{\sigma_e . \sigma_y}{\sigma_e + \sigma_y} = \frac{\sigma_y}{1 + \frac{\sigma_y}{\sigma_e}}$$

For struts with both ends pinned

$$\sigma_e = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$$

$$\sigma = \frac{\sigma_y}{1 + \frac{\sigma_y}{\pi^2 E}\left(\frac{l}{k}\right)^2}$$

$$\sigma = \frac{\sigma_y}{1 + a\left(\frac{l}{k}\right)^2}$$

Where $a = \frac{\sigma_y}{\pi^2 EI}$ and the value of 'a' is found by conducting experiments on various materials. Theoretically, but having a value normally found by experiment for various materials. This will take into account other types of end conditions.

Therefore $$\text{Rankine load} = \frac{\sigma_y . A}{1 + a\left(\frac{l}{k}\right)^2}$$

Typical values of 'a' for use in Rankin formulae are given below in table.

Material	**σ_y or σ_c MN/m^2**	**Value of a**	
		Pinned ends	**Fixed ends**
Low carbon steel	315	1/7500	1/30000
Cast Iron	540	1/1600	1/64000
Timber	35	1/3000	1/12000

note a = 4 x (a for fixed ends)

Since the above values of 'a' are not exactly equal to the theoretical values , the Rankin loads for long struts will not be identical to those estimated by the Euler theory as estimated.

Strut with initial Curvature :

As we know that the true conditions are never realized, but there are always some imperfections. Let us say that the strut is having some initial curvature. i.e., it is not perfectly straight before loading. The situation will influence the stability. Let us analyze this effect.

by a differential calculus

$$R_0 \approx \frac{1}{d^2y_0/dx^2} \text{(Approximately)}$$

$$\text{Futher } \frac{E}{R} = \frac{M}{I} \text{ and } \frac{EI}{R} = M$$

$$\text{But for this case } EI\left[\frac{1}{R} - \frac{1}{R_0}\right] = M$$

since strut is having some initial curvature

Now putting

$$\frac{1}{R} = \frac{d^2y}{dx^2} \text{ and } \frac{1}{R_0} = \frac{d^2y_0}{dx^2}$$

Where ' y_0' is the value of deflection before the load is applied to the strut when the load is applied to the strut the deflection increases to a value 'y'. Hence

$$EI\left[\frac{d^2y}{dx^2} - \frac{d^2y_0}{dx^2}\right] = M$$

$$EI\frac{d^2y}{dx^2} - EI\frac{d^2y_0}{dx^2} = M$$

$$EI\frac{d^2y}{dx^2} = M + EI\frac{d^2y_0}{dx^2}$$

$$EI\frac{d^2y}{dx^2} = -Py + EI\frac{d^2y_0}{dx^2}$$

If the pinended strut is under the action of a load P then obviously the BM would be' – py'

Hence

$$EI\frac{d^2y}{dx^2} + Py = EI\frac{d^2y_0}{dx^2}$$

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{d^2y_0}{dx^2}$$

Again letting

$$\frac{P}{EI} = n^2$$

$$\frac{d^2y}{dx^2} + n^2y = \frac{d^2y_0}{dx^2}$$

The initial shape of the strut y_0 may be assumed circular, parabolic or sinusoidal without making much difference to the final results, but the most convenient form is

$y_0 = C.\sin\frac{\pi x}{l}$ where C is some constant or here it is amplitude

Which satisfies the end conditions and corresponds to a maximum deviation 'C'. Any other shape could be analyzed into a Fourier series of sine terms. Then

$$\frac{d^2y}{dx^2} + n^2y = \frac{d^2y_0}{dx^2} = \frac{d^2}{dx^2}\left[C.\sin\frac{\pi x}{l}\right] = \left(-C.\frac{\pi^2}{l^2}\right)\sin\left(\frac{\pi x}{l}\right)$$

The computer solution would be therefore be

$$y_{general} = y_{complementry} + y_{PI}$$

$$y = A\cos nx + B\sin nx + \frac{C.\frac{\pi^2}{l^2}}{\left(\frac{\pi^2}{l^2}\right) - n^2}\sin\left(\frac{\pi x}{l}\right)$$

Boundary conditions which are relevant to the problem are

at x = 0 ; y = 0 thus B = 0

Again

when x = l ; y = 0 or x = l / 2 ; dy/dx = 0

the above condition gives B = 0

Therefore the complete solution would be

$$y = \frac{C.\frac{\pi^2}{l^2}}{\left\{\left(\frac{\pi^2}{l^2}\right) - n^2\right\}}\sin\left(\frac{\pi x}{l}\right)$$

Again the above solution can be slightly rearranged. since

$$P_e = \frac{\pi^2 EI}{l^2}$$

hence the term $\frac{\frac{\pi^2}{l^2}}{\frac{\pi^2}{l^2} - n^2}$ after multiplying the denominator & numerator by EI is equal to

$$\frac{\frac{\pi^2 EI}{l^2}}{\frac{\pi^2 EI}{l^2} - n^2 EI} = \left[\frac{P_e}{P_e - P}\right]$$

Since $n^2 = \frac{P}{EI}$

where P_e = Euler's load P=applied load

Thus

$$y = \frac{C.\frac{\pi^2}{l^2}}{\left\{\left(\frac{\pi^2}{l^2}\right) - n^2\right\}}\sin\left(\frac{\pi x}{l}\right)$$

$$y = \left\{\frac{C.P_e}{P_e - P}\right\}\sin\left(\frac{\pi x}{l}\right)$$

The crippling load is again

$$P = P_e = \frac{\pi^2 EI}{l^2}$$

Since the BM for a pin ended strut at any point is given as

M = -Py and

Max BM = P y_{max}

Now in order to define the absolute value in terms of maximum amplitude let us use the symbol as '^'.

$$\hat{M} = P.\hat{y}$$

$$= C.\frac{P.P_e}{(P_e - p)}$$

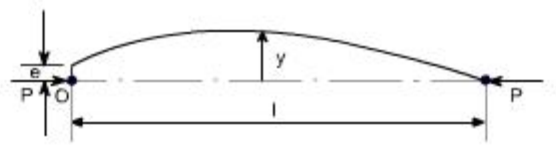
$$\text{Therefore } \hat{M} = \frac{C\ P\ P_e}{[P_e - p]} \text{ since } y_{max^m} = \frac{P_e}{[P_e - p]}$$

$$\sin\frac{\pi x}{l} = 1 \text{ when } \frac{\pi x}{l} = \frac{\pi}{2}$$

$$\text{Hence } \hat{M} = \frac{C\ P\ P_e}{[P_e - p]}$$

Strut with eccentric load

Let 'e' be the eccentricity of the applied end load, and measuring y from the line of action of the load.



Then $EI\frac{d^2y}{dx^2} = -Py$

or $(D^2 + n^2)\,y = 0$ where $n^2 = P / EI$

Therefore $y_{general} = y_{complementary}$

$= A\sin nx + B\cos nx$

applying the boundary conditions then we can determine the constants i.e.

at x = 0 ; y = e thus B = e

at x = l / 2 ; dy / dx = 0

Therefore

$$A\cos\frac{nl}{2} - B\sin\frac{nl}{2} = 0$$

$$A\cos\frac{nl}{2} = B\sin\frac{nl}{2}$$

$$A = B\tan\frac{nl}{2}$$

$$A = e\tan\frac{nl}{2}$$

Hence the complete solution becomes

$y = A\sin(nx) + B\cos(nx)$

substituting the values of A and B we get

$$y = e\left[\tan\frac{nl}{2}\sin nx + \cos nx\right]$$

Note that with an eccentric load, the strut deflects for all values of P, and not only for the critical value as was the case with an axially applied load. The deflection becomes infinite for tan (nl)/2 = ∞ i.e. nl = ? giving the same crippling load $P_e = \frac{\pi^2 EI}{l^2}$. However, due to additional bending moment set up by deflection, the strut will always fail by compressive stress before Euler load is reached.

Since

$$y = e\left[\tan\frac{nl}{2}\sin nx + \cos nx\right]$$

$$y_{max^m}\Big|_{at\ x = \frac{l}{2}} = e\left[\tan\left(\frac{nl}{2}\right)\sin\frac{nl}{2} + \cos\frac{nl}{2}\right]$$

$$= e\left[\frac{\sin^2\frac{nl}{2} + \cos^2\frac{nl}{2}}{\cos\frac{nl}{2}}\right]$$

$$= e\left[\frac{1}{\cos\frac{nl}{2}}\right] = e\ \sec\frac{nl}{2}$$

Hence maximum bending moment would be

$$M_{max^m} = P\ y_{max^m}$$

$$= P\,e\ \sec\frac{nl}{2}$$

Now the maximum stress is obtained by combined and direct strain

$$\sigma = \frac{P}{A} + \frac{M}{Z}\ \text{stress due to bending}\ \frac{\sigma}{y} = \frac{M}{I};$$

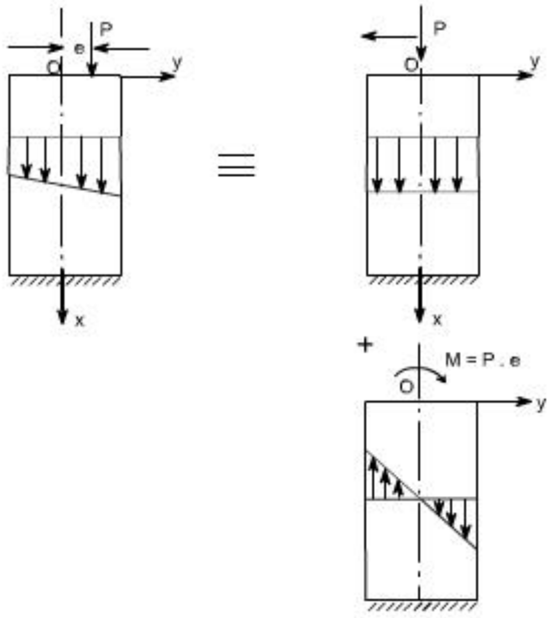
$$M = \sigma\frac{I}{y};\ \sigma_{max} = \frac{M}{Z}\ \text{Wher } Z = I/y \text{ is section modulus}$$

The second term is obviously due the bending action.

Consider a short strut subjected to an eccentrically applied compressive force P at its upper end. If such a strut is comparatively short and stiff, the deflection due to bending action of the eccentric load will be negligible compared with eccentricity ‘e' and the principal of super-imposition applies.

If the strut is assumed to have a plane of symmetry (the xy - plane) and the load P lies in this plane at the distance ‘e' from the centroidal axis ox.

Then such a loading may be replaced by its statically equivalent of a centrally applied compressive force ‘P' and a couple of moment P.e



1. The centrally applied load P produces a uniform compressive $\sigma_1 = \frac{P}{A}$ stress over each cross-section as shown by the stress diagram.

2. The end moment 'M' produces a linearly varying bending stress $\sigma_2 = \frac{My}{I}$ as shown in the figure. Then by super-imposition, the total compressive stress in any fibre due to combined bending and compression becomes,

$$\sigma = \frac{P}{A} + \frac{My}{I}$$

$$\sigma = \frac{P}{A} + \frac{M}{I/y}$$

$$\boxed{\sigma = \frac{P}{A} + \frac{M}{Z}}$$

Column Analysis and Design

Introduction

Columns are usually considered as vertical structural elements, but they can be positioned in any orientation (e.g. diagonal and horizontal compression elements in a truss). Columns are used as major elements in trusses, building frames, and sub-structure supports for bridges (e.g. piers).

2. Columns support compressive loads from roofs, floors, or bridge decks.
3. Columns transmit the vertical forces to the foundations and into the subsoil. The work of a column is simpler than the work of a beam.
3. The loads applied to a column are only axial loads.
4. Loads on columns are typically applied at the ends of the member, producing axial compressive stresses.

5. However, on occasion the loads acting on a column can include axial forces, transverse forces, and bending moments (e.g. beam-columns). Columns are defined by the length between support ends. Short columns (e.g. footing piers). Long columns (e.g. bridge and freeway piers).
6. Virtually every common construction material is used for column construction. Steel, timber, concrete (reinforced and pre-stressed), and masonry (brick, block, and stone). The selection of a particular material may be made based on the following.
   Strength (material) properties (e.g. steel vs. wood).
   Appearance (circular, square, or I-beam).
   Accommodate the connection of other members.
   Local production capabilities (i.e. the shape of the cross section).
7. Columns are major structural components that significantly affect the building's overall performance and stability.
8. Columns are designed with larger safety factors than other structural components. Failure of a joist or beam may be localized and may not severely affect the building's integrity (e.g. there is redundancy with girders and beams, but not with columns).
9. Failure of a strategic column may be catastrophic for a large area of the structure.
10. Failure may be due to overstressed, loss of section (deterioration), accident/sabotage (terrorism). Safety factors for columns are used to account for the following.
11. Material irregularities (e.g. out of straightness).
12. Support fixity at the column ends.
13. Construction inaccuracies (e.g. out of plumpness).
14. Workmanship.
15. Unavoidable eccentric (off-axis) loading.

Short and Long Columns – Modes of Failure

Column slenderness and length greatly influence a column's ability to carry load. Very short, stout columns fail by crushing due to material failure. Failure occurs once the stress exceeds the elastic (yield point) limit of the material. Long, slender columns fail by buckling – a function of the column's dimensions and its modulus of elasticity.

Buckling is the sudden uncontrolled lateral displacement of a column at which point no additional load can be supported.

Failure occurs at a lower stress level than the column's material strength due to buckling (i.e. lateral instability).

Short columns

Short columns fail by crushing at very high stress levels that are above the elastic limit of the column material.

Compressive stress for short columns is based on the basic stress equation developed at the beginning of Chapter. If the load and column size (i.e. cross-sectional area) are known, the compressive stress may be computed as

$$fa = Pactual/A \quad \leq \quad Fa$$

Where

fa = actual compressive stress (psi or ksi)

A = cross-sectional area of the column (in^2)

Pactual = actual load on the column (pounds or kips)

Fa = allowable compressive stress per code (psi or ksi)

This stress equation can be rewritten into a design form to determine the required short column size when the load and allowable material strength are known.

Arequired = Pactual/Fa

Where

Arequired = minimum cross-sectional area of the column

Long Columns – Euler Buckling

Long columns fail by buckling at stress levels that are below the elastic limit of the column material.

1. Very short column lengths require extremely large loads to cause the member to buckle.
2. Large loads result in high stresses that cause crushing rather than buckling.
3. Buckling in long, slender columns is due to the following.
4. Eccentricities in loading.
5. Irregularities in the column material.

Buckling can be avoided (theoretically) if the loads were applied absolutely axially, the column material was totally homogeneous with no imperfections, and construction was true and plumb.

A Swiss mathematician named Leonhard Euler (1707 – 1783) was the first to investigate the buckling behavior of slender columns within the elastic limit of the column's material.

Euler's equation shows the relationship between the load that causes buckling of a (pinned end) column and the material and stiffness properties of the column.

The critical buckling load can be determined by the following equation.

$$Pcritical = \pi^2 EImin/L^2$$

Where

Pcritical = critical axial load that causes buckling in the column (pounds or kips)

E = modulus of elasticity of the column material (psi or ksi)

Imin = smallest moment of inertia of the column cross-section (in^2)

(Most sections have Ix and Iy; angles have Ix, Iy and Iz.)

L = column length between pinned ends (inches)

a. As the column length increases, the critical load rapidly decreases (since it is proportional to L^2), approaching zero as a limit.
b. The critical load at buckling is referred to as *Euler's critical buckling load*.

Euler's equation is valid only for long, slender columns that fail due to buckling.

C Euler's equation contains no safety factors.
D Euler's equation results in compressive stresses developed in columns that are well below the elastic limit of the material.

Slenderness Ratios

The **radius of gyration** is a geometric property of a cross section that was first introduced in Chapter 6.

$I = Ar^2$ and $r = (I/A)^{1/2}$

Where

r = radius of gyration of the column cross section (in)

I = least (minimum) moment of inertia (in^4)

A = cross-sectional area of the column (in^2)

The radius of gyration is geometric property that is used in the analysis and design of columns.

Using the radius of gyration, the critical stress developed in a long column at buckling can be expressed by the following equation.

$F\ critical = Pcritical/A = \pi^2 EImin/AL^2 = \pi^2 E(Ar^2)/AL^2 = \pi^2 E/(L/r)^2$

The term "L/r" is known as the *slenderness ratio*.

- A higher slenderness ratio means a lower critical stress that will cause buckling.
- Conversely, a lower slenderness ratio results in a higher critical stress (but still within the elastic range of the material).

Column sections with large r-values are more resistant to buckling.

10. Compare the difference in rmin values and slenderness ratios for the three column cross sections shown below.
11. All three cross sections have relatively equal cross-sectional areas but very different radii of gyration about the critical buckling axis. The most efficient column sections for axial loads are those with almost equal rx and ry values.
12. Circular pipe sections and square tubes are the most effective shapes since the radii of gyration about both axes are the same (rx = ry).
13. Circular pipe sections and square tubes are often used as columns for light to moderate loads. Wide-flange shapes may be preferred despite the structural advantages of closed cross-sectional shapes (like tubes and pipes).
14. The practical considerations of wide-flange shapes include the following.
    Wide-flange sections support heavy loads.
    Wide-flange sections accommodate beam connections.

Unit-4.Trusses

Truss structures constitute a special class of structures in which individual straight members are connected at joints. The members are assumed to be connected to the joints in a manner that permit rotation, and thereby it fol-lows from equilibrium considerations, to be detailed in the following, that the individual structural members act as bars, i.e. structural members that can only carry an axial force in either tension or compression. Often the joints do not really permit free rotation, and the assumption of a truss structure then is an approximation. Even if this is the case the layout of a truss structure implies that it can carry its loads under the assumption that the individual members act as bars supporting only an axial force. This greatly simplifies the analysis of the forces in the structure by hand calculation and undoubtedly contributed to their popularity e.g. for bridges, towers, pavilions etc. up to the middle of the twentieth century. The layout of the structural members in the form of a truss structure also finds use with rigid or semi-rigid joints, e.g. space truss roofs, girders for suspension bridges, or steel offshore structures. The rigid joints introduce bending effects in the structural members, but these effects are easily included by use of numerically based computational methods.

In a statically determinate truss all the bar forces can be determined by the equilibrium equations, applied to the bars and joints of the truss. There are several strategies for carrying out the corresponding calculations, and three of these will be described in this chapter. The first and conceptually simplest method consists in considering each joint as an isolated body, for which the equilibrium equations must be satisfied. As there are no moment equations due to the hinge property of the joint this gives two equilibrium equations for a joint in a planar truss and three equilibrium equations for a joint in a space truss. The calculation of the bar forces proceeds by considering the individual joints sequentially as explained in Section Alternatively, the bar forces can be calculated by using sections to separate larger parts of the structure and then applying suitable equilibrium equations for these larger parts. This method is dealt with in Section.

It is characteristic of the classic methods of joints and of sections, that they are arranged to determine the bar forces sequentially, and thus are convenient for calculation of the bar forces or a subset of these by hand. However, in their basic form these methods are limited to statically determinate trusses, and even for this class of structures the calculations may become quite elaborate for space trusses and larger planar trusses. Alternatively, a general system-atic method can be developed for elastic trusses, irrespective of whether they are statically determinate or indeterminate. The method consists of setting up the equilibrium equations of all joints in a systematic way, using the elastic property of the bars. This method is a special case of the Finite Ele-ment Method, used in its general form for a wide range of problems within structural mechanics, see e.g. Cook et al. (4004) and Zienkiewicz and Taylor (4000). The formulation of the Finite Element Method takes on a particularly systematic form, when using the principle of virtual work, already mentioned for rigid bodies in Section 1.6. The extension of the principle of virtual work to truss structures is described in Section and is used to formulate the Finite Element Method for elastic truss structures in Section

4.1 Basic principles

A truss structure consists of a number of joints, connected by bars. This is illustrated in Fig. showing a truss structure consisting of the joints numbered as 1*,* 4*,* · · · , 7, connected by bars indicated by numbers in a circle. The joints are assumed to act like hinges, permitting free rotation of the bars around the joint. It is furthermore assumed that the truss structure is only loaded by concentrated forces acting at the joints. As a consequence of the assumption of hinges the bar elements can only support an axial force. This is easily demonstrated by considering Fig.. Due to the hinge there can be no moment at the ends of the bar element. Furthermore, if the force N in the bar were not aligned along the direction of the bar, there would be a non-zero moment about the hinge at the other end of the bar. Thus, the bar can only support a force of magnitude N , aligned along the direction of the bar. By convention the force in the bars of a truss structure are defined as positive when corresponding to tension, and they are then negative when representing compression.

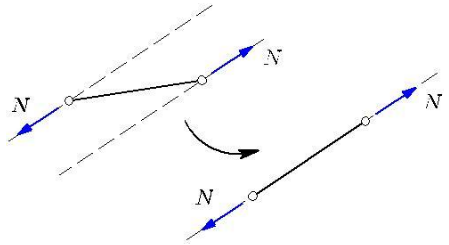


Fig. 4.4: Force in a bar along the bar axis.

A basic principle in the analysis of structures is the section. A section is used to represent a hypothetical separation of a part of the structure from the rest. This hypothetical separation enables a concise discussion of the forces exchanged between the parts on the two sides of the section. The situation is illustrated in simple form in Fig. The figure shows the hypothetical situation in which the bar number 6 is separated from the structure by a section right next to the joints at the ends of the bar. The bar will be acted on by a force along the bar axis, and it follows from equilibrium, that the force at the sections at the ends of the bar must be of equal magnitude but opposite direction. The magnitude is denoted N_6 and is shown in the figure as positive corresponding to tension.

The figure also shows the effect of separating the joint 3 from the structure. In order to maintain the same state as in the structure the joint is acted upon by forces from each of the connected bars. The bar forces are defined as positive in tension, and positive bar forces therefore appear as forces N_4, N_3,

N_5 and N_6 pointing away from the joint. It is important to note that when the bar 6 has a tension force N_6, the bar is acted on by a force of magnitude N_6 pointing away from the bar towards the connecting joint. The connecting joints will similarly be acted on by a force of magnitude N_6 pointing away from the node. By this sign convention positive forces will point away from the member – joint or bar – on which they act. When making a sketch of a structural part, i.e. a joint or a bar, the forces will always be shown corresponding to their positive direction, i.e. as tension forces. If a bar force is determined to be compressive, this corresponds to a negative value of the magnitude N , and the sign of the arrow in the figure will be retained in the direction corresponding to tension.

4.1.1 Building with triangles

The triangle plays an important role in the geometric layout of truss structures. The reason for this is illustrated by the three planar trusses shown in Fig. To be specific they can be envisaged to have a simple fixed support at the left end, and a simple support permitting horizontal motion at the right end. At first glance they may look as 'plausible' candidates for a truss, but are they satisfactory structures?

The truss shown in Fig. 4.3a consists of two triangles, connected by a quadrilateral at the center. The original structure is shown in full line, while the dotted line shows a possible deformation mechanism, by which the central quadrilateral changes shape without need for changing the length of any of the bars in the structure. Thus, even for perfectly rigid bar members the structure has a deformation mechanism. The existence of one or more free deformation mechanisms within a structure is termed kinematic indeterminacy. In the present case the implication is that the structure cannot be used with the prescribed support conditions, but will need an extra support preventing the mechanism.

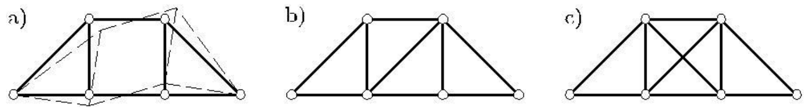


Fig. 4.3: From kinematic to static indeterminacy.

The mechanism can be locked by introducing a diagonal bar in the center quadrilateral as shown in Fig. 4.3b. It is seen that this prevents the free deformation mode, and also leads to a structure formed by triangles. The structure thereby becomes kinematic ally determinate. It is demonstrated be-low that this structure with supports providing three reaction components permits determination of all bar forces by use of the equilibrium conditions only. This property is termed static determinacy.

The equilibrium conditions imply that the force at the two ends of each bar must be of identical magnitude but opposite direction. The remaining equilibrium conditions then express force equilibrium at each joint, illustrated e.g. as equilibrium of the four forces acting on the joint 3 of the truss in Fig. 4.1. It follows from this principle that introduction of an extra bar in a truss, as shown in Fig. 4.3c, will introduce a new undetermined force in this bar. However, for a statically determinate truss the equilibrium equations are precisely sufficient to determine the forces in all bars, and consequently the introduction of an extra bar will leave the number of equilibrium conditions one short. A truss structure, in which the number of equilibrium equations is insufficient to determine all bar forces, is termed statically indeterminate. In contrast to structures with deformation mechanisms, that are generally unsuitable, statically indeterminacy does not constitute a limitation of the potential usefulness of the structure. It just implies that the specific distribution of the forces between the bars, or some of the bars, depends on the deformation properties of these bars. In the present example the two crossing diagonals in Fig. 4.3c share in preventing the deformation mechanism of the quadrilateral of the original structure. However, due to the static indeterminacy the precise ratio in which they share depends on their relative stiffness. Thus, the analysis of statically indeterminate structures requires additional in-formation about the stiffness of the structural members. In this chapter hand calculation type methods are developed for statically determinate trusses, while statically indeterminate trusses are left as part of the Finite Element formulation developed in Section 4.5.

4.1.4 Counting joints and bars

Some typical planar trusses are shown in Figure 4.4. It is seen that they are formed by triangles, and this suggests that they are statically determinate, when supported appropriately by three independent reaction components. It is now demonstrated by a common method for planar trusses, that they are indeed statically determinate. The method leads to a necessary relation between the number of joints and the number of bars. However, and probably equally important, it identifies a rational way of thinking about a truss structure, in which a process is constructed by which the structure is ex-tended joint by joint, simulating an actual construction of the truss from bar elements connected by joints.

For a planar truss the hypothetical construction process starts from a simple triangle, and in order to be specific this triangle is supported by a fixed and a movable support as shown in Fig. 4.5a. Equilibrium of the nodes can be established by two projection equations for the unsupported node,

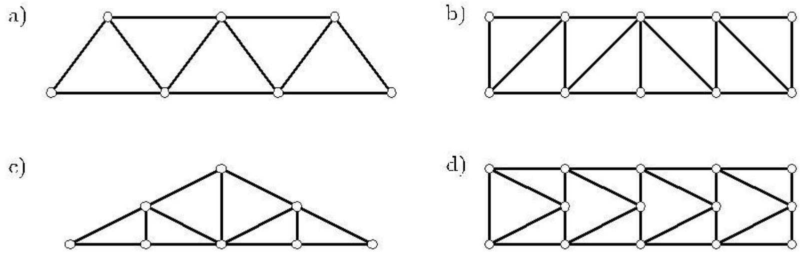


Fig. 4.4: a) V-truss, b) N-truss, c) Roof truss, d) K-truss.

a vertical projection equation of the forces on the node with the moveable support. This gives three equations, corresponding to the three bar forces to be determined. Thus, the initial triangle is statically determinate.

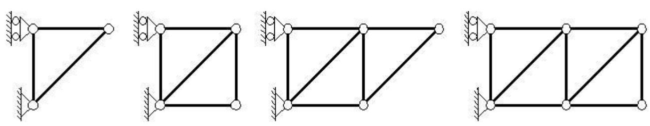


Fig. 4.5: Construction of plane truss girders by triangles.

The process is continued by attaching a new joint by two new bars as illustrated in Fig. 4.5. If the bars are not parallel, they will uniquely determine the position of the new joint, and two projection equations for the forces on the new joint will determine the bar forces. This step , in which a new joint is added and fastened by two new bars, can be continued as illustrated in the figure. The process defines a simple relation between the number of bars *b* and the number of joints *j* in a statically determinate planar truss:

At first sight it may appear that the process is dependent on the supports being applied to the initial triangle. However, this is not the case. The result is independent of the specific support conditions as long as they provide three independent reaction components. After completing the truss structure, the supports can be moved as illustrated in Fig. 4.6.

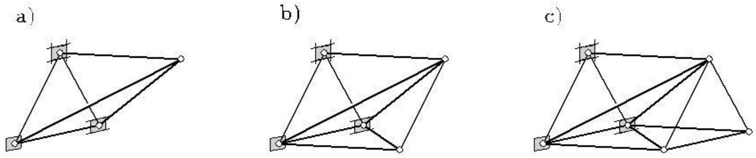


Fig. 4.7: Construction of space truss by addition of tetrahedra.

The results for planar trusses are easily extended to space trusses as illus-trated in Fig. 4.7. The starting point is a tetrahedron (pyramid), formed by 4 joints and 6 connecting bars. The tetrahedron is supported by 6 independent reaction components. This leaves $4 \cdot 3 - 6 = 6$ equilibrium conditions from the 4 nodes for determination of the 6 bar forces. The process is continued in steps consisting in the addition of 1 new joint connected by 3 new bars. The three bars keep the joint fixed in space, and the three force projection equations associated with equilibrium of the new joint determine the three new bar forces. The figure shows the two first steps in this process leading to a truss girder of a type typically used for building cranes. This leads to the following relation between the number of bars *b* and the number of joints *j* of a statically determinate space truss:

Also in this case the relation is necessary but not sufficient, and the imaginary process of constructing the space truss constitutes an important part.

4.1.3 Qualitative tension-compression considerations

It is often possible to identify whether a bar member in a truss is loaded in tension or compression by a simple qualitative argument involving an estimate of the actual deformation of the loaded truss, or by constructing the mechanism that would result if the member were removed from the truss. Figure a simply supported N-truss girder consisting of the 'head', the 'foot', the 'verticals' and the 'diagonals'. It is observed, that with *j* = 10 joints and *b* = 17 bars the truss satisfies the condition (4.1) for a statically determinate truss. Figure 4.8b shows a sketch of the deformed girder after loading by distributed downward forces. It is clearly seen that the bars in the foot are extended, indicating tension, while the bars in the head become shorter, indicating compression. However, it is more difficult to identify elongation or shortening of the verticals and the diagonals.

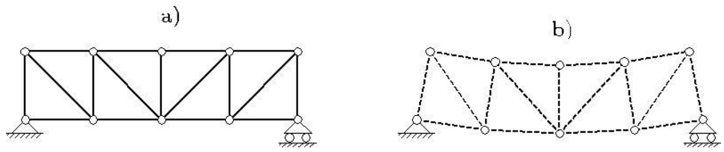


Fig. 4.8: Tension and compression members in truss girder.

A somewhat different and more precise way of estimating whether a bar is in tension or compression consists in imagining that the bar were removed from the truss. For a statically determinate structure this would create a mechanism. Figure 4.9a illustrates the mechanism generated by removing the second bar in the head, while Fig. 4.9b illustrates the mechanism associated with re-moval of the third bar in the foot. The mechanisms are shown corresponding to a downward load. It is clearly seen that the distance between the two joints constituting the end points of the removed bar approach each other in the case of the bar in the head, while they become further separated in the case of the bar in the foot. Thus, the bar in the head will experience compression, while the bar in the foot will experience tension, when the structure carries a downward load.

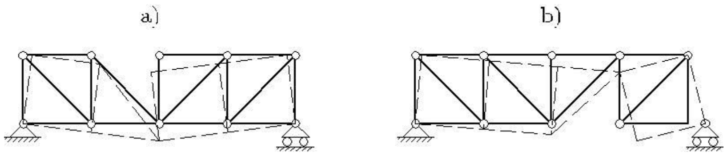


Fig. 4.9: Mechanisms by removing a bar in the head or in the foot.

A similar geometric argument can be used to identify the sign of the force in the diagonals and the verticals. Figure 4.10a illustrates the deformation mechanism generated when removing the second diagonal from the left. It is seen that diagonal would be extended by the illustrated mechanism. Thus there will be tension in the diagonal when the loads perform positive work

Method of joints

Through the mechanism. This would be the case for a downward load at the central or right nodes of the head or the foot. However, a downward load in the first set of node to the right of the left support would create negative work and therefore contribute a compressive force. Figure 4.10b illustrates the mechanism generated by removing the second vertical from the left. A vertical downward load at any of the three inner nodes of the head would lead to compression in this vertical.

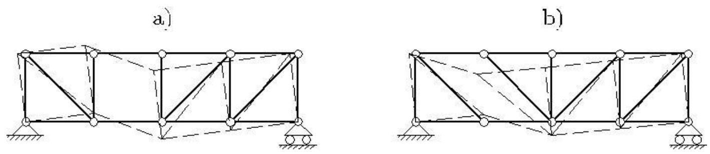


Fig. 4.10: Mechanisms by removing a diagonal or a vertical bar.

The qualitative arguments used to explain the implication of the mechanisms generated by removing a single bar from the truss can be made precise if the geometry of the infinitesimal motion of the mechanism is described exactly and used within the context of virtual work, described in Section

4.4 Method of joints

The magnitude of the forces in the bars of a statically determinate truss structure can be determined by the method of joints. The idea of the method of joints is to consider each joint as separated from the rest of the truss structure by the introduction of a virtual section. The parts on the two sides of the section will exchange identical but opposite forces, and by introducing the section and identifying these forces explicitly, they can be analyzed by the equilibrium equations. The principle is illustrated in its simplest form in Fig. 4.11. The left part of the figure shows a joint C in a planar truss loaded by the vertical force P and connected to the rest of the truss by the two bars AC and BC. A section is now introduced, separating the joint from the rest of the structure. The forces N_{AC} and N_{BC} , by which the bars act on the joint, are indicated as acting on the joint together with the load P . Thus, the joint C is acted on by three forces. The forces in the bars are considered as positive, when representing tension in the bar. Thus, the effect on the joint is a force directed away from the joint. By the law of action and reaction equal but opposite forces act on the bars. As seen, these forces represent tension in the bars. It is noted that the forces N_{AC} and N_{BC} are uniquely defined as being positive in tension. A representation in terms of vectors is less direct, as it would require identification of the part on which the force acts.

Equilibrium of the joint C requires that two force projection equations are satisfied. Vertical projection gives

By taking a vertical projection, the force N_{AC} in the horizontal bar AC does not contribute to the equilibrium equation.

The remaining bar force N_{AC} can be determined by projection on the direction orthogonal to BC. The present case is simple due to the angle 45°, and gives $N_{AC} = P$ directly. In many cases it will be more convenient to use a horizontal projection, whereby

$$\leftarrow \quad N_{AC} + N_{BC} \cos 45^\circ = 0 \;\Rightarrow\; N_{AC} = -N_{BC} \cos 45^\circ = P\,.$$

Thus, there is compression in the inclined bar BC, while the horizontal bar AC is in tension to ensure horizontal equilibrium.

In this simple illustration there were only two bar forces, and thus they could be determined directly by the two equilibrium equations available for the planar joint C. Most joints in truss structures are connected by more bars than there are equilibrium equations available for the particular joint. The bar forces can therefore only be determined sequentially, if the joints are considered in a certain order. This is illustrated in the following examples.

4.4.1 Planar truss structures

Many truss structures can conceptually be broken down into planar parts, and this section illustrates the calculation of bar forces for some simple planar trusses.

Example 4.1. Double triangle. Figure 4.14 shows a planar truss consisting of two tri-angles. There are 4 joints, providing 4 × 4 = 8 equilibrium equations, that determine the three reaction components R_A, R_D and R_{D^-} , plus the forces in the five bars in the truss.

In principle the analysis could be carried out completely on a joint by joint basis, starting from *C* and then proceeding through *B*, *A* and *D*. At each node there would be two

unknown forces, and at the end all bar forces and reactions will be determined. However, for many truss structures it is not possible to start from a node with only two unknown forces, unless appropriate reaction components are determined first. Therefore, the analysis of a statically determinate truss usually starts with determination of the reactions, using equilibrium of the full truss or parts of the truss as discussed in Section 1.5 on reactions. In the present case the reaction components R_{D^-} , R_D and R_A are determined by horizontal projection and moment about *A* and *D*, respectively:

4.4.4 Space trusses

In the case of space trusses hand calculation methods typically make use of special features of the truss geometry, and rapidly become fairly impractical for larger structures. Most space structures are therefore analyzed by the numerical Finite Element Method described in Section 4.5. A glimpse of the hand calculation procedure is provided by the following example.

4.3 Method of sections

The idea of introducing a section, whereby a structure is separated into two parts dates several hundred years back. The method of joints is a special case, in which the section is introduced in such a way that it separates precisely one joint. Hereby, all forces identified via the section pass through the released joint, and therefore are governed by force equilibrium of the joint. The idea of a section to identify the interaction of two parts of a structure is much more general and plays a central role in the theory of structures including beams and frames, but also in the general theory of continuous bodies as discussed in Chapter 8.

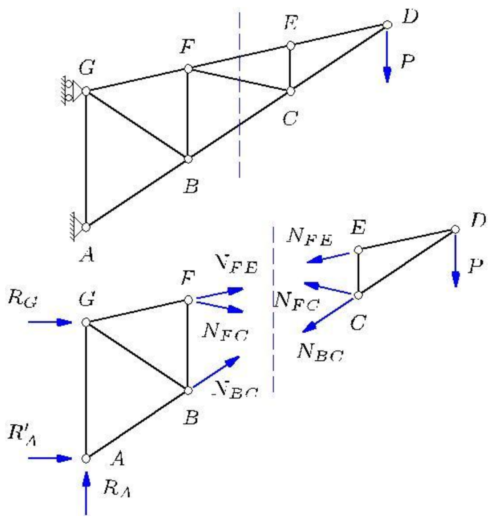


Fig. 4.40: Truss divided by vertical section.

4.3.1 Bar forces via the method of sections

The use of the method of sections to determine the bar forces in a planar truss is first illustrated by a simple example, and then summarized in concise form.

Example 4.5. V-truss by method of sections. Figure 4.41 shows a planar V-truss that has already been analyzed by the method of joints in Example 4.4. It is here analyzed by the method of sections to demonstrate the principles involved. First the reactions are determined by equilibrium of the full truss:

$$R_{A'} = 0\,, \quad R_A = \tfrac{1}{4} P\,, \quad R_C = \tfrac{1}{4} P\,.$$

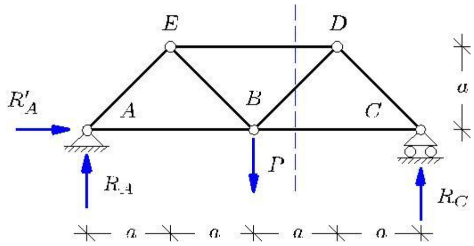


Fig. 4.41: Simply supported V-truss with load P .

In the method of joints the analysis would start from a node with two unknown bar forces – in the present case either of the joints A and C. This can also be used in the method of sections by introducing a vertical section, isolating the supported node. However, the method of sections also permits direct determination of the forces in the central bars. To illustrate the general procedure in the method of sections, a vertical section is introduced just to the right of the joint B as shown in the figure. This section intersects the bars BC, BD and DE and is used for calculating the corresponding bar forces N_{BC} , N_{BD} and N_{DE} .

Equilibrium of either the left or the right part of the structure is now used to determine the three bar forces N_{BC} , N_{BD} and N_{DE} . Note, that in contrast to the case of a single node the two parts have finite extent, and equilibrium therefore involves three equilibrium equations, and not just the two force projection equations associated with a single node. In the present problem equilibrium of the right part is the simpler, because it involves only one reaction component and not the load. The right part is shown in Fig. 4.44 together with all the forces acting on this part.

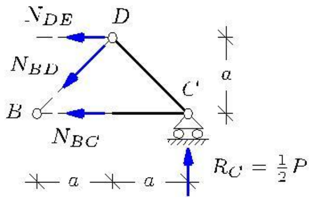


Fig. 4.44: Equilibrium in section.

The calculation of the bar forces proceeds in a systematic fashion by the following steps. First it is observed that two of the bar forces to be determined, namely N_{DE} and N_{BC} , are parallel. The remaining force N_{BD} in the inclined diagonal can then be determined by use of vertical equilibrium: This force intersects the two still unknown bar forces N_{BC} and N_{DE} in B and D, respectively. Thus, a moment equation about any of these two points will involve only one unknown bar force. The bar force N_{BC} is determined by moment about D:

$$a\, N_{BC} - a\, R_C = 0 \quad \Rightarrow \quad N_{BC} = R_C = \tfrac{1}{4} P$$

D .

Finally, the bar force N_{DE} is determined by moment about B:

$$B \qquad a\, N_{DE} + 4a\, R_C = 0 \quad \Rightarrow \quad N_{DE} = -4R_C = -P\,.$$

It is seen that each of the three forces N_{BD} , N_{BC} and N_{DE} has been calculated from an equilibrium equation, that does not involve any of the other two forces.

The method of sections for planar trusses can be formalized by the following procedure.

i) Determine the reactions on the truss structure.
ii) Divide the truss structure into two parts by a section, intersecting two or three bars.
iii) Consider each of the bar forces in turn and determine the bar force by: moment about the point of intersection of the other two forces, or projection on the transverse direction, if they are parallel.

It follows from the independence of the calculation of each bar force associated with a given section, that the order of the calculations can be changed, and indeed any of the bar forces can be calculated without calculating the others. As a consequence the method of sections can often be used to calculate isolated bar forces of a truss structure, without need for calculating the forces in adjoining bars. In addition to its computational simplicity, the method of sections often provides direct insight into the systematic variation of the forces in e.g. diagonals or verticals of regular truss structures.

Method of sections 57

4.3.4 Special types of planar trusses

Planar trusses appear in many contexts and often in the form of truss girders e.g. in bridges and cranes. Typical examples were illustrated in Fig. 4.4. The following examples illustrate the analysis for four types of trusses – three typical truss girders and a roof truss. The truss girder examples illustrate the method of analysis by introducing a typical section and calculating the bar forces associated with that section. The full analysis requires a sequence of similar sections, and omitting the repetitions associated with the typical section, the full results are summarized to illustrate how the girder layout determines the distribution of bar forces in the girder. The roof girder is more an individual type, where modification of the inner bars leads to modification of the analysis.

Example 4.6. N-truss girder. N-truss girders have constant or moderately changing height, filled with interchanging vertical and inclined bars. They find application in bridges, both traditional steel truss bridges and more recently in girders carrying the load on sus-pension bridges and cable stayed bridges. They also find use as supporting structure for roofs in industrial buildings where an important load component is a distributed vertical load. A simple illustration is shown as Fig. 4.43, where equal vertical forces P are applied to the 7 joints in the foot of a simply supported N-girder. For simplicity of analysis the horizontal spacing of the joints is taken equal to the height of the girder. While simplifying the expressions appearing in the analysis this has no principal impact on the procedure. The effect of the height of a regular truss girder under distributed load is discussed in the following example.

The reactions are determined by horizontal projection and moments about nodes I and A:

$$R_A = 0, \qquad R_A = R_I = \tfrac{7}{4} P .$$

The forces in the bars are then determined by introducing vertical and inclined sections as illustrated in Fig. 4.44.

The vertical section shown in the figure intersects the diagonal *SC* and the corresponding bars *SQ* and *BC* in the head and the foot, respectively. The bar forces in the head and the foot is parallel, and the force N_{SC} in the diagonal is therefore determined by vertical projection of all forces on the left part of the truss girder:

The force N_{BC} then follows from moment about *S*:

Finally, the force N_{SQ} in the head follows via moment about *C*:

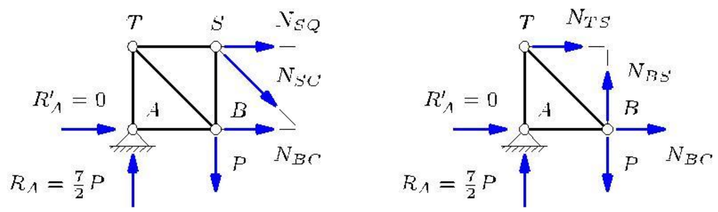


Fig. 4.44: Equilibrium at section.

The forces in the verticals are determined by use of inclined sections as shown in Fig. 4.44b. Vertical equilibrium determines the force N_{BS} in the vertical via

$$\uparrow \quad N_{BS} + R_A - P = 0 \quad \Rightarrow \quad N_{BS} = -\tfrac{5}{4}P.$$

The force N_{TS} in the head can be determined via moment about *B*, while the force N_{BC} in the foot has already been determined above by the vertical section. The procedure used for the four bar forces here is repeated along the left half of the girder, and the remaining bar forces then follow by symmetry. The resulting bar forces are shown in Fig. 4.45.

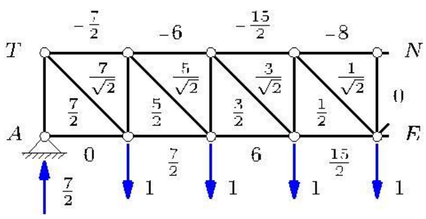


Fig. 4.45: Bar forces in N-truss.

4.4 Stiffness and deformation of truss structures

In most structures strength and stiffness play important roles, even if it implies just having 'enough strength' and 'sufficient stiffness'. Basically the concepts of strength and stiffness are material properties, and their effect in a structure depends on how the materials are used to form the structure. The concepts of strength and stiffness will be introduced gradually, when needed. Thus, the present section is devoted to stiffness of bars – the so-called uniaxial stiffness – while a general description of material stiffness and strength is given in Chapter 8.

4.4.1 Axial stress and strain

The stiffness of a bar relates the elongation *u* to the axial force *N* in the bar. The problem of relating these properties of the bar to material properties was discussed by Galileo Galilei in 1638. The essential part of this discussion is given here in a more modern form with reference to Fig. 4.35a.

The figure shows a homogeneous bar of length _ and cross-section area A. The bar is loaded by application of an axial force of magnitude N , which is considered positive in tension. This force leads to an elongation of the bar of magnitude u.

Now, a thought experiment is conducted, in which the bar is split lengthwise into two parts, each of area $^{1}{}_{4}$ A as shown in Fig. 4.35b. Each of these parts support half the load, while maintaining the original elongation. Therefore

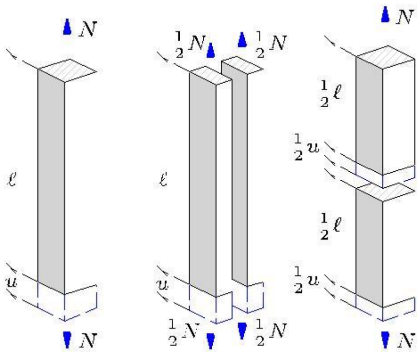


Fig. 4.35: Bar of length _ and cross-section area A.

In this formula the units are shown in square brackets. When the force is expressed in Newton [N] and the area in square meters [m^4] the resulting unit for the stress is Pascal [Pa]. The stress is an expression of the magnitude of the loading of the material.

4.4.4 Linear elastic bars

The operating stress level of a structure is normally considerably below the stress level that would lead to irreversible processes and failure. For many materials used in structures this implies that there is proportionality between the stress σ and the strain ε in any part of the structure. This behavior is called linear elasticity, and is described by the relation

$$\sigma = E\,\varepsilon\ ,E \text{ - Pa -}.$$

This relation is often called Hooke's law after Robert Hooke (1635–1703), who proposed it in 1675 and demonstrated it experimentally for several mechanical systems in 1678. The parameter E is called the modulus of elasticity. It is the factor of proportionality between the axial stress and axial strain in an experiment, where the loading is purely axial. Generally such an experiment leads to transverse contraction in addition to the axial elongation. The transverse contraction is not central to the present use in connection with trusses and will be dealt with in Chapter 8 in connection with the general discussion of elastic materials. The value of the elastic modulus varies between different materials as illustrated in Table 4.1 – from Gordon (4003).

It is seen that the elastic stiffness of the bar, AE, is the product of the material parameter E and the area A of the structural member. Thus, there are two contributing factors to the stiffness of a bar: its material stiffness, represented by E, and a geometric parameter of the structural element, here represented by the area A. This product form for the stiffness is general for beams and frames and plays an important role e.g. in design against column instability, discussed in Chapter 5.

4.4.3 Virtual work for truss structures

$$\mathrm{a} \cdot \mathrm{b} = \mathrm{a}^T \mathrm{b} = [\, a_x, a_y , a_z \,]\ b_y = a_x b_x + a_y\, b_y + a_z\, b_z \;. \quad (4.8)\ {}^{b}z$$

In the matrix product the components of the rows of the first factor are multiplied by the components of the columns of the second factor, and the terms are then added. In the present case of the scalar product this leads to the indicated summation, of which the result is a scalar, i.e. a number. A special case is the scalar product of a vector with itself,

$$|a|^4 = a^T a = [\, a_x, a_y , a_z \,] \begin{matrix} a_x \\ a_y \\ z \end{matrix} = a_x^{\,4} + a_y^{\,4} + a_z^{\,4} \, .$$

The result of this operation is the square of the length of the vector, indicated as $|a|$.

In matrix products the order of the factors is typically important. Thus, a^T a is the scalar product, while a a^T is the matrix

formed by products of the original vector components. Both the products a^T a and a a^T find application in the following theory of the elastic bar element.

Virtual work for a truss structure

The equality between external and internal virtual work for a bar only re-quires equilibrium of the bar. It must therefore apply to all bars of a truss structure, and therefore also to the sum of the contributions from each bar,

$$\underset{\text{bars}}{\delta V_{\text{ex}}} = \underset{\text{bars}}{\delta V_{\text{in}}} \, .$$

The external virtual work is now rewritten in terms of the external loads on the nodes. The basic principle is illustrated in Fig. 4.38 with reference to a two-dimensional truss structure. However, the principles are general and apply to three-dimensional trusses as well as to other structures.

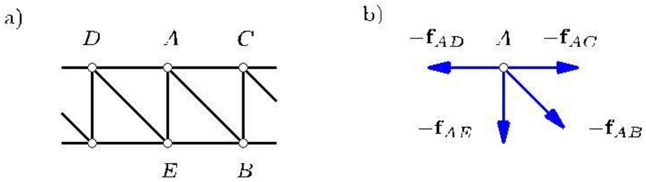
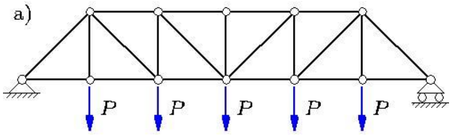


Fig. 4.38: Bar forces f_{A*} acting on node *A*.

Node *A* is acted on by all forces f_{AB} , f_{AC} , · · · from bars attached to this node. Thus, by the principle of action and reaction the similar forces but in opposite direction, $-f_{AB}$, $-f_{AC}$, · · · , are the forces acting on the node. In addition to these internal forces the node may also be acted on by an external force P_A corresponding to a load. Equilibrium of the node requires the vector sum of all forces on the node to vanish. When arranging internal forces on the left side of the equation and the external force on the right, the equilibrium condition reads where f_{A*} is the force in the bar $A*$, with $*$ denoting an arbitrary node connected to *A* by a bar – in the present example the nodes *B*, *C*, *D*, *E*.

4.4.4 Displacements of elastic truss structures

The procedure for calculation of the displacement of a node of an elastic truss structure is illustrated in Fig. 4.39. The top figure shows the structure with the actual loads – here consisting of vertical concentrated forces of magnitude *P* at all nodes in the girder foot. The bar forces corresponding to this load are calculated and denoted N_1^0, · · · , N_i^0, where the superscript 0 indicates that these are the actual forces in the bars. The node displacements corresponding to the actual load are similarly denoted $u^0_1, \cdots, u^0_j$.



The lower figure shows the same truss, but now loaded with only a single force $P^1 = 1$. This is an assumed load, used to determine the displacement component corresponding to the force P^1. The assumed concentrated load generates the bar forces $N_1^1, \cdots, N_i^1$.

Example 4.10. Displacement of node in a truss.

The calculation of node displacements in elastic trusses is illustrated by considering the simple cantilever truss in Fig. 4.40a, supporting a single vertical force *P* at node *C*. All bars are assumed to have identical elastic stiffness parameter *EA*. In this example it is desired to calculate both the vertical and the horizontal displacement components of node *C*. This is done by considering two independent load cases shown in Fig. 4.40b: a vertical unit test force P^1 and a horizontal unit test force P^4, both acting at node *C*.

The total computation consists of calculating three sets of bar forces: N_i^0 for the actual load, and N_i^1 and N_i^4 for the two test load cases. It is convenient to collect the bar lengths and forces in a table as illustrated by Table 4.4. The bar lengths are denoted by the symbol _ here, because *a* has been used for a specific dimension of the truss. If the bars had different elastic stiffness, the values $(EA)_i$ should also be included in the table.

The vertical displacement associated with bending of the truss girder is much larger than the axial displacement associated directly with the elongation of the bars *ED* and *DC*. This behavior will also be seen in beams, where most of the displacement is usually associated with bending.

4.5 Finite element analysis of trusses

The analysis methods developed so far in this chapter for trusses have mainly been based on statics, i.e. use of equilibrium conditions for the full truss and the individual bars. This approach works well for smaller structures and analysis carried out by hand. For larger truss structures and analysis carried out by computer a systematic approach in which the individual bar elements and nodes are treated in a repetitive way is desirable. In order to isolate an individual bar element from the rest of the structure it is desirable to consider the structure as flexible and to use the displacements of the nodes as the primary variables in the analysis. This represents a change in the point of view relative to the previous methods of nodes and sections, where the forces in the bars were the primary variables.

The basic idea of using the displacement of each of the nodes is illustrated in Fig. 4.41. Consider a flexible bar *AB* connecting the nodes *A* and *B*. Loading of the structure introduces a displacement u_A of node *A* and u_B of node *B*. These displacements may introduce a change of length of the bar *AB* from *a* to $a + \Delta a$. This change of length corresponds to a force N_{AB} in the bar.

An efficient analysis method, particularly suited for computer implementation, can be developed by expressing all bar forces in terms of the displacements u_A, u_B, $\cdots$ of the nodes, and then formulating and solving the equilibrium conditions for all the nodes of the structure. This task requires:

i) A constitutive relation between the elongation of each bar and the developed bar force.

ii) A general formulation for the elongation of a bar with arbitrary orientation in space, expressed in terms of the displacement of the two nodes of the bar.

iii) A systematic formulation of the equilibrium conditions for each node in terms of the relevant bar forces.
iv) Introduction of suitable support conditions.

These tasks are described in the following subsections, leading to the development of a small computer program Mini Truss.

4.5.1 Elastic bar element

The derivation of the elastic bar element consists in first determining the strain in the element in terms of the displacements of the element nodes, and then expressing the forces in the nodes in terms of this strain.

Strain in bar element

4.5.4 Finite Element Method for trusses

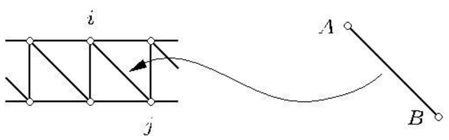
The next step is to use the information about the individual bar elements to set up conditions for all the nodes of the truss structure. The principle was illustrated in Fig. 4.38, where it was demonstrated that equilibrium of a node A requires the sum of the forces f_{A*} from all connecting bars to balance the external load f_A^{ex} at node A,

The forces in the individual bar elements are available from a element matrix relation of the form (4.34), and a central point in the formulation of the finite element method is the procedure used to assemble the contributions from the individual elements into a model for the structure.

Assembling the global stiffness matrix

The structure of the element stiffness matrix (4.33), where the force contribution at the element nodes is given in terms of the displacements of the nodes via a block matrix, leads to the following simple procedure to create a model of the complete truss structure.

i) Identify all nodes of the structure by numbers $1, \cdots, n$. Denote the corresponding coordinate set of the nodes by $x_1, \cdots, x_n$.
ii) Associate each bar element with two nodes, e.g. bar element *AB* with the nodes *i* and *j* of the structure, as illustrated in Fig. 4.44. This association between the element nodes *A, B* and the global structural nodes *i, j* is called the topology of the model.



iii) The contribution of the forces from the individual elements can now be obtained by placing the sub matrices from the element stiffness relation (4.33) in the global format as shown here,

When placed in this global format the displacements u_i and u_j contribute in the correct way to the internal forces f_i and f_j at nodes *i* and *j*. Adding the contributions from all elements to form the global stiffness matrix of the structure is seen to correspond to adding the internal forces at each of the nodes as prescribed in (4.33).

Support conditions

The model must also include provisions for supports, typically in the form of constraints on the displacements of certain nodes. Constraint of a node can typically be introduced by imposing one or more relations between the displacement components $u_i = [\, u_x, u_y , u_z \,]^T_i$ at the corresponding node. The introduction of such a constraint reduces the number of unknown displace-ment components in the model. Before presenting the implementation of gen-eral constraints two simple alternative methods of implementing the support conditions are discussed.

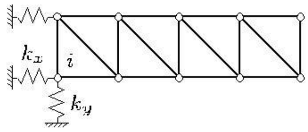


Fig. 4.45: Support springs attached to node *i*.

A simple method is to constrain the supported nodes by introducing stiff springs as illustrated in Fig. 4.45. The springs connect the node to a rigid support. They act essentially as bar elements, but because the 'other end' of the spring is fully constrained, the corresponding stiffness matrix to be included in the model is just a block matrix K_s appearing in the diagonal position corresponding to the supported node as illustrated by the corresponding global force stiffness matrix contribution

i

A support consisting of springs with stiffness constants k_x, k_y , k_z in the co-ordinate directions correspond to the diagonal block stiffness matrix

$$K_s = \begin{bmatrix} k_x & & \\ & k_y & \\ & & k_z \end{bmatrix} . \qquad (4.37)$$

This format permits some of the springs to have vanishing stiffness. The stiffness of an inclined spring with spring constant k_n along a direction described by the unit vector $n = [n_x, n_y , n_z\,]$ follows from the bar element stiffness matrix (4.34) as

Several springs can be applied to a node, simply be adding their stiffness contributions.

The use of stiff springs to represent constraints involves a compromise. Ideally the springs should be infinitely stiff, relative to the stiffness components of the structure itself. This would lead to ill-conditioning of the equations, and numerical round off errors in the solution of the equation system sets a limit on the magnitude of the spring constants that can be used without compromising the accuracy of the solution procedure. A simple modification of the idea of springs can be used when the involved degrees of freedom are constrained to zero. In that case the rows and columns corresponding to the constrained degrees of freedom can be set to zero, except for the diagonal element that is retained. When removing any loads associated with these degrees of freedom, the equations for the constrained degrees of freedom are uncoupled from the unconstrained degrees of freedom, and the equations can be solved directly, retaining the original size and organization of the matrix and displacement components.

A third more general alternative exists, that has a particularly elegant implementation in Matlab. The first step is to separate the displacement vector into two parts: a vector u_c containing the constrained displacement compo-nents, and a vector u_u containing the remaining unconstrained displacement components. Rearrangement of the equilibrium equations gives the block matrix equation format

The stiffness sub-matrices follow from rearrangement of the original stiffness matrix K. At the constrained degrees of freedom the total force consists of any imposed load f_c plus the reaction force components r produced by the support. In this format u_c represents imposed displacements that may be non-zero. The solution proceeds in two steps. First the unconstrained displacements u_u are obtained from the top part of the equations, and then the reaction forces r follow from the lower part as

Node data. The W-truss is described in an *xy*-coordinate system with origo at the center of the truss foot, and the *y*-axis vertical upwards. The width *a* and height *h* of the truss are given in parametric form in terms of variables a and h. The node coordinates are given in the form of an array X, with each node corresponding to one row. The first part of the data file then is

% Width 'a' and height 'h' of truss a = 14.0; h = 4.0;

The node coordinates [x,y] are given in the order of the node number, starting with node 1. Thus, the node number is not given explicitly, but implied by the row number in the node coordinate matrix X.

The program identifies the truss structure as being 4 or 3-dimensional by counting the number of columns in the node coordinate matrix X. If the truss is 3-dimensional the displacement vector for a node has 3 components, while a 4-dimensional structure has 4 displacement components per node.

Element data. The truss elements are defined in the topology matrix T. Each row of this matrix defines an element, by listing its two nodes by their node number, and by giving a third number identifying a set of element properties, area *A* and elasticity modulus *E*, given in a material property matrix H.

```
% Element property matrix H = [ A E ], H = [ 1.0 100.0
    1.0  100.0
    0.8  100.0 ];
```

In this example there are three element types: type 1 for the foot, type 4 for the head, and type 3 for the diagonals.

Loads. The loads are specified in the load matrix P. This matrix contains a row for each loaded node. The data line specifies the node number and the force components. In the present example node 6 is loaded by a force with components $[f_x, f_y] = [0.000, -1.000]$.

```
% Prescribed loads P = [node fx fy (fz)]
P = [ 6   0.000 -1.000 ];
```

Support conditions. The support conditions are given in the constraint matrix C. The constraint matrix contains a row for each constrained displacement component. In the present example there are 3 constrained displacement components u_1 and v_1 at node 1 and v_4 at node 4.

The optional parameter uc indicates the magnitude of a prescribed displace-ment component. If not included as a third column in C, constrained dis-placement components are set to zero.

Graphics. The MiniTruss program produces two plots of the structure: a plot of the initial undeformed geometry including node numbers, and an-other plot of the deformed structure after application of the load. For most real structures it is necessary to scale the displacements to be able to see the deformation. The coordinate window used for the plots is controlled via definition of the plot axes, specified in the array

```
% Axes used for geometry plots [Xmin Xmax Ymin Ymax] PlotAxes = [-0.55*a
0.55*a -0.5*h 1.4*h];
```

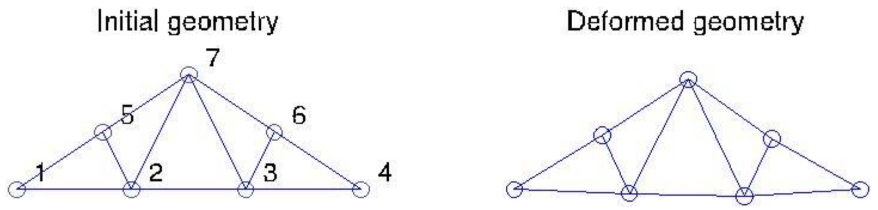


Fig. 4.47: Plots of initial and deformed geometry of W-truss.

The default in the program MiniTruss is to use the two top subplots in a 4×4 plot layout. For long trusses, as e.g. bridges etc., a 4×1 plot format can be introduced by changing to subplot (4,1).

Analysis process. The first step in an analysis with the MiniTruss program is to read the appropriate data file into memory. This is done either by writing W Truss in the command window, or by uploading W Truss to the Matlab editor and pressing the F5 key from the editor. The data is now available in active memory and the analysis is carried out by activating the script file MiniTruss.m, either by writing MiniTruss in the command window or by pressing the F5 key with MiniTruss.m in the editor.

The program activates the following processes. First the global load vector f is formed. Then the global stiffness matrix of the structure K is formed by the function kbar by collecting the stiffness contributions from all bar-elements. The function solves then solves the constrained equations, accounting for the support conditions. Finally the displacements are reshaped into vector components for each node in the matrix Un, and the internal forces s and strains e are obtained by post-processing.

Unit 5.THEORY OF ELASTISITY

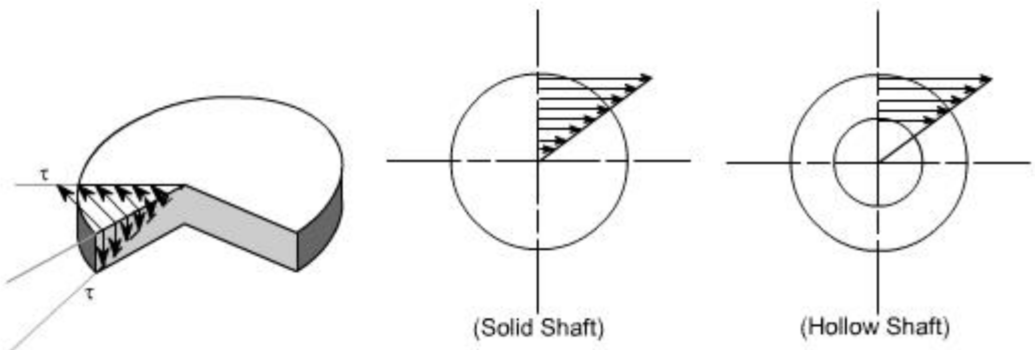
Distribution of shear stresses in circular Shafts subjected to torsion:
The simple torsion equation is written as

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

or

$$\tau = \frac{G\theta.r}{L}$$

This states that the shearing stress varies directly as the distance ‘r' from the axis of the shaft and the following is the stress distribution in the plane of cross section and also the complementary shearing stresses in an axial plane.



Hence the maximum strear stress occurs on the outer surface of the shaft where r = R
The value of maximum shearing stress in the solid circular shaft can be determined as

$$\frac{\tau}{r} = \frac{T}{J}$$

$$\tau_{max}\Big|_{r=d/2} = \frac{T.R}{J} = \frac{T}{\frac{\pi d^4}{32}} \cdot d/2$$

where d=diameter of solid shaft

$$\text{or} \quad \tau_{max^m} = \frac{16T}{\pi d^3}$$

From the above relation, following conclusion can be drawn
Power Transmitted by a shaft:
In practical application, the diameter of the shaft must sometimes be calculated from the power which it is required to transmit.
Given the power required to be transmitted, speed in rpm ‘N' Torque T, the formula connecting
These quantities can be derived as follows

$$P = T.\omega$$

$$= \frac{T.2\pi N}{60} \text{watts}$$

$$= \frac{2\pi NT}{60 \times 10^3} (kw)$$

Torsional stiffness: The torsional stiffness k is defined as the torque per radian twist.

$$k = \frac{T}{\theta}$$

$$\text{i.e} = \frac{GJ}{L}$$

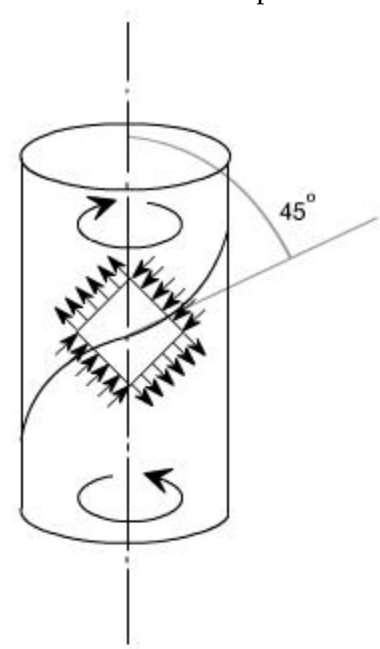
$$\text{or } k = \frac{G.J}{L}$$

For a ductile material, the plastic flow begins first in the outer surface. For a material which is weaker in shear longitudinally than transversely – for instance a wooden shaft, with the fibres parallel to axis the first cracks will be produced by the shearing stresses acting in the axial section and they will upper on the surface of the shaft in the longitudinal direction.

In the case of a material which is weaker in tension than in shear. For instance a, circular shaft of cast iron or a cylindrical piece of chalk a crack along a helix inclined at 45^0 to the axis of shaft often occurs.

Explanation: This is because of the fact that the state of pure shear is equivalent to a state of stress tension in one direction and equal compression in perpendicular direction.

A rectangular element cut from the outer layer of a twisted shaft with sides at 45^0 to the axis will be subjected to such stresses; the tensile stresses shown will produce a helical crack mentioned.



TORSION OF HOLLOW SHAFTS:

From the torsion of solid shafts of circular x – section, it is seen that only the material at the outer surface of the shaft can be stressed to the limit assigned as an allowable working stresses. All of the material within the shaft will work at a lower stress and is not being used to full capacity. Thus, in these cases where the weight reduction is important, it is advantageous to use hollow shafts. In discussing the torsion of hollow shafts the same assumptions will be made as in the case of a solid shaft. The general torsion equation as we have applied in the case of torsion of solid shaft will hold good

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

For the hollow shaft

$$J = \frac{\pi(D_0^4 - d_i^4)}{32} \quad \text{where } D_0 = \text{Outside diameter}$$

$$d_i = \text{Inside diameter}$$

$$\text{Let } d_i = \frac{1}{2}.D_0$$

$$\tau_{max^m}\Big|_{solid} = \frac{16T}{\pi D_0^3} \qquad (1)$$

$$\tau_{max^m}\Big|_{hollow} = \frac{T.D_0/2}{\frac{\pi}{32}(D_0^4 - d_i^4)}$$

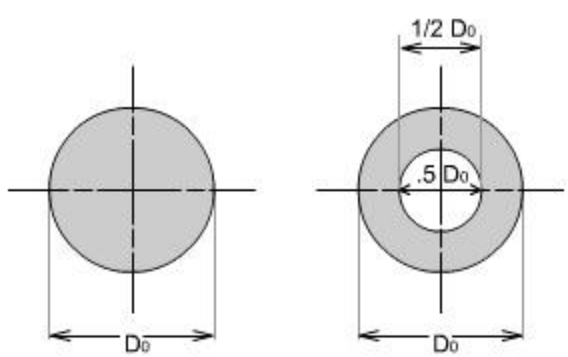
$$= \frac{16T.D_0}{\pi D_0^4\left[1-(d_i/D_0)^4\right]}$$

$$= \frac{16T}{\pi D_0^3\left[1-(1/2)^4\right]} = 1.066.\frac{16T}{\pi D_0^3} \qquad (2)$$

Hence by examining the equation (1) and (2) it may be seen that the in the case of hollow shaft is 6.6% larger then in the case of a solid shaft having the same outside diameter.

Reduction in weight:

Considering a solid and hollow shafts of the same length 'l' and density '□ ' withd_i = 1/2 D_o



Weight of hollow shaft

$$= \left[\frac{\pi D_0^{\,2}}{4} - \frac{\pi (D_0/2)^2}{4}\right] l \times \rho$$

$$= \left[\frac{\pi D_0^{\,2}}{4} - \frac{\pi D_0^{\,2}}{16}\right] l \times \rho$$

$$= \frac{\pi D_0^{\,2}}{4}\left[1 - 1/4\right] l \times \rho$$

$$= 0.75\,\frac{\pi D_0^{\,2}}{4}\, l \times \rho$$

Weight of solid shaft $= \frac{\pi D_0^{\,2}}{4} . l . \rho$

Reduction in weight $= (1 - 0.75)\frac{\pi D_0^{\,2}}{4} l \times \rho$

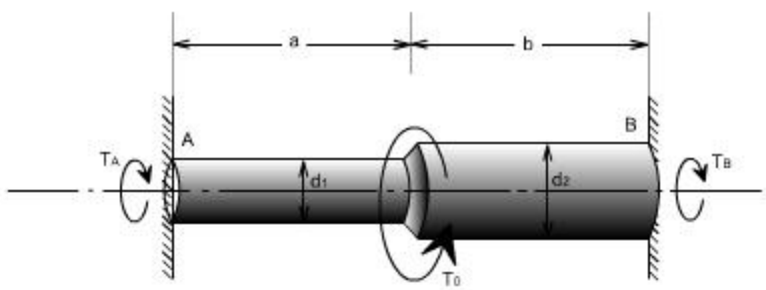
$$= 0.25\,\frac{\pi D_0^{\,2}}{4}\, l \times \rho$$

Hence the reduction in weight would be just 25%.

Illustrative Examples:

Problem 1

A stepped solid circular shaft is built in at its ends and subjected to an externally applied torque. T_0 at the shoulder as shown in the figure. Determine the angle of rotation $\square_0$ of the shoulder section where T_0 is applied ?



Solution: This is a statically indeterminate system because the shaft is built in at both ends. All that we can find from the statics is that the sum of two reactive torque T_A and T_B at the built – in ends of the shafts must be equal to the applied torque T_0

Thus $T_A + T_B = T_0$ ------ (1)

[From static principles]

Where T_A ,T_B are the reactive torque at the built in ends A and B. whereas T_0 is the applied torque

From consideration of consistent deformation, we see that the angle of twist in each portion of the shaft must be same.

$$\frac{T}{J} = \frac{G.\theta}{l}$$

$$\text{or } \theta_A = \frac{T_A a}{J_A G}$$

$$\theta_B = \frac{T_B a}{J_B G}$$

Using the relation for angle of twist $\Rightarrow \frac{T_A a}{J_A G} = \frac{T_B b}{J_B G} = \theta_0 \quad \text{or} \quad \frac{T_A}{T_B} = \frac{J_A}{J_B}.\frac{b}{a} \quad (2)$

N.B: Assuming modulus of rigidity G to be same for the two portions

So the defines the ratio of T_A and T_B

So by solving (1) & (2) we get

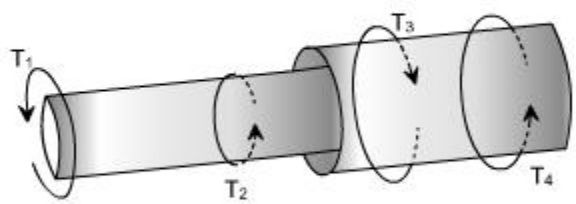
$$T_A = \frac{T_0}{1 + \frac{J_B a}{J_A b}}$$

$$T_b = \frac{T_0}{1 + \frac{J_a b}{J_b a}}$$

Using either of these values in (2) we have the angle of rotation θ_0 at the junction

$$\theta_0 = \frac{T_0.a.b}{[J_A.b + J_B.a]G}$$

Non Uniform Torsion: The pure torsion refers to torsion of a prismatic bar subjected to torques acting only at the ends. While the non uniform torsion differs from pure torsion in a sense that the bar / shaft need not to be prismatic and the applied torques may vary along the length.



Here the shaft is made up of two different segments of different diameters and having torques applied at several cross sections. Each region of the bar between the applied loads between changes in cross section is in pure torsion, hence the formula's derived earlier may be applied. Then form the internal torque, maximum shear stress and angle of rotation for each region can be calculated from the relation

$$\frac{T}{J} = \frac{\tau}{r} \text{ and } \frac{T}{J} = \frac{G\theta}{L}$$

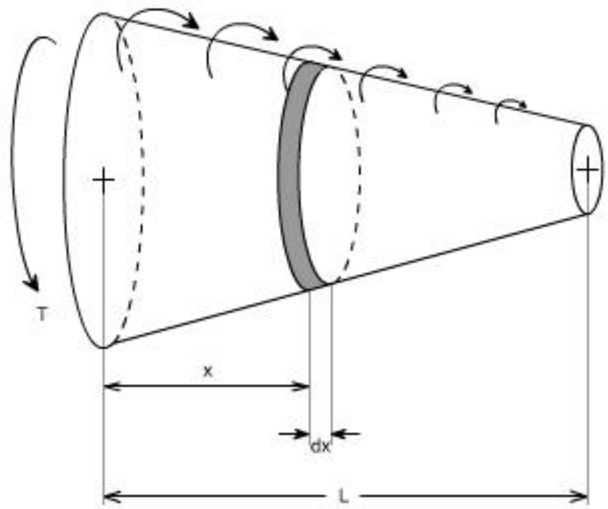
The total angle to twist of one end of the bar with respect to the other is obtained by summation using the formula

$$\theta = \sum_{i=1}^{n} \frac{T_i L_i}{G_i J_i}$$

i = index for no. of parts

n = total number of parts

If either the torque or the cross section changes continuously along the axis of the bar, then the □ (summation can be replaced by an integral sign (∫). i.e We will have to consider a differential element.



After considering the differential element, we can write $d\theta = \frac{T_x dx}{G I_x}$

Substituting the expressions for T_x and J_x at a distance x from the end of the bar, and then integrating between the limits 0 to L, find the value of angle of twist may be determined.

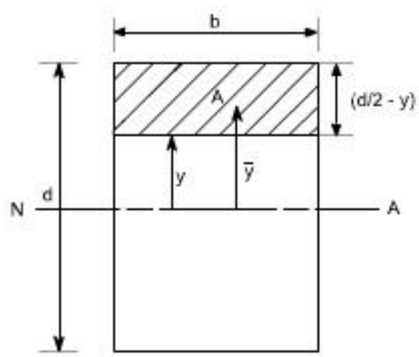
$$\theta = \int_0^L d\theta = \int_0^L \frac{T_x dx}{G I_x}$$

Shearing stress distribution in typical cross-sections:

Let us consider few examples to determine the sheer stress distribution in a given X- sections

Rectangular x-section:

Consider a rectangular x-section of dimension b and d



A is the area of the x-section cut off by a line parallel to the neutral axis. $\bar{y}$ is the distance of the centroid of A from the neutral axis

$$\tau = \frac{F.A.\bar{y}}{I.z}$$

for this case, $A = b\left(\frac{d}{2} - y\right)$

While $\bar{y} = \left[\frac{1}{2}\left(\frac{d}{2} - y\right) + y\right]$

i.e $\bar{y} = \frac{1}{2}\left(\frac{d}{2} + y\right)$ and $z = b\,;\, I = \frac{b.d^3}{12}$

substituting all these values, in the formula

$$\tau = \frac{F.A.\bar{y}}{I.z}$$

$$= \frac{F.b.\left(\frac{d}{2} - y\right).\frac{1}{2}.\left(\frac{d}{2} + y\right)}{b.\frac{b.d^3}{12}}$$

$$= \frac{\frac{F}{2}.\left\{\left(\frac{d}{2}\right)^2 - y^2\right\}}{\frac{b.d^3}{12}}$$

$$= \frac{6.F.\left\{\left(\frac{d}{2}\right)^2 - y^2\right\}}{b.d^3}$$

This shows that there is a parabolic distribution of shear stress with y.
The maximum value of shear stress would obviously beat the location y = 0.

Such that $\tau_{max} = \frac{6.F}{b.d^3}.\frac{d^2}{4}$

$$= \frac{3.F}{2.b.d}$$

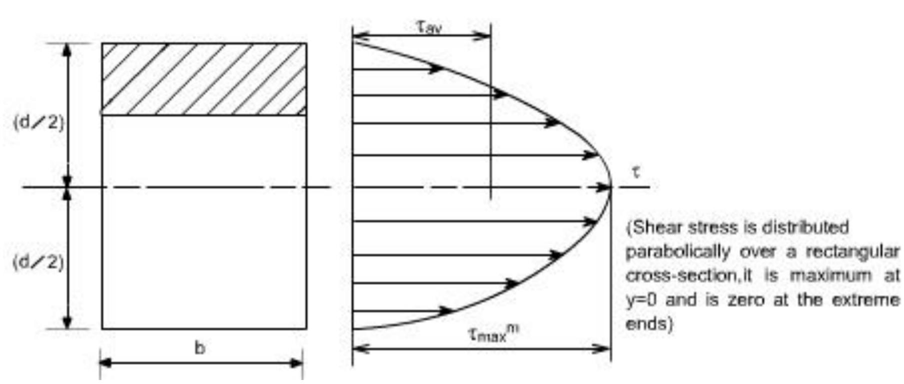
So $\boxed{\tau_{max} = \frac{3.F}{2.b.d}}$ The value of τ_{max} occurs at the neutral axis

The mean shear stress in the beam is defined as

$$\tau_{mean} \text{ or } \tau_{avg} = F/A = F/b.d$$

So $\tau_{max} = 1.5\,\tau_{mean} = 1.5\,\tau_{avg}$

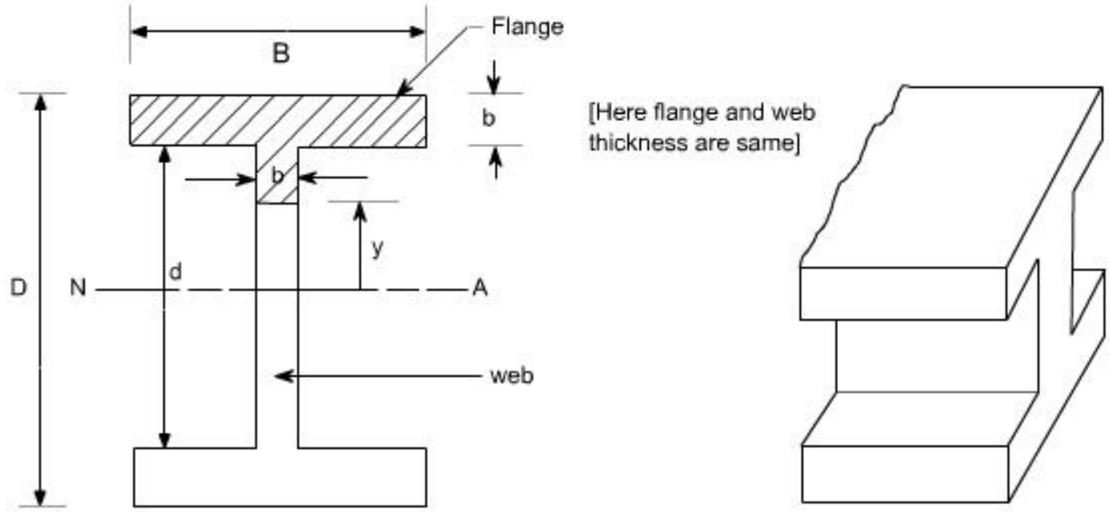
Therefore the shear stress distribution is shown as below.



It may be noted that the shear stress is distributed parabolic ally over a rectangular cross-section, it is maximum at y = 0 and is zero at the extreme ends.

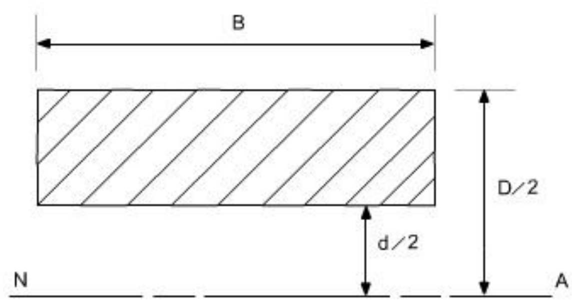
I - section:

Consider an I - section of the dimension shown below.



The shear stress distribution for any arbitrary shape is given as $\tau = \frac{F A \bar{y}}{Z I}$

Let us evaluate the quantity $A\bar{y}$, the $A\bar{y}$ quantity for this case comprise the contribution due to flange area and web area



Flange area

$$\text{Area of the flange} = B\left(\frac{D-d}{2}\right)$$

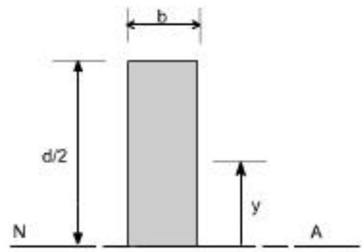
Distance of the centroid of the flange from the N.A

$$\bar{y} = \frac{1}{2}\left(\frac{D-d}{2}\right) + \frac{d}{2}$$

$$\bar{y} = \left(\frac{D+d}{4}\right)$$

Hence,

$$A\bar{y}\big|_{Flange} = B\left(\frac{D-d}{2}\right)\left(\frac{D-d}{4}\right)$$



Web Area

Area of the web

$$A = b\left(\frac{d}{2} - y\right)$$

Distance of the centroid from N.A

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} - y\right) + y$$

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} + y\right)$$

Therefore,

$$A\bar{y}\big|_{web} = b\left(\frac{d}{2} - y\right)\frac{1}{2}\left(\frac{d}{2} + y\right)$$

Hence,

$$A\bar{y}\big|_{Total} = B\left(\frac{D - d}{2}\right)\left(\frac{D + d}{4}\right) + b\left(\frac{d}{2} - y\right)\left(\frac{d}{2} + y\right)\frac{1}{2}$$

Thus,

$$A\bar{y}\big|_{Total} = B\left(\frac{D^2 - d^2}{8}\right) + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

Therefore shear stress,

$$\tau = \frac{F}{bI}\left[\frac{B\left(D^2 - d^2\right)}{8} + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)\right]$$

To get the maximum and minimum values of substitute in the above relation.

y = 0 at N. A. And y = d/2 at the tip.

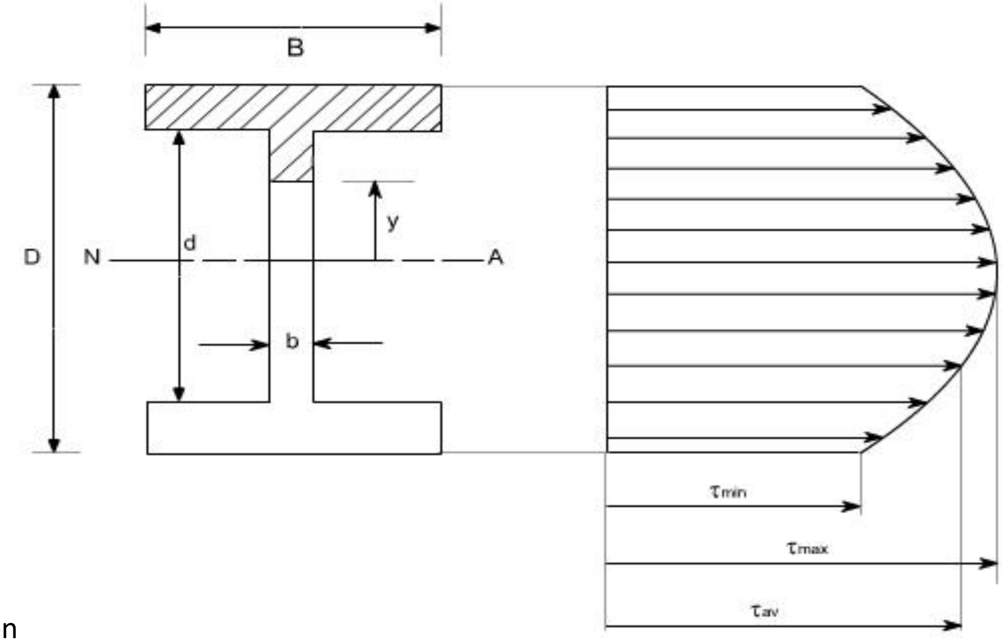
The maximum shear stress is at the neutral axis. I.e. for the condition y = 0 at N. A.

Hence, τ_{max} at $y = 0 = \frac{F}{8bI}\left[B\left(D^2 - d^2\right) + bd^2\right]$ .....(2)

The minimum stress occur at the top of the web, the term bd 2 goes off and shear stress is given by the following expression

τ_{min} at $y = d/2 = \frac{F}{8bI}\left[B\left(D^2 - d^2\right)\right]$ .... (3)

The distribution of shear stress may be drawn as below, which clearly indicates a parabolic

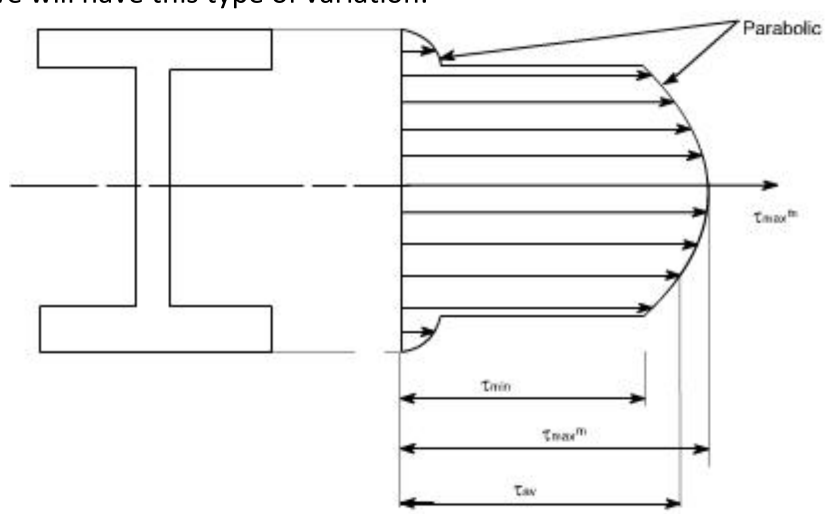


distribution

$$\tau_{max^m} = \frac{F}{8bI}\left[B\left(D^2 - d^2\right) + bd^2\right]$$

Note: from the above distribution we can see that the shear stress at the flanges is not zero, but it has some value, this can be analyzed from equation (1). At the flange tip or flange or web interface y = d/2.Obviously than this will have some constant value and than onwards this will have parabolic distribution.

In practice it is usually found that most of shearing stress usually about 95% is carried by the web, and hence the shear stress in the flange is neglible however if we have the concrete analysis i.e. if we analyze the shearing stress in the flange i.e. writing down the expression for shear stress for flange and web separately, we will have this type of variation.

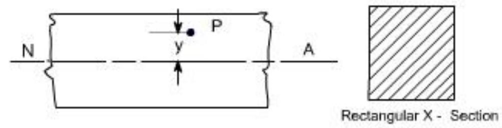


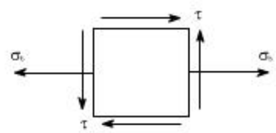
This distribution is known as the “top – hat” distribution. Clearly the web bears the most of the shear stress and bending theory we can say that the flange will bear most of the bending stress.

Principal Stresses in Beams

It becomes clear that the bending stress in beam σ_x is not a principal stress, since at any distance y from the neutral axis; there is a shear stress (or we are assuming a plane stress situation)

In general the state of stress at a distance y from the neutral axis will be as follows.





At some point 'P' in the beam, the value of bending stresses is given as

$$\sigma_b = \frac{My}{I} \text{ for a beam of rectangular cross - section of dimensions b and d; } I = \frac{bd^3}{12}$$

$$\sigma_b = \frac{12\,My}{bd^3}$$

whereas the value shear stress in the rectangular cross - section is given as

$$\tau = \frac{6F}{bd^3}\left[\frac{d^2}{4} - y^2\right]$$

Hence the values of principle stress can be determined from the relations,

$$\sigma_1, \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau_{xy}^2}$$

Letting $\sigma_y = 0;\ \sigma_x = \sigma_b$,the values of σ_1 and σ_2 can be computed as

$$\text{Hence } \sigma_1 / \sigma_2 = \frac{1}{2}\left(\frac{12My}{bd^3}\right) \pm \frac{1}{2}\sqrt{\left(\frac{12\,My}{bd^3}\right)^2 + 4\left(\frac{6\,F}{bd^3}\left(\frac{d^2}{4} - y^2\right)\right)^2}$$

$$\sigma_1, \sigma_2 = \frac{6}{bd^3}\left[My \pm \sqrt{\left\{M^2 y^2 + F^2\left(\frac{d^2}{4} - y^2\right)^2\right\}}\right]$$

Also,

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \text{putting } \sigma_y = 0$$

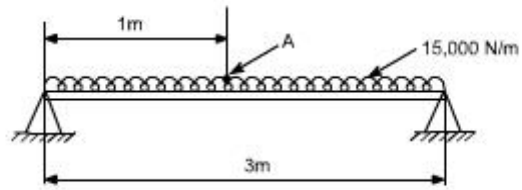
we get,

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x}$$

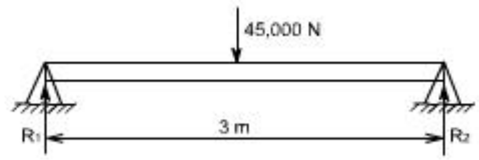
After substituting the appropriate values in the above expression we may get the inclination of the principal planes.

Illustrative examples: Let us study some illustrative examples, pertaining to determination of principal stresses in a beam

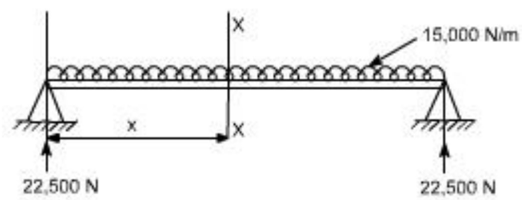
1. Find the principal stress at a point A in a uniform rectangular beam 200 mm deep and 100 mm wide, simply supported at each end over a span of 3 m and carrying a uniformly distributed load of 15,000 N/m.



Solution: The reaction can be determined by symmetry



$R_1 = R_2 = 22{,}500$ N



Consider any cross-section X-X located at a distance x from the left end.
Hence,
S. F $_{\text{at XX}}$ =22,500 – 15,000 x
B.M $_{\text{at XX}}$ = 22,500 x – 15,000 x (x/2) = 22,500 x – 15,000. x^2 / 2
Therefore,
S. F $_{\text{at X}}$ = 1 m = 7,500 N
B. M $_{\text{at X}}$ = 1 m = 15,000 N

$$S.F|_{x=1m} = 7{,}500\,N$$

$$B.M|_{x=1m} = 15{,}000\,N.m$$

$$\sigma_x = \frac{My}{I}$$

$$= \frac{15{,}000 \times 5 \times 10^{-2} \times 12}{10 \times 10^{-12} \times \left(20 \times 10^{-2}\right)^3}$$

$$\sigma_x = 11.25\,MN/m^2$$

For the compution of shear stresses

$$\tau = \frac{6F}{bd^3}\left[\frac{d^2}{4} - y^2\right] \qquad \text{putting } y = 50\,mm,\ d = 200\,mm$$

$$F = 7500\,N$$

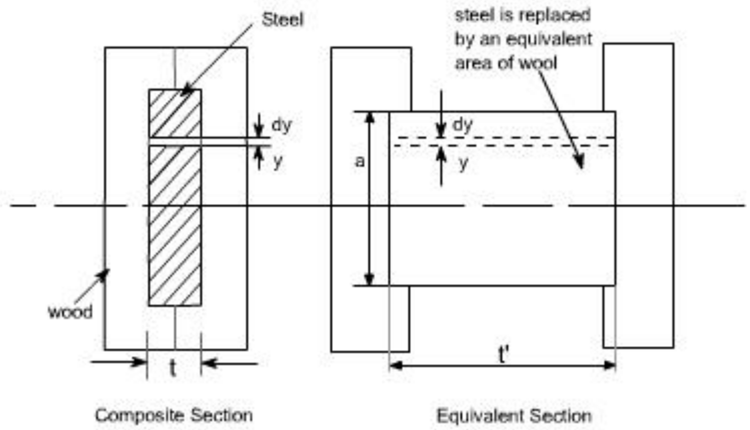
$$\tau = 0.422\,MN/m^2$$

Now substituting these values in the principal stress equation,
We get 11.27 $MN/m^{2,}$ - 0.025 MN/m^2

Bending Of Composite or Flitched Beams

A composite beam is defined as the one which is constructed from a combination of materials. If such a beam is formed by rigidly bolting together two timber joists and a reinforcing steel plate, then it is termed as a flitched beam.

The bending theory is valid when a constant value of Young's modulus applies across a section it cannot be used directly to solve the composite-beam problems where two different materials, and therefore different values of E, exist. The method of solution in such a case is to replace one of the materials by an equivalent section of the other.



Consider, a beam as shown in figure in which a steel plate is held centrally in an appropriate recess/pocket between two blocks of wood .Here it is convenient to replace the steel by an equivalent area of wood, retaining the same bending strength. i.e. the moment at any section must be the same in the equivalent section as in the original section so that the force at any given dy in the equivalent beam must be equal to that at the strip it replaces.

$$\sigma . t = \sigma ' . t' \text{ or } \boxed{\frac{\sigma}{\sigma'} = \frac{t'}{t}}$$

recalling $\sigma = E.\varepsilon$

Thus

$$\varepsilon E t = \varepsilon' E' t'$$

Again, for true similarity the strains must be equal,

$$\varepsilon = \varepsilon' \text{ or } E t = E' t' \text{ or } \boxed{\frac{E}{E'} = \frac{t'}{t}}$$

Thus, $\boxed{t' = \frac{E}{E'} . t}$

Hence to replace a steel strip by an equivalent wooden strip the thickness must be multiplied by the modular ratio E/E'.

The equivalent section is then one of the same materials throughout and the simple bending theory applies. The stress in the wooden part of the original beam is found directly and that in the steel found from the value at the same point in the equivalent material as follows by utilizing the given relations.

$$\frac{\sigma}{\sigma'} = \frac{t'}{t}$$

$$\frac{\sigma}{\sigma'} = \frac{E}{E'}$$

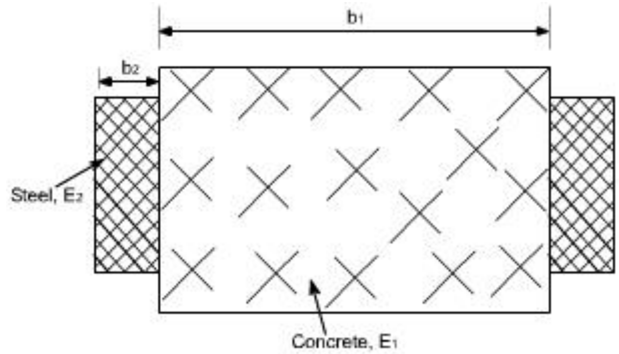
Stress in steel = modular ratio x stress in equivalent wood

The above procedure of course is not limited to the two materials treated above but applies well for any material combination. The wood and steel fletched beam was nearly chosen as a just for the sake of convenience.

Assumption

In order to analyze the behavior of composite beams, we first make the assumption that the materials are bonded rigidly together so that there can be no relative axial movement between them. This means that all the assumptions, which were valid for homogenous beams are valid except the one assumption that is no longer valid is that the Young's Modulus is the same throughout the beam.

The composite beams need not be made up of horizontal layers of materials as in the earlier example. For instance, a beam might have stiffening plates as shown in the figure below.



Again, the equivalent beam of the main beam material can be formed by scaling the breadth of the plate material in proportion to modular ratio. Bearing in mind that the strain at any level is same in both materials, the bending stresses in them are in proportion to the Young's modulus.